On properties of Sugeno Fuzzy Measure space

Receved :13/9/2015 **Accepted :11/11/2015**

Firas Hussean Maghool

AL- Qadisiya University

College of Computer science and Mathematic

Department of mathematics.

h.mfiras@yahoo.com

Abstract:

In this paper, first study some properties of a Sugeno fuzzy measure as a set function and Continunity of sugeno fuzzy measure space, discuss pesuduometric generating properties of this measure and prove some theory.

Mathematic Classification QA 299.6-433

Keyword:- σ -field , Sugeno fuzzy measure, pesuduometric generating properties

Introduction:

Fuzzy measure is generalization of the notion of measure in mathematical analysis . In 1974, the Japanese Scholar Sugeno [1] presented a kind of typical non additive measure Sugeno fuzzy measure. The properties and application of Sugeno measure have been studied by many authors as [2]and [4] , in this paper we present definition sugeno fuzzy measure as a monotone and continuous set function and study pesuduometric generating properties of sugeno fuzzy measure space and prove some important theory

Preliminaries

Definition(2.1)[1]:

A family F of subsets of a set X is called a σ -field on a set Ω , if

(1) $X \in F$

(2) If $A \in F$, then $A^c \in F$

(3) If $A_n \in F, n=1,2,...$ then $\bigcup_{n=1}^{\infty} A_n \in F$

A measurable space is a pair (X ,F),

where X is a set and F is σ -field on X

A subset A of X is called measurable

(measurable with respect to the

σ -field F) , if $A \in F$ i.e. any member of F is called a measurable set.

Definition(2.2)[1][2]:-

Let X is nonempty set and F is $\sigma -$ field , a set function $\mu : F \to [0,1]$ that is satisfy the following axioms:

(1) $\mu(X)=1$

$$(2)\mu(\bigcup_{i=1}^{\infty} A_i) = \begin{cases} \frac{1}{\lambda}[\prod_{i=1}^{\infty}(1+\lambda \cdot \mu(A_i))]-1 & , \quad \lambda \neq 0 \\ \sum_{i=1}^{\infty}\mu(A_i) & , \quad \lambda = 0 \end{cases}$$

where $\lambda \in (-1,\infty)$ is called λ - Sugeno fuzzy measure and atripe (X,F,μ) λ - Sugeno fuzzy measure space , X is a set.

Example: Let $\mu : X \to [0,1]$

$X=\{1,2\}, \quad F=\{\phi,\{1\},\{2\},X\}$

$$\mu(A)=\begin{cases} 0, & A=\phi \\ 0.4, & A=\{1\} \\ 0.2, & A=\{2\} \\ 1 & A=X \end{cases}$$

Is λ - Sugeno fuzzy measre where $\lambda = 5$

Solve:-

$\mu(\{1\} \cup \{2\}) = \mu(X)=1$

$\mu(\{1\} \cup \{2\}) = \frac{1}{5}\{1+5\mu\{1\})(1+5\mu(\{2\})-1\}$

$$=\frac{1}{5}[(1+2)(1+1)]-1$$

$$=\frac{6}{5}-\frac{1}{5}=1$$

In the following we called λ- Sugeno fuzzy measure is Sugeno fuzzy measure

Main Result

1- Properties of Sugeno fuzzy measure space:

Theorem(2.3):

Let (X,F,μ) be Sugeno fuzzy measure space then

(1) $\mu(\phi)=0$

(2) if $A_1, A_2 \in F$ then
$\mu(A_1 \cup A_2)=\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)$

(3) if $A_1, A_2,..., A_n \in F$ then

$$\mu(\bigcup_{k=1}^{n} A_i)=\begin{cases}\frac{1}{\lambda}[\prod_{k=1}^{n}(1+\lambda\cdot\mu(A_k))-1 & , \lambda\neq 0\\ \sum_{k=1}^{n}\mu(A_k) & , \lambda=0\end{cases}$$

(4) if $A\in F$

Then $\mu(A)+\mu(A^c)=1-\lambda\mu(A)\mu(A^c)$

Proof :

