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GENERALIZED STABILITY OF AN ADDITIVE-QUADRATIC FUNCTIONAL EQUATION IN VARIOUS SPACES

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ABSTRACT. In this paper, using the direct and fixed point methods, we have established the generalized Hyers-Ulam stability of the following additive-quadratic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+2[f(x)+f(-x)]-[f(y)+f(-y)];$$

in non-Archimedean and intuitionistic random normed spaces.

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Keywords. generalized Hyers-Ulam stability; additive mapping; quadratic mapping; non-Archimedean random normed spaces; intuitionistic random normed spaces; fixed point.

1. INTRODUCTION

In classical analysis, norm of a vector is determined by a non-negative real number. However in reality, associating an exact value to the norm is not possible. In such cases, random norms are useful substitutes. The concept of random normed space extended by Alsina, Schweizer and Sklar in [1].

Ulam [23] in 1940 proposed a stability problem between a group and a metric group. In fact Ulam's stability problem, in the theory of functional equation, states that: if a map $f: G_1 \to G_2$ where G_1 is a group and G_2 is a metric group, satisfies a functional equation approximately, when is it close to an exact solution of that functional equation?

A partial answer to this question was given by Hyers [10] for Banach spaces.

Since then, many mathematicions generalized Hyers's theorem for different kinds of functional equations in several spaces and also by using fixed point method (see, e.g., [2–4, 24]).

The generalized Hyers-Ulam stability of different mixed type functional equations in random normed spaces, intuitionistic random normed spaces and non-Archimedean random normed spaces has been studied by many authors. (see, e.g., [3,11,13–15,17, 22]). In this paper we present the generalized Hyers-Ulam stability of the following mixed type additive and quadratic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+2[f(x)+f(-x)]-[f(y)+f(-y)] \quad (1.1)$$

under arbitrary t-norms by direct method in non-Archimedean random normed spaces and intuitionistic random normed spaces and under min t-norm by fixed point method in intuitionistic random normed spaces and provide an examples. Our research is a generalized of the Ravi and Suresh work [17] to various spaces

2. Preliminaries

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In this section we recall some definitions and results which will be used later inthe paper.

Definition 2.1. [5, 21]Acontinuous triangular norm (briefly a t-norm)is amapping $T: [0,1] \times [0,1] \to [0,1]$,such that, T satisfies the following conditions:

(1) T is commutative and associative;

(2) T is continuous;

(3) $T(a,1) = a \ \forall a \in [0,1]$;

(4) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$.

Example1 ([5, 7]). The following are the four basic t-norms:

(1) Minimum$t$-norm $T_M$given by $T_M(a,b) = \min(a,b)$;

(2) product t-norm$T_p$given by$T_p(a,b) = ab$;

(3) Lukasiewicz t-norm$T_L$given by$T_L(a,b) = \max(a+b-1,0)$;

(4) Weakest t-norm (drastic product) $T_D$given by

$$T_D(x,y) := \begin{cases} \min(x,y) & \text{if} \max(x,y) = 1 \\ 0 & \text{o.w} \end{cases}.$$

If T is a t-norm, then $x_T^{(n)}$is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$by 1; if $n = 0$ and $T(x_T^{n-1}, x)$ $if\, n \ge 1$. A t-norm$T$ is said to be of Had˘zi´c-type (denoted by $T \in H$)if the family$\{x_T^{(n)}\}_{n\in N}$is equicontinuous at$x = 1$,that is, for any$\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$such that

$$x > 1-\delta \Longrightarrow x_T^n > 1-\varepsilon \quad \forall\, n \ge 1$$

The$t$-norm T_M is a trivial example ofHad˘zi´c-type but T_p is not ofHad˘zi´c-type (see[5,7]).

Other important triangular norms are (see[8]):

(1) The Sugeno-Weber family $\{T_\lambda^{SW}\}_{\lambda\epsilon[-1,\infty]}, defined\ by\ T_{-1}^{SW} = T_D, T_\infty^{SW} = T_P$
and $T_\lambda^{SW}(x,y) = \max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right), \lambda \in (-1,\infty)$.

(2) The Domby family $\{T_\lambda^D\}_{\lambda\epsilon[0,\infty]}$, defined by $T_D, if\ \lambda = 0,\ T_M, if\ \lambda = \infty\ and$

and $T_\lambda^D(x,y) = \frac{1}{1+[(\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda]^{\frac{1}{\lambda}}}$, $\lambda \in (0,\infty)$.

(3) The Aczel-Alsina family $\{T_\lambda^{AA}\}_{\lambda\epsilon[0,\infty]}$,defined by $T_D, if\ \lambda = 0,\ T_M, if\ \lambda = \infty\ and T_\lambda^{AA}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{\frac{1}{\lambda}}}, \lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ the value $T(x_1, x_2, \ldots, x_n)$ defined by

$$T_{i=1}^0 x_i = 1, \qquad T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, \ldots, x_n)$$

T can also be extended to a countable operation taking for any sequence $\{x_n\}_{n\in N}$ in $[0,1]$. Moreover

$$T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^n x_i$$

The limit on the right-hand side of (2.1) exists since the sequence $\{T_{i=1}^n x_i\}_{n\in N}$ is nonincreasing and bounded from below.

PRPOPSITION 1([7,8]).

"(1)for $T \ge T_L$ the following implication hold:

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1-x_n) < \infty$$

(2) if T is of Had˘zi´c-type, then $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$for every sequence $\{x_n\}_{n\in N}$ in$[0,1]$ such that $\lim_{n\to\infty} x_n = 1$.

(3) if $T \in \{T_\lambda^{AA}\}_{\lambda\epsilon(0,\infty)} \cup \{T_\lambda^D\}_{\lambda\epsilon(0,\infty)}$, then

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1-x_n)^\lambda < \infty$$

(4) if $T \in \{T_\lambda^{SW}\}_{\lambda\epsilon[-1,\infty)}$,then

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1-x_n) < \infty.$$

Let Δ^+ denote the spaces of all distribution functions, that is, the spaces of all mappings $f: \mathbb{R} \cup \{-\infty, +\infty\} \to [0,1]$such that f monotone, nondecreasing, left continuous, $f(x) = 0$ and $f(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $f \in \Delta^+$ for which $\mathcal{L}^- f(+\infty) = 1$, where $\mathcal{L}^- f(x)$ denotes the left limitof the function f at the point x, that is, $\mathcal{L}^- f(x) = \lim_{t\to x^-} f(t)$.

The space Δ^+ is partially ordered by the usual point wise ordering of functions ,i,e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function H_0 given by

$$H_0(t) := \begin{cases} 0 \text{ if } t \le 0 \\ 1 \text{ if } t > 0 \end{cases}.$$

It is obvious that $H_0 \ge f$ for all $f \in D^+$."

3. non-Archimedean random normed spaces

"By a non-Archimedean field we mean a field K equipped with a function (valuation)|. |fromK in to $[0,\infty)$ such that

1.$|r| = 0$ if and only if $r = 0$;

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2. $|rs|=|r||s|$;

ly $|1| = |-1| = 1$ and $|n| \le 1 \ \forall\, n \ge 1$.

