Some properties of a new subclass of multivalent analytic functions with negative coefficients Involoing the generalized Noor integral operator

Received :27/1/2016 Accepted :4/4/2016

AHMED MOHSIN MAHDI
Department of mathematics
College of Computer Science and Information Technology
University of Al-Qadisiya
Diwiniya- Iraq
E-mail: ahmed.Mohsin@qu.edu.iq

Abstract. In this paper, we study a new subclass $A_{k,p,m}^{\lambda,q}(\gamma,\mu,\eta,a,b,c)$ of multivalent analytic functions with negative coefficients defined in the unit disk by making use of the generalized Noor integral operator. We obtain some geometric properties for this class, like coefficient estimate, extreme points, inclusive property,radii of starlikeness and convexity, Hadamard product and weighted mean.

Keywords and phrases : Multivalent function, Extreme points, Radii of starlikeness and convexity, Hadamard product

Mathematics Classification QA 299.6 - 4.33

1. Introduction

Let $A(p,m)$ denote the class of all functions of the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p}, \quad (p,m \in N = 1,2,\cdots) \quad \ldots(1)$$

which are analytic and multivalent in the open unit disk $U = \{z \in C : |z| < 1\}$.

Let $k(p,m)$ denote the subclass of $A(p,m)$ cosisting of functions analytic and multivalent which can be expressed in the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p}, \quad (a_{n+p} \geq 0; p,m \in N = 1,2,\cdots) \quad \ldots(2)$$

For the functions $f \in k(p,m)$ given by (1.2) and $g \in k(p,m)$ defined by:

$$g(z) = z^p - \sum_{n=m}^{\infty} b_{n+p} z^{n+p}, \quad (b_{n+p} \geq 0)$$

we define the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by:

$$(f * g)(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z)$$

For real or complex number $a,b,e \notin \{0,-1.-2,\ldots\}$, the hypergeometric series is defined by

$${}_2F_1(a,b;e;z) = 1 + \frac{ab}{e}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{e(e+1)} \times \frac{z^2}{2!} + \ldots \quad \ldots(3)$$

We note that the series in(3)converges absolutely for all $z \in U$ so that it represents and analytic function in U.

The authors [2] introduced a function $\left(z^p {}_2F_1(a,b;e;z)\right)^{-1}$ given by

$$\left(z^p {}_2F_1(a,b;e;z)\right) * \left(z^p {}_2F_1(a,b;e;z)\right)^{-1} = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p), \quad \ldots(4)$$

which leads us to the following family of linear operators:

$$I_{p,m}^{\lambda}(a,b,e)\, f(z) = \left(z^p {}_2F_1(a,b;e;z)\right)^{-1} * f(z), \quad \ldots(5)$$

where

$f(z)\in S_p; a,b,e\in R\setminus z_0^- =\{0,-1,-2,\ldots\},$

$\lambda > -p, z\in U.$

It is evident that $I_{1,1}^{1}(k+1,e;e)=I_k$ is the Noor integral operator. The operator $I_{p,1}^{\lambda}(a,1e)=I_p^{\lambda}(a;e)$ was define recently by Cho et al. [1], $I_{p,1}^{1}(k+p,e;e)=I_{k,p}$ was introduced by Liu and Noor [3] (see also[4]), and $I_{p,1}^{\lambda}(a,\lambda+p;e)=I_p(a;e)$ was investigation by Saitoh [5]. By some easy calculations we obtain

$$I_{p,k}^{\lambda}(a,b,e)\, f(z)=z^p-\sum_{n=m}^{\infty}\frac{(c)_n(\lambda+p)_n}{(a)_n(b)_n}a_{p+n}z^{p+n}, \qquad \ldots (6)$$

where $(x)_k$ denote the pochhammer symbol defined by

$(x)_0=1 and (x)_k=x(x+1)(x+2)\ldots(x+k-1),$

$k\in N.$

It is easily verified from (6)that

$$z(I_{p,k}^{\lambda}(a,b,e)\, f(z))'=(\lambda+p)I_{p,k}^{\lambda+1}(a,b,e)\, f(z) -\lambda I_{p,k}^{\lambda}(a,b,e)\, f(z)$$

Differentiating above, q -times, we get

$$z(I_{p,k}^{\lambda}(a,b,e)\, f(z))^{(q+1)}=(\lambda+p)I_{p,k}^{\lambda+1}(a,b,e)\, f(z)^{(q)} -(\lambda+q)(I_{p,k}^{\lambda}(a,b,e)\, f(z))^{(q)}$$

where $q\in N_0, q<p$ and for each $f\in k(p,m)$ we have

$$f^{(q)}(z)=S(p,q)z^{p-q}-\sum_{n=m}^{\infty}S(n+p,q)a_{n+p}z^{n+p-q},$$

where $S(p,q)=\dfrac{p!}{(p-q)!}$.