(1) since $X\cup\phi=X$

$\mu(X\cup\phi)=\mu(X)=1$

$\mu(X\cup\phi)=\mu(X)+\mu(\phi)+\lambda\mu(X)\mu(\phi)$

$\Rightarrow 1+\mu(\phi)+\lambda\mu(\phi)=1$

$\Rightarrow \mu(\phi)+\lambda\mu(\phi)=0$

$\Rightarrow (1+\lambda)\mu(\phi)=0$

(2) $\mu(A_1\cup A_2)=\frac{1}{\lambda}(\prod_{i=1}^{2}(1+\lambda\mu(A_i))-1)$

$$=\frac{1}{\lambda}[(1+\lambda\mu(A_1)(1+\lambda\mu(A_2)-1)]$$

$$=\frac{1}{\lambda}[(1+\lambda\mu(A_1+\lambda\mu(A_2)+\lambda^2\mu(A_1)\mu(A_2)-1]$$

$$=\frac{1}{\lambda}[(\lambda\mu(A_1+\lambda\mu(A_2)+\lambda^2\mu(A_1)\mu(A_2)]$$

$$=\mu(A_1)+\mu(A_2)+\lambda\ \mu(A_1)\mu(A_2)$$

(3) put $A=\phi$ when $k\geq n$

$$\mu(\bigcup_{k=1}^{n}A_k)=\mu(\bigcup_{k=1}^{\infty}A_k)=\begin{cases}\frac{1}{\lambda}[\prod_{k=1}^{\infty}(1+\lambda\cdot\mu(A_k))-1 & , \lambda\neq 0\\ \sum_{k=1}^{\infty}\mu(A_k) & , \lambda=0\end{cases}$$

$$\mu(\bigcup_{k=1}^{n}A_k)=\begin{cases}\frac{1}{\lambda}[\prod_{k=1}^{n}(1+\lambda\cdot\mu(A_k)\prod_{k=n+1}^{\infty}(1+\lambda\mu(A_k))-1] & , \lambda\neq 0\\ \sum_{k=1}^{n}\mu(A_k)+\sum_{k=1}^{\infty}\mu(A_k) & , \lambda=0\end{cases}$$

$$\mu(\bigcup_{k=1}^{n}A_i)=\begin{cases}\frac{1}{\lambda}[\prod_{k=1}^{n}(1+\lambda\cdot\mu(A_k))-1 & , \lambda\neq 0\\ \sum_{k=1}^{n}\mu(A_k) & , \lambda=0\end{cases}$$

(4) since $\lambda\in F, A=\phi$

Then $\mu(\phi)\subseteq\mu(A)\subseteq\mu(X)\Rightarrow 0\leq\mu(A)\leq 1$

Since $A\cup A^c=X\rightarrow\mu(A\cup A^c)=\mu(X)=1$

$\mu(A)+\mu(A^c)+\lambda\mu(A)\mu(A^c)=1$

$\mu(A)=1-(\mu(A^c)+\lambda\mu(A)\mu(A^c))$

Then

$\mu(A)+\mu(A^c)+\lambda\mu(A)\mu(A^c)=1$

If $\lambda=0$

Then $\mu(A)=1-\mu(A^c)$

Proposition(2.4):

If $A_1, A_2 \in F, \quad A_1 \subset A_2$

then $\mu(A_1) \le \mu(A_2)$

Proof:

since $A_1 \subset A_2 \Rightarrow A_2^c \subset A_1^c$

$\Rightarrow \mu(A_2^c \cup A_2)=\mu(X)=1$

$\mu(A_1^c)=\mu(A_2)+\lambda\mu(A_1^c)\mu(A_2)$

$1-(\mu(A_1)+\lambda\mu(A_1)\mu(A_1^c)+\mu(A_2)+\lambda\mu(A_1^c)\mu(A_2)$

$1-\mu(A_1)[1+\lambda\mu(A_1^c)]+\mu(A_2)[1+\lambda\mu(A_1^c)]=1$

$[1+\lambda\mu(A_1^c)][\mu(A_2)-\mu(A_1)]=0$

$\mu(A_2)-\mu(A_1)\ge 0$

$\mu(A_2)\ge\mu(A_1)$

Proposition (2.5) :- Let (X,F,μ) be a Sugeno fuzzy measure space then

$$\mu(A_1\cap A_2)=\frac{1-\mu(A_1\cup A_2)}{1+\lambda\mu(A_1\cup A_2)}$$

Proof:- since $A_1\cap A_2=(A_1\cup A_2)^c$

$\mu(A_1\cap A_2)=\mu(A_1\cup A_2)^c=$

$1-\mu(A_1\cup A_2)-\lambda\mu(A_1\cup A_2)\mu(A_1\cup A_2)^c$

$\mu(A_1\cap A_2)=1-[\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)]$

$-\lambda[\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)]\mu(A_1\cap A_2)$

$\mu(A_1\cap A_2)=1-\mu(A_1)-\mu(A_2)-\lambda\mu(A_1)\mu(A_2)$

$-(\lambda\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2))\mu(A_1\cap A_2)$

$\mu(A_1\cap A_2)+(\lambda\mu(A_1)+\lambda\mu(A_2)+$

$\lambda^2\mu(A_1)\mu(A_2))\mu(A_1\cap A_2)$

$=1-\mu(A_1)-\mu(A_2)+\lambda\mu(A_1)\mu(A_2)$

$=\mu(A_1\cap A_2)[1+\lambda(\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2)]$

$=1-(\mu(A_1)+\mu(A_2)+\lambda\mu(A_1)\mu(A_2))$

$$\to \mu(A_1\cap A_2)=\frac{1-\mu(A_1\cup A_2)}{1+\lambda\mu(A_1\cup A_2)}$$

Remark:- from Proposition (2.5) we have

$$\mu(A_1-A_2)=\mu(A_1\cap A_2^c)=\frac{1-\mu(A_1\cup A_2^c)}{1+\lambda\mu(A_1\cup A_2^c)}$$

2- Continunity of sugeno fuzzy measure space:

Let $\{A_n\}$ be a sequence of subset of X The set of all points which are belong to infinitely many sets of the sequence $\{A_n\}$ is called the upper limit(or limit superior) of $\{A_n\}$ and (in symbol A^*) defined by

$A^*=\lim_{n\to\infty}\sup A_n=\{x\in A_n :$ for infinitely many $n\}=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k=\lim_{n\to\infty}\bigcup_{k=n}^{\infty}A_k$

Thus, $x\in A_n$ iff for all n , then $x\in A_k$ for some $k\ge n$

the lower limit (or limit inferior) of $\{A_n\}$ defined by A_* is the set of all points which belong to almost all sets of the sequence $\{A_n\}$, and denoted by

$A_*=\lim_{n\to\infty}\inf A_n=\{x\in A_n :$ for all but finitely many $n\}=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k=\lim_{n\to\infty}\bigcap_{k=n}^{\infty}A_k$

Thus, $x\in A_n$ iff for some n , then $x\in A_k$ for all $k\ge n$.

A sequence $\{A_n\}$ of subset of a set X is said to be converge if

$\lim_{n\to\infty} \sup A_n = \lim_{n\to\infty} \inf A_n = A$ and A is said to be the limit of $\{A_n\}$ we write

$A = \lim_{n\to\infty} A_n$ or $A_n \to A$ [4].

A sequence $\{A_n\}$ of subset of a set Ω is said to be increasing if $A_n \subset A_{n+1}$ for $n = 1,2,\dots$. And is said to be decreasing if $A_{n+1} \subset A_n$ for $n = 1,2,\dots$.

A monotone sequence of sets is one which either increasing or decreasing.

If $\{A_n\}$ is an increasing sequence of subset of a set X and $\bigcup_{n=1}^{\infty} A_n = A$,we say that A_n an increasing sequence of a set with limit A, or that A_n increase to A, write $A_n \uparrow A$, also if $\{A_n\}$ is a decreasing sequence of subset of a set X and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n a decreasing sequence of a set with limit A, or that the A_n decrease to A, write $A_n \downarrow A$ [4].