By the trivial valuation, we mean the mapping$|.|$Taking everything but 0 in to 1 and$|0| = 0$.

The most important examples of non-Archimedean spaces are P-adic numbers. In 1897, Hensel [9] discovered the P-adic numbers as a number theoretical analogue ofpower series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer$n_x \in \mathbb{Z}$such that $x = \frac{a}{b}p^{-n_x}$where a and b are integers not divisible by P. Then $|x|_p := p^{-n_x}$ defines anon-Archimedean norm on . The completion of $\mathbb{Q}\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field. Let X be a vector spaces over a field K with a non-Archimedean nontrivial valuation $|.|$, that is, there exists $a_0 \in$ Ksuch that $|a_0|$ is not in $\{0,1\}$.

A function $\|.\|: X \to [0,\infty)$is called non-Archimedean if it satisfies the following conditions:

1. $\|x\| = 0$ if and only if $x = 0$;
2. For any $r \in$ K, $x \in X$, $\|rx\| = |r|\|x\|$;
3. The strong triangle inequality(ultrametric) namely;$\|x + y\| \le \max\{\|x\|, \|y\|\} \ \forall\, x, y \in X$.

Then $(X, \|.\|)$ Is called anon-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\}$$

for all $n, m \ge 1$ with n$> m$, a sequence $\{x_n\}$ is a cauchy sequence in X if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence isconvergent."

Definition 2([5, 22]). "A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a Linear space over a non-Archimedean field K , T is a continuous t-norm, and $\mu$is a mapping from Xinto D^+ such that the following conditions hold:

(1)$\mu_x(t) = H_0(t) \forall\, t > 0$ iff $x = 0$;

(2)$\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right) \forall\, x \in X, t > 0 \text{ and } \alpha \ne 0$;

(3)$\mu_{x+y}(\max\{t,s\}) \ge T\left(\mu_x(t), \mu_y(s)\right) \forall\, x, y, z \in X \text{ and } t, s \ge 0$.

It is easy to see that, if (3) holds, then so is "(4)$\mu_{x+y}(\mathrm{t+s}) \ge T\left(\mu_x(t), \mu_y(s)\right)$.

Example2.

The triple (X, μ, T_M), where$(X, \|.\|)$ is a non-Archimedean normed linear and

$$\mu_x(t) := \begin{cases} 0 \text{ if } t \le \|x\|; \\ 1 \text{ if } t > \|x\|, \end{cases}$$

is a non-Archimedean RN-space.

Example 3. Let. Define (X, μ, T_M) is a non-Archimedean RN-space where$(X, \|.\|)$is a non-Archimedean normed linear space and $\mu_x(t) = \frac{t}{t+\|x\|} \ \forall x \in X, t > 0$.

Definition 3([5, 22]). "Let (X, μ, T_M)be a non-Archimedean RN-space. Let $\{x_n\}$be a sequence in X.

(1) The sequence $\{x_n\}$is said to be convergent if there exists $x \in X$such that

$$\lim_{n\to\infty} \mu_{x_n - x}(t) = 1,$$

for $t > 0$. In this case, the point $x$is called the limit of the sequence $\{x_n\}$.

(2) The sequence $\{x_n\}$in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and

$t > 0$, there exists $n_0 \ge 1$ such that, for all $n \ge n_0$ and $p > 0$

$$\mu_{x_{n+p} - x_n}(t) > 1 - \varepsilon.$$

(3) If each Cauchy sequence in X is convergent, then the random space is said to be complete and the non-Archimedean RN-space (X, μ, T_M) is called a non-Archimedean random Banach space."

Remark 1([5]). Let (X, μ, T_M) be a non-Archimedean RN-space. Then we have

$$\mu_{x_{n+p} - x_n}(t) \ge \min\{\mu_{x_{n+j+1} - x_{n+j}}(t) : j = 0,1,2,\ldots,p-1\}.$$

Thus, the sequence $\{x_n\}$is a Cauchy sequence in Xif, for any $\varepsilon > 0$and $t > 0$, there

exists $n_0 \ge 1$ such that, for all $n \ge n_0$,

$$\mu_{x_{n+1} - x_n}(t) > 1 - \varepsilon.$$

4. HYERS–ULAM STABILITY OF THE MIXED TYPE FUNCTIONAL

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EQUATION (1.1) IN NON-ARCHIMEDEAN RANDOM NORMED SPACES.

The functional equation (1.1) is called the additive-quadratic functional equation since the function $f(x) = ax^2 + bx$ is a solution for this equation where a and b are constants. One can easily show that an even mapping $f: X \to Y$ satisfies equation (1.1) if and only if the even mapping $f: X \to Y$ is a quadratic mapping, that is,

$$f(2x+y) + f(2x-y) = 2[f(x+y) + f(x-y)] + 4f(x) - 2f(y).$$

Also, one can easily show that an odd mapping $f: X \to Y$ satisfies equation (1.1) if and only if the odd mapping $f: X \to Y$ is an additive mapping, that is,

$$f(2x+y) + f(2x-y) = 2[f(x+y) + f(x-y)].$$

In this section we investigate the stability of the additive-quadratic functional equation (1.1), where f:$X \to Y$,$f(0) = 0$. since f is a sum of an even function and an odd function, therefore f satisfies the above functional equation ⟺it is additive-quadratic. Next we define a random approximately additive-quadratic mapping. Let ψ be a distribution function on $X \times X \times [0,\infty)$such that $\psi(x,y,.)$ is nondecreasing and

$$\psi(cx,cy,t) \geq \psi\left(x,y,\frac{t}{|c|}\right) \qquad \forall x \in X, c \neq 0$$

Definition 4. A mapping $f: X \to Y$ is said to be ψ -approximately additive-quadratic if

$$\mu_{D_s f(x,y)}(t) \geq \psi(x,y,t), \quad \forall x,y \in X, t > 0 \tag{4.1}$$

Where $D_s f(x,y) \coloneqq f(2x+y) + f(2x-y) - 2[f(x+y) + f(x-y)] - 2[f(x) + f(-x)] + [f(y) + f(-y)]$, $\forall x,y \in X, t > 0$.