For two functions f and g analytic in U, we say that the function f is subordinate to g in U, and write $f\prec g \quad (z\in U)$ if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0)=0$,$|w(z)|<1 \quad (z\in U)$

Such that

$f(z)=g(w(z)) \qquad (z\in U).$

Indeed, it is know that

$f(z)\prec g(z) \quad ,(z\in U)\Rightarrow f(0)=g(0),$

$f(U)\subset g(U).$

Furthermore, if the function g is univalent in U , then we have the following equivalence:

$f(z)\prec g(z) \quad ,(z\in U)\Leftrightarrow f(0)=g(0)$,

$f(U)\subset g(U)$

By making use of the Noor integral operator $I_{p,k}^{\lambda}(a,b;e)$ and the above mentioned principle of subordination between analytic functions, we introduce and investigate the following the new subclass of the class k_p of p-valent analytic functions.

Definition 1. A function $f\in k(p,m)$ is said to be in the class $Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$ if it satisfies

$$(\lambda+p)\frac{(I_{p,m}^{\lambda+1}(a,b,c)\, f(z))^{(q)}}{(I_{p,m}^{\lambda}(a,b,c)\, f(z))^{(q)}} -(\lambda+q)\prec\frac{p-q+(m-\eta)z}{1-\gamma z} \qquad \ldots(7)$$

where

$p,m\in N, q\in N_0, p>q, 0\le\gamma<1, 0<m\le 1,$

$0\le\eta<1, a,b,c\in R\setminus z_0^- and \lambda>-p$

By the definition of differential subordination and (6), (7) are equivalent to the following condition:

$$\left|\frac{z(I_{p,m}^{\lambda}(a,b,c)\, f(z))^{(q+1)}-(p-q)(I_{p,m}^{\lambda}(a,b,c)\, f(z))^{(q)}}{\gamma z(I_{p,m}^{\lambda}(a,b,c)\, f(z))^{(q+1)}+(m-\eta)(I_{p,m}^{\lambda}(a,b,c)\, f(z))^{(q)}}\right|<1, \qquad (z\in U) \qquad \ldots (8)$$

2. Coefficient Estimates

Theorem 2.1 Let $f\in k(p,m)$ be defined by (1.2). Then $f\in Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$ if and only if

$$\sum_{n=m}^{\infty}\frac{(n+p)!(n(1+\gamma)+m-\eta+\gamma(p-q))(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}a_{n+p}$$
$$\leq\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}, \quad ...(9)$$

where

$p,m\in N, q\in N_0, p>q, 0\leq\gamma<1, 0<m\leq 1,$

$0\leq\eta<1, a,b,c\in R\setminus z_0^- \, and \lambda>-p$

The result is sharp.

Proof. Assume that inequality (2.1) holds true and $|z|=1$. Then, we obtain

$$\left|z\left(I_{p,m}^{\lambda}(a,b,c)f(z)\right)^{(q+1)}-(p-q)\left(I_{p,m}^{\lambda}(a,b,c)f(z)\right)^{(q)}\right|$$
$$-\left|\gamma z\left(I_{p,m}^{\lambda}(a,b,c)f(z)\right)^{(q+1)}+(m,\eta)\left(I_{p,m}^{\lambda}(a,b,c)f(z)\right)^{(q)}\right|$$
$$=\left|-\sum_{n=m}^{\infty}\frac{n(n+p)!(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}a_{n+p}z^{n+p-q}\right|$$
$$-\left|\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}z^{p-q}-\sum_{n=n}^{\infty}\frac{(n+p)!(m-\eta+\gamma(n+p-q))(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n b_n}a_{n+p}z^{n+p-q}\right|$$
$$\leq\sum_{n=m}^{\infty}\frac{n(n+p)!(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}a_{n+p}\left|z^{n+p-q}\right|-\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}\left|z^{p-q}\right|+$$
$$\sum_{n=n}^{\infty}\frac{(n+p)!(m-\eta+\gamma(n+p-q))(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n b_n}\times a_{n+p}\left|z^{n+p-q}\right|-\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}\leq 0,$$

by hypothesis. Hence, by maximum modulus principle, we have $f\in Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$.

