Constrained Spline approximation

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Abstract:

In this paper, we find the relationships between the order of constrained approximation by a polynomials which is copositive with a function ($\mathbb{f}$), by modulus of smoothness τ_{k-1} , to the function $\hat{\mathbb{f}}$,which is multiply by J_c , and between the best approximation to the pairs of "intertwining splines of a polynomials" in $I = [-b, b]$, and the same of relationships but by the modulus of smoothness ω_{k-1}^{φ} to the function $\hat{\mathbb{f}}$ multiply by $n^{-1}\,|J_c|$. Also we find the order of constrained approximation to the pairs of intertwining polynomials in ($I_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1}$) .

Key words: approximation, intertwining, spline, modulus of smoothness.

1. Introduction and definitions:

In this paper we study how well can find the order of a constrained approximations by a pair of "intertwining spline" which is coming from combination of overlapping polynomial pieces enfold contaminated period . The polynomial $\mathcal{P}_{\mathbb{i}} \in \prod_r$ which is copositive with a function

$\mathbb{f} \in L_{\psi,p}(I) \cap \triangle^0(I)$ $, I \subseteq R,$ $0 < \mathrm{p} < 1$. Let f be a function which is a positive finitely many time say $\mathfrak{s} \geq 1$, on I say $-b < \mathrm{j}_{\mathfrak{s}} < \cdots < \mathrm{j}_1 < b = \mathrm{j}_0$, for $\mathrm{j}_{\mathfrak{s}} \in \mathbb{J}_{\mathfrak{s}}$.

Recall that the order "Ditizain-Totik modulus of smoothness" is given by [5] :

$$\omega_{\varphi}^{k}(\mathbb{f}, \delta, I)_{\psi,p} = \sup_{0<h\leq\delta} \left\|\triangle_h^k(\mathbb{f}, .)\right\|_{L_{\psi,p}(I).}$$

Where $\|.\|_{L_{\psi,p}(I)}$ symbolize the weighted quasi normed space [5] on an interval $[-b, b] \subseteq I \subseteq R$. "The weighted quasi normed space" $L_{\psi,p}(I), 0 < p < 1$. have form :

$$L_{\psi,p}(I) = \left\{ \mathbb{f} \ni \mathbb{f}: I \subset R \longrightarrow R: \left(\int_I \left|\frac{\mathbb{f}(x)}{\psi(x)}\right|^p dx\right)^{\frac{1}{p}} < \infty, 0 < p < 1 \right\}$$

and the quasi normed $\|\mathbb{f}\|_{L_{\psi,p}(I)} < \infty$, and

$$\Delta_h^k(\mathbb{f}, x, I)_{\psi} = \Delta_h^k(\mathbb{f}, x)_{\psi} = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \frac{\mathbb{f}(x - \frac{kh}{2} - ih)}{\psi(x + \frac{kh}{2})} & , x \pm \frac{kh}{2} \in I \\ 0 & o.w \end{cases}$$

is the symmetric difference [5] and in this paper we used new "chebyshev partition $X_j = acos\frac{j\pi}{n}$ [5] .

Let $(\delta = min|\mathrm{j}_{\mathrm{i}+1} - \mathrm{j}_{\mathrm{i}}|, 0 \leq \mathrm{i} \leq \mathfrak{s})$ where $\mathrm{j}_0 = -b$ and $\mathrm{j}_{\mathfrak{s}+1} = b$, [5] , and let $\mathbb{I}_{\mathring{\mathrm{i}}} = \left[\mathrm{j}_{\mathrm{i}}^{(v)}, \mathrm{j}_{\mathrm{i}}^{(\mathcal{K}-1)}\right], \mathring{\mathrm{i}} = 1, \dots, \mathfrak{s}$, such that $\acute{\mathrm{j}}_{\mathrm{i}} < \acute{\acute{\mathrm{j}}}_{\mathrm{t}} < \cdots < \mathrm{j}_{\mathrm{i}}^{(\mathcal{K}-1)}$, $v = 1, \dots, \mathcal{K} - 2$.