THEOREM 1. *Let* $f: X \to Y$be an even andψ-approximately additive-quadratic function. If, for some $\alpha \in \mathbb{R}, \alpha > 0$ and for some positiveinteger$k$with

$$|2^k| < \alpha$$

$$\psi(2^{-k}x, 2^{-k}y, t) \geq \psi(x,y,\alpha t), \tag{4.2}$$

And
$$\lim_{n\to\infty} T_{j=n}^{\infty} M(x, \tfrac{\alpha^j t}{|2|^{kj}}) = 1, \tag{4.3}$$

for all x ∈ X $and\ t > 0$, then there exists a unique quadratic mapping$\Phi: X \to Y$such that:

$$\mu_{f(x)-\Phi(x)}(t) \geq T_{i=1}^{\infty} M(x, \tfrac{\alpha^{i+1}t}{|2|^{ki}}), \quad \forall x \in X, t > 0 \tag{4.4}$$

Where
$M(x,t) \coloneqq T[\psi(x,0,t), \psi(2x,0,t), \dots, \psi(2^{k-1}x,0,t)] \forall x \in X, t > 0.$

Proof. First we show, by induction on *j*, that, for all x ∈ X, $t > 0$ and $j \geq 1$

$$\mu_{f(2^j x) - 4^j f(x)}(t) \geq M_j(x,t) = T[\psi(x,0,t), \psi(2x,0,t), \dots, \psi(n^{j-1}x,0,t)] \tag{4.5}$$

Putting $y = 0$ in (4.1) we have

$$\mu_{2f(2x) - 8f(x)}(t) \geq \psi(x,0,t)$$

then

$$\mu_{f(2x)-4f(x)}(t) \geq \psi(x,0,2t) \geq \psi(x,0,t) \forall x \in X, t > 0.$$

This prove (4.5) for j=1. Assume that (4.5) hold for some j >1 . Replacing y by 0 and x by $2^j x$ in (4.1) we get

$$\mu_{f(2^{j+1}x)-4f(2^j x)}(t) \geq \psi(2^j x,0,t) \forall x \in X, t > 0. \tag{4.6}$$

Since |4| ≤ 1, it follows that

$$\mu_{f(2^{j+1}x)-4^{j+1}f(x)}(t) \geq T(\mu_{f(2^{j+1}x)-4f(2^j x)}(t), \mu_{4f(2^j x)-4^{j+1}f(x)}(t))$$

$$= T(\mu_{f(2^{j+1}x)-4f(2^j x)}(t), \mu_{f(2^j x)-4^j f(x)}\left(\tfrac{t}{|4|}\right))$$
$$\geq T(\mu_{f(2^{j+1}x)-4f(2^j x)}(t), \mu_{f(2^j x)-4^j f(x)}(t))$$
$$\geq T(\psi(2^j x,0,t), M_j(x,t))$$

$=M_{j+1}(x,t), \qquad \forall x \in X, t > 0.$

So

$$\mu_{f(2^j x)-4^j f(x)}(t) \geq M(x,t)$$

Holds for all $j \geq 1$, in particular, we have

$$\mu_{f(2^k x)-4^k f(x)}(t) \geq M(x,t) \quad \forall x \in X, t > 0. \tag{4.7}$$

Replacing x by $2^{-(kn+k)}x$ in (4.7) and using the inequality (4.2), we have

$$\mu_{f\left(\frac{x}{2^{kn}}\right)-4^k f\left(\frac{x}{2^{k+kn}}\right)}(t) \geq M\left(\frac{x}{2^{k+kn}}, t\right) \geq M(x, \alpha^{n+1}t),$$

$\forall x \in X, t > 0$ and $m \geq 0$. Then we have

$$\mu_{(2^{6k})^n f\left(\frac{x}{2^{kn}}\right)-4^k(4^k)^n f\left(\frac{x}{2^{k+kn}}\right)}(t) \geq M\left(x, \frac{\alpha^{n+1}t}{|4^k|^n}\right) \geq M\left(x, \frac{\alpha^{n+1}t}{|2^k|^n}\right),$$

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$\forall x \in X, t > 0$ and $m \geq 0$. *So*

$$\mu_{(4^k)^n f\left(\frac{x}{2^{kn}}\right) - 4^{k(n+1)} f\left(\frac{x}{2^{k(n+1)}}\right)}(t) \geq M\left(x, \frac{\alpha^{n+1} t}{|2^k|^n}\right),$$

$\forall x \in X, t > 0.$

$$\mu_{(4^k)^n f\left(\frac{x}{2^{kn}}\right) - 4^{k(n+1)} f\left(\frac{x}{2^{k(n+1)}}\right)}(t)$$
$$\geq T_{j=n}^{n+(p-1)}\left(\mu_{(4^k)^j f\left(\frac{x}{2^{kj}}\right) - 4^{k(j+1)} f\left(\frac{x}{2^{k(j+1)}}\right)}(t)\right)$$
$$\geq T_{j=n}^{n+(p-1)} M\left(x, \frac{\alpha^{j+1} t}{|2^k|^j}\right)$$

$\forall x \in X, t > 0$. Since $\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^{j+1} t}{|2^k|^j}\right) = 1, \forall x \in X, t > 0$, it follows that $\{(4^k)^n f\left(\frac{x}{(2^k)^n}\right)\}$is a cauchy sequence in the non-Archimedean random Banach space (Y, μ, T). Hence, we can define a mapping $\Phi: X \to Y$ such that

$$\lim_{n\to\infty} \mu_{(4^k)^n f\left(\frac{x}{(2^k)^n}\right) - \Phi(x)}(t) = 1, \forall x \in X, t > 0.$$

Since$f: X \to Y$is even, $\Phi$is an even mapping. It follows that for all $x \in X$and $t > 0$.

$$\mu_{f(x) - (4^k)^n f\left(\frac{x}{(2^k)^n}\right)}(t)$$
$$= \mu_{\sum_{i=0}^{n-i} (4^k)^i f\left(\frac{x}{(2^k)^i}\right) - (4^k)^{i+1} f\left(\frac{x}{(2^k)^{i+1}}\right)}(t)$$
$$\geq T_{j=0}^{n-1}\left(\mu_{(4^k)^i f\left(\frac{x}{2^{ki}}\right) - 4^{k(i+1)} f\left(\frac{x}{2^{k(i+1)}}\right)}\right)(t)$$
$$\geq T_{i=0}^{n-1}\left(M\left(x, \frac{\alpha^{i+1} t}{|2^k|^i}\right)\right),$$

and so

$$\mu_{f(x) - \Phi(x)}(t)$$
$$\geq T\left[\mu_{f(x) - (4^k)^n f\left(\frac{x}{(2^k)^n}\right)}(t), \mu_{(4^k)^n f\left(\frac{x}{(2^k)^n}\right) - \Phi(x)}(t)\right]$$
$$\geq T\left(T_{i=0}^{n-1}\left(M\left(x, \frac{\alpha^{i+1} t}{|2^k|^i}\right)\right), \mu_{(4^k)^n f\left(\frac{x}{(2^k)^n}\right) - \Phi(x)}(t)\right)$$

taking $n \to \infty$ we have

$$\mu_{f(x) - \Phi(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2^k|^i}\right),$$

Which prove (4.4). Since T is continuous, from a well-known result in probabilistic metric space (see e.g., [21, Chapter 12]) it follows that

$$\lim_{n\to\infty} \mu_{D_{\Phi f(x,y)}}(t)$$
$$= \mu_{D_{\Phi(2x+y) + \Phi(2x-y) - 2[\Phi(x+y) + \Phi(x-y)] - 2[\Phi(x) + \Phi(-x)] + [\Phi(y) + \Phi(-y)]}}(t),$$