To show the converse, let $f\in Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$. Then

$$\left|\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(q+1)}-(p-q)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(q)}}{\gamma z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(q+1)}+(m-\eta)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(q)}}\right|$$
$$=\left|\frac{\sum_{n=m}^{\infty}\frac{n(n+p)!(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}a_{n+p}\left|z^{n+p-q}\right|}{\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}z^{p-q}-\sum_{n=n}^{\infty}\frac{(n+p)!(m-\eta+\gamma(n+p-q))(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n b_n}a_{n+p}z^{n+p-q}}\right|<1.$$

Since $Re(z)\leq|z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=m}^{\infty}\frac{n(n+p)!(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}a_{n+p}\left|z^{n+p-q}\right|}{\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}z^{p-q}-\sum_{n=n}^{\infty}\frac{(n+p)!(m-\eta+\gamma(n+p-q))(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n b_n}a_{n+p}z^{n+p-q}}\right\}<1. \quad \dots (10)$$

Now choosing the value of z on the real axis so that $\frac{z\left(I_{p,m}^{\lambda}(a,b,c)f(z)\right)^{(q+1)}}{z\left(I_{p,m}^{\lambda}(a,b,c)f(z)\right)^{(q)}}$ is real. Upon clearing the denominator of (10)and letting $z\to 1^-$ through real values, we obtain the inequality (9). Finally, the result (9) is sharp for the function

$$f(z)=z^p-\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\left(n(1+\gamma)+m-\eta+\gamma(p-q)\right)(c)_n(\lambda+p)_n}z^{n+p},(n\geq m, m\in N) \quad \dots (11)$$

3. Extreme points and inclusive property

In This section, we obtain extreme points and inclusive property for the class

$f\in Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$

Theorem 3. Let $f_p(z)=z^p$ and

$$f(z)=z^p-\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}z^{n+p},(n\ge m,m\in N).$$

Then $f\in Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$ if and only if can be expressed in the form

$$f(z)=\Theta p z^p+\sum_{n=m}^{\infty}\Theta_{n+p}f_{n+p}(z), \qquad \ldots (12)$$

where $\Theta_p\ge 0,\Theta_{n+p}\ge 0$ and $\Theta_p+\sum_{n=m}^{\infty}=1$

Proof. Suppose that f can be express as in (9). Then

$$f(z)=\Theta p z^p+\sum_{n=m}^{\infty}\Theta_{n+p}z^p-\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}z^{n+p},$$

$$z^p-\sum_{n=m}^{\infty}\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}\Theta_{n+p}z^{n+p}$$

Now

$$\sum_{n=m}^{\infty}\frac{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}$$

$$\times\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}\Theta_{n+p}=\sum_{n=m}^{\infty}\Theta_{n+p}=1-\Theta_p\le 1.$$

This shows that $Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$

By (), we have

Conversely, Assume that $Ak_{p,m}^{\lambda,q}(\gamma,m\eta,a,b,c)$.

$$a_{n+p}\le\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n},(n\ge m).$$

Therefore, we can set

$$\Theta_{n+p}=\frac{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p},(n\ge m)$$

and $\Theta_p=1-\sum_{n=m}^{\infty}\Theta_{n+p}$. Then

$$f(z)=Z^p-\sum_{n=m}^{\infty}a_{n+p}z^{n+p}$$

$$z^p-\sum_{n=m}^{\infty}\frac{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}\Theta_{n+p}z^{n+p}$$

$$=z^p-\sum_{n=m}^{\infty}(z^p-f_{n+p}(z))\Theta_{n+p}$$

$$=(1-\sum_{n=m}^{\infty}\Theta_{n+p})z_p+\sum_{n=m}^{\infty}\Theta_{n+p}f_{n+p}(z)$$

$$=\Theta_p z^p+\sum_{n=m}^{\infty}\Theta_{n+p}f_{n+p}(z),$$

that is the required representation.