Theorem (3.1)[4]:

Let $\{A_n\}$ be a sequence of subset of a set X and let $A \subset X$

(1) If $A_n \uparrow A$ then $A_n^c \downarrow A^c$

(2) If $A_n \downarrow A$ then $A_n^c \uparrow A^c$

Theorem(3.2):

Let (X, F, μ) be Sugeno fuzzy measure space and $A_n \in F$ then

(i) If $A_n \uparrow A$ then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(ii) If $A_n \downarrow A$ then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(iii) If $A_n \downarrow \phi$ then $\lim_{n\to\infty} \mu(A_n) = 0$

(iv) If $A_n \uparrow X$ then $\lim_{n\to\infty} \mu(A_n) = 1$

Proof:- (i) since $A_n \uparrow A$, $A_n \subset A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = A$

$$\mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(A_1) + \sum_{i=2}^{\infty} \mu(A_i \mid A_{i-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{i=2}^{\infty} \mu(A_i \mid A_{i-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{i=2}^{\infty} \mu(A_i) - \mu(A_{i-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} (\mu(A_n) - \mu(A_1))$$

$$= \lim_{n\to\infty} (\mu(A_n)$$

then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$

(ii) $A_n \downarrow A \to A_n^c \uparrow A^c$

Then from (i) $\lim_{n\to\infty} \mu(A_n^c) = \mu(A^c)$

$$\Rightarrow \lim_{n\to\infty} (1 - \mu(A_n) - \lambda\mu(A_n^c)\mu(A_n))$$
$$= 1 - \mu(A) - \lambda\mu(A)\mu(A^c)$$

$\Rightarrow 1-\lim_{n\to\infty}\mu(A_n)-\lambda\mu(A_n^c)\mu(A_n)=$
$1-\mu(A)-\lambda\mu(A)\mu(A^c)$

$=-\lim_{n\to\infty}\mu(A_n)=-\mu(A)$

$\therefore \lim_{n\to\infty}\mu(A_n)=\mu(A)$

(iii)from(ii) $A_n \downarrow \phi$ then $\lim_{n\to\infty}\mu(A_n)=\mu(\phi)$

$\lim_{n\to\infty}\mu(A_n)=0$

(iv) $A_n \uparrow X$ then from (ii)
$\lim_{n\to\infty}\mu(A_n)=\mu(X)$

$\rightarrow \lim_{n\to\infty}\mu(A_n)=1$.

Theorem(3.3):- Let (X,F,μ) be Sugeno Fuzzy measure space and $A_n \in F$, $\mu(A_n)\to 0$ as $n\to\infty$ then

$\mu(A\bigcup A_n)=\lim_{n\to\infty}\mu(A-A_n)=\mu(A)$.

Proof:-
$A\subset A\bigcup A_n \to \mu(A)\le\mu(A\bigcup A_n) \quad \forall n$

$\mu(A\bigcup A_n)=\mu(A)+\mu(A_n)+\lambda\mu(A)\mu(A_n)$

Since $\mu(A_n)\to 0$ then $\mu(A\bigcup A_n)\to\mu(A)$
and $A-A_n\subset A\subset(A-A_n)\bigcup A_n$

We have
$\mu(A-A_n)\le\mu(A)$
$=\mu(A-A_n)+\mu(A_n)+\lambda\mu(A-A_n)\mu(A_n)$

Since $\mu(A_n)\to 0$ we have

$\mu(A-A_n)\to\mu(A)$.

3-pesduometric generating properties:

In this section we will study pesduometric generating properties in sugeno measure many authors was study with another measure as [3],[5].

Definition(4.1)[4]:

Let β be a collection of all real valued function defined on a set X, and $f,\ f_n\in\beta$, $n\in N$ and $A\in X$ we say that $\{f_n\}$ uniformly convergent to f on A, if for every $\varepsilon>0$ there is $k\in Z^+$ such that $|f_n(x)-f(x)|<0 \quad \forall n>k$ and $x\in A$, we write $f_n\to f$ on A.

Definition (4.2):

A set function μ is said to be have pseudometic generating property (p.g.p) if for any $\varepsilon>0$, there exist $\gamma>0$ such for any Borel sets A and B,
$\mu(A)\vee\mu(B)<\gamma\Rightarrow\mu(A\cup B)<\varepsilon$.