$\forall x, y \in X, t > 0$, where

$$D_{\Phi f(x,y)} = 4^{nk} f\left(\frac{2x+y}{2^{kn}}\right) + 4^{nk} f\left(\frac{2x-y}{2^{kn}}\right) - 2\left[4^{nk} f\left(\frac{x+y}{2^{kn}}\right) + 4^{nk} f\left(\frac{x-y}{2^{kn}}\right)\right] - 2\left[4^{nk} f\left(\frac{x}{2^{kn}}\right) + 4^{nk} f\left(\frac{-x}{2^{kn}}\right)\right] + 4^{nk}\left(f\left(\frac{y}{2^{kn}}\right) + f\left(\frac{-y}{2^{kn}}\right)\right].$$

On the other hand, replacing x, y by $2^{-kn}x 2^{-kn}y$in (4.1) and using (4.2) we get

$$\mu_{D_{\Phi f(x,y)}}(t) \geq \psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{|4^k|^n}\right)$$
$$\geq \psi\left(2^{-kn}x, 2^{-kn}y, \frac{t}{|2^k|^n}\right)$$
$$\geq \psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right),$$

$\forall x, y \in X, t > 0$.Since $\lim_{n\to\infty} \psi\left(x, y, \frac{\alpha^n t}{|2^k|^n}\right) = 1$, we show that Φ is quadratic mapping. Finally if $\acute{\Phi}: X \to Y$ is another quadratic mapping such that

$$\mu_{\acute{\Phi} - f(x)}(t) \geq M(x, y) \forall x, y \in X, t > 0,$$

then, for all $m \in N$, $x \in X, t > 0$,

$$\mu_{\Phi(x) - \acute{\Phi}(x)}(t) \geq$$
$$T\left(\mu_{\Phi(x) - (4^k)^n f\left(\frac{x}{|2^k|^n}\right)}, \mu_{(4^k)^n f\left(\frac{x}{|2^k|^n}\right) - \acute{\Phi}(x)}(t)\right),$$

Therefore, we conclude that $\Phi = \acute{\Phi}$ this completes the proof. ∎

In theorem (1) if f is an odd mapping, then the following theorem can be proved similarly.

THEOREM 2. Let $f: X \to Y$ be an odd and ψ - approximately additive-quadratic function. If, for some $\alpha \in \mathbb{R}, \alpha > 0$ and for some positive integer k with

$$|2^k| < \alpha$$
$$\psi(2^{-k}x, 2^{-k}y, t) \geq \psi(x, y, \alpha t),$$

and $\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1$,

$\forall x \in X$ *and* $t > 0$, then there exists a unique additive mapping$\Phi: X \to Y$such that:

$$\mu_{f(x) - \Phi(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right),, \quad \forall x \in X, t > 0$$

Where

$$M(x, t) :=$$
$$T[\psi(x, 0, t), \psi(2x, 0, t), \dots, \psi(2^{k-1}x, 0, t)] \forall x \in X, t > 0..$$

COROLLARY 1."Let K be a non-Archimedean field, X be a vector space over K and (X, μ, T)be non-Archimedean random Banach space over K under the t-norm $T \in H$. Let $f: X \to Y$ be an even and ψ - approximately additive-quadratic mapping. If, for some $\alpha \in \mathbb{R}, \alpha > 0$ and for some positive integer k with

$$|2^k| < \alpha$$

$\psi(2^{-k}x, 2^{-k}y, t) \geq \psi(x, y, \alpha t)$, for all $x \in X$ *and* $t > 0$.Then there exists a uniquequadratic mapping $Q: X \to Y$ such that

$$\mu_{f(x) - Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2^k|^i}\right),, \quad \forall x \in X, t > 0,$$

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where$M(x,t) := T[\psi(x,0,t),\psi(2x,0,t),\dots,\psi(2^{k-1}x,0,t)]\forall x \in X, t > 0$.."

*Proof.*Since

$$\lim_{j\to\infty} M(x,\frac{\alpha^j t}{|2^k|^j})=1,$$

$\forall$x ∈ X and$t > 0$ and $T$is of Hadˇzi´c type, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M(x,\frac{\alpha^j t}{|2^k|^j})=1,$$

$\forall$x ∈ X and$t > 0$. Now, if we can apply theorem 1, then we can get the conclusion.

■

Example **4**. Let (X,μ,T_M) be a non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t+\|x\|}$$

$\forall$x ∈ X and$t > 0$ and (X,μ,T_M) be a complete non-Archimedean random normed space. Define

$$\psi(x,y,t) = \frac{t}{1+t}$$

It is easy to see that (4.2) holds for $\alpha = 1$. Also, since $M(x,t) = \frac{t}{1+t}$,

We have

$$\lim_{n\to\infty} T_{j,M=n}^{\infty} M(x,\frac{\alpha^j t}{|2|^{kj}})=\lim_{n\to\infty}(\lim_{i\to\infty} T_{M,j=n}^{i} M(x,\frac{t}{|2|^{kj}})),$$

$$= \lim_{n\to\infty}\lim_{i\to\infty}(\frac{t}{t+|2|^{kn}})=1$$

for all x ∈ X and$t > 0$.

Let $f: X \to Y$be an even and ψ -approximately additive-quadratic mapping. Thus all the conditions of theorem (1) hold and so there exists a unique quadratic mapping $Q: X \to Y$such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{t}{t+|2^k|}.$$

4. Intuitionistic random normed spaces

In this section we recall some definitions and results which will be used later in the paper.

Definition 5([7, 21]). If T is a t-norm, then its dual t-conorm $S: [0,1] \times [0,1] \to [0,1]$ is given by

$$S(x,y) = 1 - T(1-x, 1-y).$$

it is obvious that a t-conorm is a commutative, associative, and monotone operation on [0,1] with unit element 0.

Definition 6([5, 7]). A measure distribution function is a function $\mu: \mathbb{R} \to [0,1]$ which is monotone, nondecreasing, left continuous, $\inf_{x\in\mathbb{R}}\mu(x) = 0$and $sup_{x\in\mathbb{R}}\mu(x) = 1$.

We denote by D the collection of all measure distribution functions, and by H_0a special element of D defined by

$$H_0(t) := \begin{cases} 0 \text{ if } t \leq 0 \\ 1 \text{ if } t > 0 \end{cases}.$$

If $X$is a nonempty set, then $\mu: X \to D$is called a probabilistic measure on X and $\mu(x)$is denoted by μ_x.

Definition 7 ([5,16,20]). "A non-measure distribution function is a function $\nu: \mathbb{R} \to [0,1]$ which is non-increasing, right continuous,$\inf_{x\in\mathbb{R}}\nu(x) = 1$ and $sup_{x\in\mathbb{R}}\nu(x) = 0$. We denote by $\boldsymbol{B}$the collection of all non-measure distribution functions, and by G a special element of B defined by

$$G(t) := \begin{cases} 1 \text{ if } t \leq 0, \\ 0 \text{ if } t > 0 \end{cases}.$$

If $X$is a nonempty set, then $\nu: X \to \boldsymbol{B}$is called a probabilistic non-measure on $X$and $\nu(x)$is denoted by ν_x."