Theorem 4. Let $0\le\gamma<1, 0<m\le 1, 0\le\eta<1$, $a,b,c\in R\setminus z_0^-$ and $\lambda>-p$. Then $Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)\subset Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$, where

$$\sigma=\frac{(\lambda+p+1)(1+m-\eta+\gamma(p-q+1))-(\lambda+p)(1+m+\gamma(p-q+1))(m-\eta+\gamma(p-q))}{(\lambda+p+1)(1+m-\eta+\gamma(p-q+1))-(\lambda+p)(m-\eta+\gamma(p-q))} \qquad \ldots (13)$$

Proof. Let the function f given by (2) belong to the class $Ak_{p,m}^{\lambda,q}(\gamma,m,\eta,a,b,c)$. Then, by using Theorem 2, we get

$$\sum_{n=m}^{\infty}\frac{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p}\le 1$$

In order to prove that $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$, we must have

$$\sum_{n=m}^{\infty}\frac{(p-q)!(n+p)!\big(n(1+\gamma)+m-\sigma+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\sigma+\gamma(p-q))(a)_n(b_n)}a_{n+p}\le 1 \qquad \ldots (14)$$

Note that (14) is satisfies if

$$\frac{(p-q)!(n+p)!\big(n(1+\gamma)+m-\sigma+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\sigma+\gamma(p-q))(a)_n(b_n)}a_{n+p}$$
$$\le\frac{(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p}. \qquad \ldots (15)$$

Rewriting the inequality (15),we have

$$\sigma=\frac{(\lambda+p+1)(1+m-\eta+\gamma(p-q+1))-(\lambda+p)(1+m+\gamma(p-q+1))(m-\eta+\gamma(p-q))}{(\lambda+p+1)(1+m-\eta+\gamma(p-q+1))-(\lambda+p)(m-\eta+\gamma(p-q))}$$
$$(n\ge m), m\in N \qquad \ldots (16)$$

Since the right -hand side of (16) is an increasing function of n, thus we get (13) and this completes the proof.

4. Radii of starlikeness and convexity

Theorem 4. If $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$, then f is starlike of order $\rho(0\le\rho<p)$ in the disk $|z|<r_1$, where

$$r_1=\inf_n\left\{\frac{(p-\rho)(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{(n+p-\rho)\,p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p}\right\}^{\frac{1}{n}}.,(n\ge m).$$

The result is sharp for the function f given by (11).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\le p-\rho\;\forall|z|<r_1. \qquad \ldots (17)$$

But

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{-\sum_{n=m}^{\infty}na_{n+p}z^{n+p}}{z^p-\sum_{n=m}^{\infty}na_{n+p}z^{n+p}}\right|\le$$
$$\frac{\sum_{n=m}^{\infty}na_{n+p}|z^n|}{1-\sum_{n=m}^{\infty}na_{n+p}|z^n|}.$$

Thus (17) will be satisfied if

$$\frac{\sum_{n=m}^{\infty}na_{n+p}|z^n|}{1-\sum_{n=m}^{\infty}na_{n+p}|z^n|}\le p-\rho,$$

or if

$$\sum_{n=m}^{\infty}\frac{(n+p-\rho)}{(p-\rho)}a_{n+p}|z|^n\le 1, \qquad \ldots (18)$$

with the said of (9), (18) is true if

$$|z|\le\left\{\frac{(p-\rho)(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{(n+p-\rho)p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p}\right\}^{\frac{1}{n}}.,(n\ge m).,$$

which follows the result.

Theorem 5. If $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$, then f is convex of order $\rho (0\le\rho<p)$ in the disk $|z|<r_1$, where

$$r_2=inf_n\left\{\frac{p(p-\rho)(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{(n+p)(n+p-\rho)p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p}\right\}^{\frac{1}{n}}.,(n\ge m).$$

The result is sharp for the function f given by (11).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|\le p-\rho\,\forall|z|<r_2. \quad \dots (19)$$

But

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|=\left|\frac{-\sum_{n=m}^{\infty}(n+p)na_{n+p}z^{n+p}}{pz^{p-1}-\sum_{n=m}^{\infty}(n+p)a_{n+p}z^{n+p-1}}\right|\le\frac{\sum_{n=m}^{\infty}n(n+p)a_{n+p}|z^n|}{p-\sum_{n=m}^{\infty}(n+p)a_{n+p}|z^n|}$$

Thus (19)will be satisfied if

$$\frac{\sum_{n=m}^{\infty}n(n+p)a_{n+p}|z^n|}{p-\sum_{n=m}^{\infty}(n+p)a_{n+p}|z^n|}\le p-\rho,$$

or if

$$\sum_{n=m}^{\infty}\frac{((n+p)n+p-\rho)}{p(p-\rho)}a_{n+p}|z|^n\le 1, \quad \dots (20)$$

with the aid of (9),(20) is true if

$$|z|\le\left\{\frac{p(p-\rho)(p-q)!(n+p)!\big(n(1+\gamma)+m-\eta+\gamma(p-q)\big)(c)_n(\lambda+p)_n}{(n+p)(n+p-\rho)p!(n+p-q)!(m-\eta+\gamma(p-q))(a)_n(b_n)}a_{n+p}\right\}^{\frac{1}{n}}.,(n\ge m).,$$

which follows the result.