Theorem(4.3):

A set function μ has p.g.p if and only if there exist a sequence $\{\gamma_n\}_n$ of real number such that $\gamma_n\downarrow 0$ and, for any sequence $\{A_n\}_n$ with $\mu(A_n)<\gamma_n$, the following inequalities hold

$$\mu(\bigcup_{k=n+1}^{+\infty}A_k)\le\gamma_n,\quad n\ge 1 .$$

Proof:

Let a set function μ has p.g.p then there exist $\gamma_1 \in (0, \frac{1}{2})$ such that

$\mu(A) \vee \mu(B) < \gamma_1$ implies $\mu(A \cup B) < \frac{1}{2}$

For above γ_1 there exist $\gamma_2 \in (0, \frac{1}{2^2} \wedge \gamma_1)$ to satisfy that $\mu(A) \vee \mu(B) < \gamma_2$ implies $\mu(A \cup B) < \gamma_1$ and, similarly there exist

$\gamma_3 \in (0, \frac{1}{2^3} \wedge \gamma_2)$ to satisfy that $\mu(A) \vee \mu(B) < \gamma_3$ implies $\mu(A \cup B) < \gamma_2$

Repeating this procedure,we can obtain a sequence $\{\gamma_n\}_n$ such that

$0 < \gamma_{n+1} < \frac{1}{2^{n+1}} \wedge \gamma_n \quad \forall n \geq 1$.

If $\mu(A_n) < \gamma_n, \quad \forall n \geq 1$

then we have $\mu(\bigcup_{k=n+1}^{n+r} A_k) \leq \gamma_n, \quad n \geq 1$

So that $\theta(\bigcup_{k=n+1}^{+\infty} A_k) \leq \gamma_n, \quad n \geq 1$

Conversely, for any $\varepsilon > 0$, there exist $n_0 \geq 1$ such that $\gamma_{n_o} < \varepsilon$

if we choose $\gamma = \gamma_{n_o+2}$ and , when $\mu(A) \vee \mu(B) < \gamma$ then we have

$\theta(A \cup B) = \theta(\bigcup_{k=n_0+1}^{+\infty} A_n) \leq \gamma_{n_o} < \varepsilon$

If we choose $\gamma = \gamma_{n_o+2}$ and when $\mu(A) \vee \mu(B) < \gamma$ then we have

$\mu(A \cup B) = \mu(\bigcup_{k=n+1}^{+\infty} A_k) \leq \gamma_n < \varepsilon$ where $A_{n_o+1} = A, \quad A_{n_o+2} = B,$ and otherwise $A_n = \phi$

Thus a set function μ has p.g.p.

autocontinuity of fuzzy measure " , ONR Grant No.N00014-94-1-0263,(1997).

[4] Ash. R, "Probability and Measure Theory ", Second edition, Academic press (2000).

[5] Q.Jiang, "On the exhaustivity and property (p.g.p) of null-additive set function " , www.orsc.edu.cn.com.

References

[1]M.Sugeno,"Theory of fuzzy integrals and its application ", ph.D.dissertation, Tokyo institute of Technology,1974.

[2] Z.Y.Wang,G.J.Klir, Fyzzy Measure Theory, Plenum Press, New York, 1992.

[3] Q.Jiang, S.Wang and D.Ziou, "Pseudometric generating property and

خصائص فضاء sugeno الضبابي القابل للقياس

تاريخ الاستلام 2015/9/13 تاريخ القبول 2015/11/11

فراس حسين مجهول

جامعه القادسية

كليه علوم الحاسوب والرياضيات| قسم الرياضيات

الايميل h.mfiras@yahoo.com

الخلاصة :

في هذا البحث اولا ندرس بعض خصائص فضاء **sugeno** الضبابي القابل للقياس كدالة مجموعه , و الاستمرارية للفضاء **sugeno** الضبابي القابل للقياس ثم نناقش خصائص الشبه متري المولد لهذا القياس ونبرهن بعض النظريات .

الكلمات المفتاحية:

الحقل المتكامل, قياس **sugeno** الضبابي , خصائص الشبه متري المولد.