LEMMA 1. ([5]). Consider the set L^* and the operation $\leq_{L^*}$defined by

$$L^* = \{(x_1,x_2): (x_1,x_2) \in [0,1]^2, x_1 + x_2 \leq 1\}$$

$$(x_1,x_2) \leq_{L^*} (y_1,y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1,x_2),(y_1,y_2) \in L^*$$

Then $(L^*,\leq_{L^*})$ is a complete lattice ([18, 19]). We denote the units by $0_{L^*} = (0,1)$and $1_{L^*} = (1,0)$.

Definition 8([20]). "A triangular norm (t-norm) on L^* is a mapping $\tau: (L^*)^2 \to L^*$satisfying the following conditions:

1. $\forall\, x \in L^*, \tau(x,1_{L^*}) = x$(boundary condition);
2. $\forall\,(x,y) \in (L^*)^2, \tau(x,y) = \tau(y,x)$ (commutativity);
3. $\forall (x,y,z) \in (L^*)^3, \tau(x,\tau(y,z)) = \tau(\tau(y,x),z)$(associativity);
4. $\forall (x,\acute{x},y,\acute{y}) \in (L^*)^4, x \leq_{L^*} \acute{x},\ y \leq_{L^*} \acute{y} \Longrightarrow \tau(x,y) \leq_{L^*} \tau(\acute{x},\acute{y})$(monotoncity).

If $(L^*,\leq_{L^*}, \tau)$is an Abelian topological monoid with unit 1_{L^*} then τ is said to be a continuous t-norm."

Definition 9([20]). A continuous t-norm τon $L^*$is said to be continuous t-representable if there exists a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on [0,1]such that for all$x = (x_1,x_2),\ y = (y_1,y_2) \in L^*$

$$\tau(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\tau(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$
$$and$$
$$M(a,b) = (\min\{a_1,b_1\}, \max\{a_2,b_2\})$$

for all

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$a=(a_1,a_2), b=(b_1,b_2)\in L^*$are continuous t-representable. Now, we define a sequence τ^n recursively by $\tau^1=\tau$as

$$\tau^n\left(x^{(1)},\ldots,x^{(n+1)}\right)=\tau\left(\tau^{n-1}\left(x^{(1)},\ldots,x^{(n)}\right),x^{(n+1)}\right),\forall n\geq 2, x^i\in L^*.$$

Definition 10"([16, 20]). A negator on $L^*$is any decreasing mapping $\aleph: L^*\to L^*$satisfying $\aleph(0_{L^*})=1_{L^*}$and $\aleph(1_{L^*})=0_{L^*}$.If $\aleph(\aleph(x))=x$for all $x\in L^*$ then $\aleph$is called an involutive negator. A negator on $[0,1]$is a decreasing mapping $\aleph::[0,1]\to[0,1]$satisfying $\aleph(0)=1$and $\aleph(1)=0$.$\aleph_s$denotes the standard negator on$[0,1]$defined by $\aleph_s(x)=1-x$ $x\in[0,1]$."

Definition 11 ([5]).If μ and ν be measure and non-measure distribution functions $\mu,\nu: X\times(0,+\infty)\longrightarrow[0,1]$where $\mu_x(t)+\nu_x(t)\leq 1\forall x\in X,t>0$.The triple$(X,\rho_{\mu,\nu},\tau)$ is said to be an intuitionistic random normed spaces (briefly IRN-spaces) if X is a vector spaces, τ is a continuous t-representable, and $\rho_{\mu,\nu}: X\times(0,+\infty)\to L^*$satisfying the following conditions: $\forall x,y\in X,t,s>0$,

1. $\rho_{\mu,\nu}(x,0)=0_{L^*}$;
2. $\rho_{\mu,\nu}(x,t)=1_{L^*}\Leftrightarrow x=0$;
3. $\rho_{\mu,\nu}(\alpha x,t)=\rho_{\mu,\nu}\left(x,\frac{t}{|\alpha|}\right)\forall\alpha\neq 0$;
4. $\rho_{\mu,\nu}(x+y,t+s)\geq_{L^*}\tau\left(\rho_{\mu,\nu}(x,t),\rho_{\mu,\nu}(y,s)\right)$.

In this case,$\rho_{\mu,\nu}$ is called an IR-norm.

Here $\rho_{\mu,\nu}(x,t)=\left(\mu_x(t),\nu_x(t)\right)$.

Example **5** ([5]). Let $(X,\|.\|)$ be a normed space. But $\tau(a,b)=(a_1b_1,\min(a_2+b_2,1))\forall$ a= $(a_1,a_2), b=(b_1,b_2)\in L^*$ and $\rho_{\mu,\nu}(x,t)=\left(\mu_x(t),\nu_x(t)\right)=\left(\frac{t}{t+\|x\|}\ ,\ \frac{t}{t+\|x\|}\right)\ \forall\ t\in\mathbb{R}$. Then $(X,\rho_{\mu,\nu},\tau)$ is an IRN-space.

Definition 12 ([20]). But$(X,\rho_{\mu,\nu},\tau)$be an IRN-space.

1. A sequence $\{x_n\}$in $X$is said to be convergent to a point $x\in X$denoted by $(\{x_n\}\overset{\rho_{\mu,\nu}}{\to}x)$ if, $\rho_{\mu,\nu}(x_n-x),t)\longrightarrow 1_{L^*}$ as $n\longrightarrow\infty$ $\forall t>0$.

2. A sequence $\{x_n\}$in $X$is called a Cauchy sequence if,$\forall\varepsilon>0$. and $t>0$, there exists a positive integer $n_0\in\mathbb{N}$ such that

$$\rho_{\mu,\nu}(x_n-x_m,t)>_{L^*}(N_s(\varepsilon),\varepsilon)$$
$$\forall\ n,m\geq n_0 where\ N_s\ is\ a\ standard\ negator.$$

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3. An IRN-space $(X,\rho_{\mu,\nu},\tau)$ is said to be complete $\Leftrightarrow$ every Cauchy sequence in X is convergent to a point in X.

6. HYERS-ULAM STABILITY OF THE MIXED TYPE ADDITIVE-QUADRATIC FUNCTIONAL EQUATION(1.1) IN IRN-SPACES BY DIRECT METHOD

Here, by the direct method, we prove the generalized stability of the AQ- FE(1.1) in CIRN-spaces. Also, we present an illustrative example.

For a given mapping $f: X\to Y$, we define

$$D_sf(x,y):=f(2x+y)+f(2x-y)-2[f(x+y)+f(x-y)]-2[f(x)+f(-x)]+[f(y)+f(-y)]$$

$\forall x,y\in X$, $t>0$.