5. Weighted mean

Definition 2. Let f and g be in the class $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$. Then the Weighted mean h_j of f and g is given by

$$h_j(z)=\frac{1}{2}[(1-j)\,f(z)+(1+j)\,g(z)],0<j<1.$$

Theorem 6. Let f and g be in the class $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$. Then the Weighted mean h_j of f and g is also in the class $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$

Proof. By Definition 2, we have

$$h_j(z)=\frac{1}{2}[(1-j)\,f(z)+(1+j)\,g(z)],0<j<1.$$

$$=\frac{1}{2}[(1-j)(z^p-\sum_{n=m}^{\infty}a_{n+p}z^{n+p})+$$

$$(1+j)(z^p-\sum_{n=m}^{\infty}b_{n+p}z^{n+p})],0<j<1.$$

$$=z^p-\sum_{n=m}^{\infty}\frac{1}{2}((1-j)a_{n+p}+(1+j)b_{n+p})z^{n+p}$$

since f and g are in the class $Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$, then by Theorem (2), we get

$$\sum_{n=m}^{\infty}\frac{(n+p)!(n(1+\gamma)+m-\eta+\gamma(p-q)(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}$$

$$\times a_{n+p}\le\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!},$$

and

$$\sum_{n=m}^{\infty}\frac{(n+p)!(n(1+\gamma)+m-\eta+\gamma(p-q)(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}$$

$$\times b_{n+p}\le\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!},$$

Hence

$$\sum_{n=m}^{\infty}\frac{(n+p)!(n(1+\gamma)+m-\eta+\gamma(p-q)(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}(\frac{1}{2}((1-j)a_{n+p}+(1+j)b_{n+p})z^{n+p})$$

$$=\frac{1}{2}(1-j)\sum_{n=m}^{\infty}\frac{(n+p)!(n(1+\gamma)+m-\eta+\gamma(p-q)(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}a_{n+p}$$

$$+\frac{1}{2}(1+j)\sum_{n=m}^{\infty}\frac{(n+p)!(n(1+\gamma)+m-\eta+\gamma(p-q)(c)_n(\lambda+p)_n}{(n+p-q)!(a)_n(b)_n}b_{n+p}$$

$$\le\frac{1}{2}(1-j)\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}+\frac{1}{2}(1+j)\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}=\frac{p!(m-\eta+\gamma(p-q))}{(p-q)!}.$$

This shows that $h_j\in Ak_{p,m}^{\lambda,q}(\gamma,m,\sigma,a,b,c)$

References

- N.E. Cho, O.S. Kwon and H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl. 292 (2004) 470â€“483.
- X. L. Fu, M. S. Liu, Some subclasses of analytic functions involving the generalized Noor integral operator, J. Math. Anal. Appl.323 (2006) 190-208.
- J.-L. Liu, K. I. Noor, Some properties of Noor integral operator, J. Nat. Geom. 21 (2002), 81â€“90.
- J. Patel, N.E. Cho, Some classes of analytic functions involving Noor integral operator, J. Math. Anal. Appl. 312 (2005) 564â€“575.
- H. Saitoh, A linear operator and its application of first order differential subordinations, Math. Japon. 44 (1996) 31â€“38.

بعض الخصائص الجديدة للدوال التحليلية متعددة التكافؤ مع معاملات سالبة مرتبطة مع تعميم مؤثر التكاملي Noor

تاريخ الاستلام 2016/1/27 **تاريخ القبول 2016/4/4**

احمد محسن مهدي

قسم الرياضيات الطبية

كلية علوم الحاسوب وتكنلوجيا المعلومات

جامعة القادسية

ديوانية -العراق

E-mail: ahmadiraqmath@yahoo.com

الخلاصة :

في هذا البحث، دراسة فئة فرعية جديدة من الدوال التحليلية متعددة التكافؤ مع معاملات السلبية المحددة في القرص الوحدة من خلال الاستفادة من مؤثر Noor التكاملي. حصلنا على بعض الخصائص الهندسية لهذه العوائل ، مثل تقدير معامل، ونقاط الحرجة ، نصف قطر starlikeness ونصف قطر التحدب و ضرب هادمارد.

الكلمات الافتتاحية:

دوال متعددة التكافؤ، النقاط الحرجة ، انصاق الاقطار ، ضرب Hadamard.