THEOREM 3. Let$X$ be a real linear space and$(Y,\rho_{\mu,\nu},\tau)$be a complete IRN-space and $f: X\to Y$ be an even mapping with $f(0)=0$ for which $\exists$ a map $\xi: X^2\to D^+$and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x,y)$ is denoted by $\xi_{x,y}$, $\zeta(x,y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t),\zeta_{x,y}(t))$ denoted by$Q_{\xi,\zeta}(x,y,t)$with the property

$$\rho_{\mu,\nu}(D_sf(x,y),t)\geq_{L^*}Q_{\xi,\zeta}(x,y,t),(6.1)$$

If

$$\lim_{j\to\infty}\tau_{i=1}^{\infty}\left(Q_{\xi,\zeta}\left(2^{i+j-1}x,0,2^{2j+i+1}t\right)=1_{L^*},\right.\qquad(6.2)$$

and

$$\lim_{m\to\infty}Q_{\xi,\zeta}(2^mx,2^my,2^{2m}t)=1_{L^*},\qquad(6.3)$$

$\forall x,y\in X$, $t>0$, then $\exists!$ quadratic mapping $S: X\to Y$

$$\rho_{\mu,\nu}(f(x)-S(x),t)\geq_{L^*}\tau_{i=1}^{\infty}\left(2^{i-1}x,0,2^{i+1}t\right),\qquad(6.4)$$

$\forall x\in X$ and $t>0$.

Proof. Buty = 0 in (6.1) we get

$$\rho_{\mu,\nu}(2f(2x)-8f(x)),t)\geq_{L^*}Q_{\xi,\zeta}(x,0,t),(6.5)$$

$\forall x\in X$. Then we get

$$\rho_{\mu,\nu}\left(\frac{f(2x)}{4}-f(x),t\right)\geq_{L^*}Q_{\xi,\zeta}(x,0{,}8t),(6.6)$$

Therefore,

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$\rho_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{2k+2}}-\frac{f(2^k x)}{2^{2k}},t\right)\geq_{L^*} Q_{\xi,\zeta}(2^k x,0,2^{2k+3}t),$ (6.7)

That is

$$\rho_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{2k+2}}-\frac{f(2^k x)}{2^{2k}},\frac{t}{2^{k+1}}\right)\geq_{L^*} Q_{\xi,\zeta}(2^k x,0,2^{k+2}t), \quad (6.8)$$

$\forall k\in N, t>0$. As

$$1>\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^k}$$

by the triangle inequality for $x\in X$, $t>0$, it follows:

$$\rho_{\mu,\nu}\left(\frac{f(2^n x)}{2^{2n}}-f(x),t\right)\geq_{L^*}\rho_{\mu,\nu}\left(\frac{f(2^n x)}{2^{2n}}-f(x),\sum_{k=0}^{n-1}\frac{1}{2^{k+1}}t\right)$$

$$\geq_{L^*}\tau_{k=0}^{n-1}\left(\rho_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{2k+2}}-\frac{f(2^k x)}{2^{2k}},\frac{1}{2^{k+1}}t\right)\right.$$

$$\geq_{L^*}\tau_{k=0}^{n-1}(Q_{\xi,\zeta}(2^k x,0,2^{k+2}t))$$

$$=\tau_{i=1}^{n}(Q_{\xi,\zeta}(2^{i-1}x,0,2^{i+1}t)) \quad (6.9)$$

$x\in X, t>0$.In order to prove the convergence of the sequence $\left\{\frac{f(2^j x)}{2^{2j}}\right\}$,

we replace $x$with $2^j x$and multiplying the left hand of (6.9) by $\frac{2^{2j}}{2^{2j}}$,

$$\rho_{\mu,\nu}\left(\frac{f(2^{n+j}x)}{2^{2n+2j}}-\frac{f(2^j x)}{2^{2j}},t\right)\geq_{L^*}\tau_{i=1}^{n}\left(Q_{\xi,\zeta}(2^{i+j-1}x,0,2^{2j+i+1}t)\right). \quad (6.10)$$

Since the right hand side of the inequality (6.10) →1 as $i,j\to\infty$,

the sequence $\left\{\frac{f(2^j x)}{2^{2j}}\right\}$is a Cauchy sequence.

Therefore, we may define

$$S(x)=\lim_{j\to\infty}\frac{f(2^j x)}{2^{2j}}$$

$\forall x\in X$. Since $f:X\to Y$ is even, $S:X\to Y$ is an even mapping.

Replacing $x,y$with $2^m x$and $2^m y$, respectiveiy, in (6.1) then multiplying the right hand side by$\frac{2^{2m}}{2^{2m}}$,it follows that:it follows that:

$\rho_{\mu,\nu}\left(\frac{1}{2^{2m}}D_s f(2^m x,2^m y),t\right)\geq_{L^*} Q_{\xi,\zeta}(2^m x,2^m y,2^{2m}t),$

$\forall x,y\in X$.Taking the limit as $m\to\infty$ we find that S satisfies (1.1), that is, S is a quadratic map. To prove (6.4) take the limit as $n\to\infty$ in (6.9).

Finally, to prove the uniqueness of the quadratic function S, let us assume that there exists a quadratic function r which satisfies (6.4) and equation (1.1). Therefor

$$\rho_{\mu,\nu}(r(x)-S(x),t)=\rho_{\mu,\nu}\left(r(x)-\frac{f(2^j x)}{2^{2j}}+\frac{f(2^j x)}{2^{2j}}-s(x),t\right)$$

$$\geq_{L^*}\tau\left(\rho_{\mu,\nu}\left(r(x)-\frac{f(2^j x)}{2^{2j}},\frac{t}{2}\right),\rho_{\mu,\nu}\left(\frac{f(2^j x)}{2^{2j}}-S(x),\frac{t}{2}\right)\right).$$

Taking the limit as $j\to\infty$, we find $\rho_{\mu,\nu}(r(x)-S(x),t)=1$. Therefore $r=s$. ■

In Theorem (3) if $f$is an odd mapping, then the following theorem can be proved Similarly.

THEOREM 4. Let X be a real linear space and$(Y,\rho_{\mu,\nu},\tau)$be a complete *IRN*-space and$f:X\to Y$be an odd mapping with f (0) = 0 for which there is a map$\xi:X^2\to D^+$:
and a mapζfromX^2 to the space of non-measure distribution functions.$\xi(x,y)$ is denoted by $\xi_{x,y}$, $\zeta(x,y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t),\zeta_{x,y}(t))$denoted by$Q_{\xi,\zeta}(x,y,t)$with the property

$$\rho_{\mu,\nu}(D_s f(x,y),t)\geq_{L^*} Q_{\xi,\zeta}(x,y,t), \quad (6.11)$$

If

$$\lim_{j\to\infty}\tau_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{i+j-1}x,0,2^{i+1}t)=1_{L^*}, \quad (6.12)$$

and

$$\lim_{m\to\infty}Q_{\xi,\zeta}(2^m x,2^m y,2^m t)=1_{L^*}, \quad (6.13)$$

$\forall x,y\in X$, $t>0$, then there exists a unique quadratic mapping $S:X\to Y$

$$\rho_{\mu,\nu}(f(x)-S(x),t)\geq_{L^*}\tau_{i=1}^{\infty}(2^{i-1}x,0,2t), \quad (6.14)$$

$\forall x\in X$, $t>0$.

COROLLARY 2. *Let* $(X,\acute{\rho}_{\acute{\mu},\acute{\nu}},\tau)$be an *IRN*- space and$(Y,\rho_{\mu,\nu},\tau)$be a complete *IRN*-space. *If* $f:X\to Y$be an even mapping satisfying

$$\rho_{\mu,\nu}(D_s f(x,y),t)\geq_{L^*}\acute{\rho}_{\acute{\mu},\acute{\nu}}(x+y,t), \quad (6.15)$$

$\forall x,y\in X$,$t>0$in which

$$\lim_{j\to\infty}\tau_{i=1}^{\infty}(\acute{\rho}_{\acute{\mu},\acute{\nu}}(x,0,2^{j+2}t))=1_{L^*}, \quad (6.16)$$

$\forall x,y\in X$,$t>0$.

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Then $\exists!$ quadratic mapping$S: X \to Y$such that

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \tau_{i=1}^{\infty}(\acute{\rho}_{\mu,\acute{\nu}}(x, 0{,}4t),$$

$\forall x \in X$,$t > 0$.

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \acute{\rho}_{\mu,\acute{\nu}}(x + y, t)$$

$\forall x, y \in X$, $t > 0$, the corollary immediate from Theorem (3). ■

COROLLARY 3.*Let* $(X, \acute{\rho}_{\mu,\acute{\nu}}, \tau)$be an *IRN*- space and$(Y, \rho_{\mu,\nu}, \tau)$ be a complete *IRN*-space. *If* $f: X \to Y$ be an odd mapping satisfying

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geq_{L^*} \acute{\rho}_{\mu,\acute{\nu}}(x + y, t), \quad (6.17)$$

$\forall x, y \in X$,$t > 0$in which

$$\lim_{j\to\infty} \tau_{i=1}^{\infty}(\acute{\rho}_{\mu,\acute{\nu}}(x, 0, 2^{2-j}t)) = 1_{L^*}, \quad (6.18)$$

$\forall x, y \in X, t > 0$.

Then $\exists!$additive mapping$S: X \to Y$such that

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \tau_{i=1}^{\infty}(\acute{\rho}_{\mu,\acute{\nu}}(x, 0, 2^{2-i}t),$$

$\forall x \in X$, $t > 0$.

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \acute{\rho}_{\mu,\acute{\nu}}(x + y, t)$$

$\forall x, y \in X$,$t > 0$, the corollary immediate from Theorem (4). ■

***Example* 6**. Let $(X, \|.\|)$ be a Banach algebra space and$(X, \acute{\rho}_{\mu,\acute{\nu}}, M)$ be an IRN-space in which

$$\acute{\rho}_{\mu,\acute{\nu}}(x, t) = \left(\frac{t}{t + 4(\|x\| + 1)}, \frac{4(\|x\| + 1)}{t + 4(\|x\| + 1)}\right),$$

$\forall x, y \in X$, $t > 0$ and let $(Y, \rho_{\mu,\nu}, \tau)$be a complete IRN-space in which

$$\rho_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right),$$

$\forall x, y \in X$, $t > 0$.Define the mapping $f: X \to Y$by $f(x) = x^2 + x_0$for all $x \in X$where $x_0$is a unit vector in X. A straightforward computation shows that

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geq_{L^*} \acute{\rho}_{\mu,\acute{\nu}}(x + y, t),$$

$\forall x, y \in X$, $t > 0$. Also we have

$$\lim_{j\to\infty} M_{i=1}^{\infty}\left(\acute{\rho}_{\mu,\acute{\nu}}(x, 0, 2^{j+2}t)\right)$$
$$= \lim_{j\to\infty}\lim_{m\to\infty} M_{i=1}^{\infty}(\acute{\rho}_{\mu,\acute{\nu}}(x, 0, 2^{j+2}t))$$
$$= \lim_{j\to\infty}\lim_{m\to\infty}(\acute{\rho}_{\mu,\acute{\nu}}(x, 0, 2^{j+2}t)$$
$$= \lim_{j\to\infty}\acute{\rho}_{\mu,\acute{\nu}}(x, 0, 2^{j+2}t) = 1_{L^*}$$

$\forall x \in X$,$t > 0$. Therefore, $\exists!$quadratic mapping $S: X \to Y$ suchthat

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \acute{\rho}_{\mu,\acute{\nu}}(x, 0{,}4t),$$

$\forall x \in X$, $t > 0$.

7. HYERS-ULAM STABILITY OF THE ADDITIVE-QUADRATIC FUNCTIONAL EQUATION (1.1) IN IRN-SPACES BY FIXED POINT METHOD

By the fixed point method, we prove the generalized stability of the mixed type F-E (1.1) in complete IRN-spaces. Before giving the main result, we present a definition and a theorem will be used later.

Definition 13 ([6]). Let X be a set. A function $d: X \times X \to [0, \infty]$is called ageneralized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

THEOREM 5([6]). Let(X, d)be a complet generalized metric spaces and let be a strictly contractive mapping with $J : X \to X$ Lipschitz constant $\alpha < 1$. Then foreach given element$x \in X$,either

$$d(j^n x, j^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1)$d(J^n x, J^{n+1} x) < \infty, \forall n \geq n_0$;
(2) the sequence$\{J^n x\}$converges to a fixed pointy^* *of* J;
(3)$y^*$is the unique fixed ponit of J in the set $Y = \{y \in X \setminus d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \dfrac{1}{1 - \alpha} d(y, Jy)$for all $y \in Y$.

Now we present the main result in this section

THEOREM 6. Let$X$ be a real linear space and$(Y, \rho_{\mu,\nu}, \tau)$be a complete IRN-space and $f: X \to Y$ be an even mapping with $f(0) = 0$ for which there is a map $\xi: X^2 \to D^+$and a map ζ from X^2 to the space of non-measure distribution functions.

$\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta\ (x, y)$ is denotedby $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ denoted by$Q_{\xi,\zeta}(x, y, t)$with the property

$$Q_{\xi,\zeta}(2x, 2y, \alpha t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t), 0 < \alpha < 4$$

and

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t) \quad (7.1)$$

$\forall x, y \in X$, $t > 0$.Then there exists a unique quadratic mapping $g: X \to Y$ such that

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$$\rho_{\mu,\nu}(f(x)-g(x),t) \geq_{L^*} Q_{\xi,\zeta}(x,0,2(4-\alpha)t) \quad (7.2)$$

$\forall x \in X,\, t>0$. Moreover, we have

$$g(x) := \lim_{n\to\infty} f\left(\frac{2^n x}{4^n}\right).$$

Proof. Let $y = 0$ in (7.1); we get

$$\rho_{\mu,\nu}(2f(2x)-8f(x),t) \geq_{L^*} Q_{\xi,\zeta}(x,0,t) \quad (7.3)$$

$\forall\, x \in X$, and $t>0$ and hence

$$\rho_{\mu,\nu}\left(\frac{f(2x)}{4}-f(x),t\right) \geq_{L^*} Q_{\xi,\zeta}(x,0,8t) \quad (7.4)$$

Consider the set

$$E := \{g: X \to Y: g(0)=0\},$$

and the mapping d_G defined on $E\times E$ by

$d_G(g,h) = \inf\{\epsilon > 0 : \rho_{\mu,\nu}(g(x)-h(x),\epsilon t) \geq ,Q_{\xi,\zeta}(x,0,8t)\}$

$\forall x \in X$, *and* $t>0$. Then (E,d_G) is a complete generalized metric space (see the proof of [12, lemma 2.1]). Now, let us consider the linear mapping $J : E\times E$ defined by

$$Jg(x) = \frac{g(2x)}{4}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \frac{\alpha}{4}$. Indeed, let $g,h \in E$ be the mappings such that $d_G(g,h) < \varepsilon$. Then we have

$$\rho_{\mu,\nu}(g(x)-h(x),\epsilon t) \geq_{L^*} Q_{\xi,\zeta}(x,0,8t)$$

$$\rho_{\mu,\nu}\left(Jg(x)-Jh(x),\frac{\varepsilon\alpha t}{4}\right) = \rho_{\mu,\nu}\left(\frac{g(2x)}{4}-\frac{h(2x)}{4},\frac{\varepsilon\alpha t}{4}\right) = \rho_{\mu,\nu}(g(2x)-h(2x),\varepsilon\alpha t) \geq_{L^*} Q_{\xi,\zeta}(2x,0,\alpha 8t)$$

$$Q_{\xi,\zeta}(2x,2y,\alpha t) \geq_{L^*} Q_{\xi,\zeta}(x,y,t), 0<\alpha<4$$

we have

$$\rho_{\mu,\nu}\left(Jg(x)-Jh(x),\frac{\varepsilon\alpha t}{4}\right) \geq_{L^*} Q_{\xi,\zeta}(2x,0,8t)$$

that is,

$$d_G(g,h) < \varepsilon \Longrightarrow d_G(Jg,Jh) < \frac{\alpha}{4}\varepsilon.$$

This means that

$$d_G(Jg,Jh) < \frac{\alpha}{4} d_G(g,h),$$

$\forall g,h \in E$. Next, from

$$\rho_{\mu,\nu}\left(\frac{f(2x)}{4}-f(x),t\right) \geq_{L^*} Q_{\xi,\zeta}(x,0,8t)$$

follows that $d_G(f,Jf) \leq 1$. Using the Theorem (5), there exists a fixed point of J, that is, there is a mapping $g: X\to Y$ such that $g(2x) = 4g(x)\ \forall x \in X$.

Since, for all $x \in X$ and $t>0$, $d_G(u,v) < \varepsilon \Longrightarrow \rho_{\mu,\nu}(u(x)-v(x),t) \geq_{L^*} Q_{\xi,\zeta}\left(2x,0,\frac{8t}{\varepsilon}\right)$.

It follows from $d_G(J^n f,g) \to 0$ that $\lim_{m\to\infty} f\left(\frac{2^n x}{4^n}\right) = g(x)$ for all $x \in X$.

Since $f: X\to Y$ is even, $g: X\to Y$ is an even mapping. Also from

$$d_G(f,g) \leq \frac{1}{1-L} d(f,Jf),$$

$\forall g,h \in E$. Then $d_G(f,g) \leq \frac{1}{1-\frac{\alpha}{4}}$. It immediately follows that

$$\rho_{\mu,\nu}\left(g(x)-f(x),\frac{4}{4-\alpha}t\right) \geq_{L^*} Q_{\xi,\zeta}(x,0,8t)$$

$\forall x \in X$ and $t>0$. This means that

$$\rho_{\mu,\nu}(g(x)-f(x),t) \geq_{L^*} Q_{\xi,\zeta}(x,0,2(4-\alpha)t)$$

$\forall x \in X$ and $t>0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists such that $C \in (0,\infty)$ such that

$$\rho_{\mu,\nu}(g(x)-f(x),Ct) \geq_{L^*} Q_{\xi,\zeta}(x,0,8t)$$

$\forall x \in X$ and $t>0$. This completes the proof. ■

In Theorem (6) if f is an odd mapping, then the following theorem can be proved similarly.

THEOREM 7..Let X be a real linear space and $(Y,\rho_{\mu,\nu},\tau)$ be a complete IRN-space and $f: X\to Y$ be an odd mapping with $f(0)=0$ for which there is a map $\xi: X^2 \to D^+$ and a map ζ from X^2 to the space of non-measure distribution functions.

$\xi(x,y)$ is denoted by $\xi_{x,y}$, $\zeta(x,y)$ is denotedby $\zeta_{x,y}$ and $(\xi_{x,y}(t),\zeta_{x,y}(t))$ denoted by $Q_{\xi,\zeta}(x,y,t)$ with the property

$$Q_{\xi,\zeta}(2x,2y,\alpha t) \geq_{L^*} Q_{\xi,\zeta}(x,y,t), 0<\alpha<2$$

and

$$\rho_{\mu,\nu}(D_s f(x,y),t) \geq_{L^*} Q_{\xi,\zeta}$$

$\forall x,y \in X,\, t>0$. Then there exists a unique quadratic mapping $g: X\to Y$ such that

$$\rho_{\mu,\nu}(f(x)-g(x),t) \geq_{L^*} Q_{\xi,\zeta}(x,0,2(2-\alpha)t)$$

$\forall x \in X, t>0$. Moreover, we have

$$g(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}.$$

COROLLARY4.*Let* $(X,\acute{\rho}_{\mu,\acute{\nu}},\tau)$ be an *IRN*- space and $(Y,\rho_{\mu,\nu},\tau)$ be a complete *IRN*-space. *If* $f: X\to Y$ be an even mapping satisfying

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$$\rho_{\mu,\nu}(D_s f(x,y),t) \geq_{L^*} \left(\frac{t}{t+\|x+y\|}, \frac{\|x\|}{t+\|x+y\|}\right),$$
$\forall\, x, y \in X, t > 0$

Then there exists a unique quadratic mapping $S: X \to Y$ satisfying(1.1) and

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \left(\frac{2(4-\alpha)t}{2(4-\alpha)t+\|x\|}, \frac{\|x\|}{2(4-\alpha)t+\|x\|}\right),$$
$\forall x \in X,\ t > 0$.Moreover, we have

$$S(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x,y,t) = \left(\frac{t}{t+\|x+y\|}, \frac{\|x\|}{t+\|x+y\|}\right),$$
$\forall x \in X$, and $t > 0$ in theorem 6. Then we can choose $2 \leq \alpha < 4$ and so we get the desired result.■

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