$\mathbb{J}_{\mathfrak{s}}^* = \left\{\mathrm{j}_{\mathrm{i}}^{(v)}, v = 1, \dots, (\mathcal{K} - 1): \mathbb{X}_{\mathrm{j(i)}+1} = \acute{\mathrm{j}}_{\mathrm{t}} < \cdots < \mathrm{j}_{\mathrm{i}}^{(\mathcal{K}-1)} = \mathbb{X}_{\mathrm{j(i)}}, 0 < \mathrm{i} \leq \mathfrak{s}, \mathrm{j} = 1, \dots, n\right\}$.The simple $\Delta^0(\mathbb{J}_{\mathfrak{s}}^*)$, denoted to the set of all functions $(\mathbb{f}), \ni (-1)^{\mathfrak{s}-\mathrm{i}}\mathbb{f}(x) \geq 0$.

We call the period $J_c = [b - \mu|I| , b]$, $J_k = [-b + \mu|I|, b]$, contaminated period if $(b - \mu|I| < \mathrm{j}_{\mathrm{i}} \leq b)$, $(-b + \mu|I| < \mathrm{j}_{\mathrm{i}} \leq b)$, respectively for some point $\mathrm{j}_{\mathrm{i}} \in \mathbb{J}_{\mathfrak{s}}$, and $\mu > 0$ be affixed .

2. Auxiliary Result:

In the following theorems we referred to the relationships when $\{\overline{S} , S\}$, appear a pair of "intertwining spline" has an order (r), on the knot sequence $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$, and $\{\mathcal{P}_{\mathrm{i}}\}$, $\mathrm{i} = 1, 2$, a pair of "intertwining polynomials" has an order $< r$, and a constant C, in all relation in prove of the following theorems are dependent on (p, κ).

Theorem (1): Let $\mathbb{f} \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_{\mathfrak{s}}), 0 < p < 1, \mathbb{J}_{\mathfrak{s}} = \{\mathrm{j}_1, \dots, \mathrm{j}_{\mathfrak{s}} : -b < \mathrm{j}_{\mathfrak{s}} < \cdots < \mathrm{j}_1 < b = \mathrm{j}_0\}, \mathfrak{s} \geq 0$.

Let $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$, be a knot sequence, then there are an intertwining pair of spline $\{\bar{S}, S\}$, of order (r), on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$, and $\bar{S} - f, S - f \in \Delta^0(\mathbb{J}_{\mathfrak{s}}) \ni$

$$\|\bar{S} - S\|_{L_{\psi,p}(I)} \leq C|J_c|\tau^{k-1}(\acute{f}, |I|I)_{\psi,p} .$$

Proof:

By (Theorem (2.1.3) [5]), $\exists$ a polynomials $\mathcal{P}_{\mathrm{i}} \in \prod_r \cap \Delta^0(\mathbb{J}_{\mathfrak{s}}) \ni$

$$\|\mathbb{f} - \mathcal{P}_{\mathrm{i}}\|_{L_{\psi,p}(J_c)} \leq C\tau^{k-1}(\mathbb{f}, |J_c|J_c)_{\psi,p} .$$

Also $\exists$ a polynomials $Q_{\mathrm{i}} \in \prod_r \cap \Delta^0(\mathbb{J}_{\mathfrak{s}}) \ni$

$$\|\mathbb{f} - Q_{\mathrm{i}}\|_{L_{\psi,p}(J_c)} \leq C\tau^{k-1}(\mathbb{f}, |J_c|J_c)_{\psi,p} .$$

Now, for one sided polynomial approximation on an period J_c, $\exists$ two polynomials $\mathcal{P}_{\mathrm{i}}$, Q_{i}, $\mathrm{i} = 1, \dots, \mathfrak{s}$ of degree $< r$, $\ni$ $\mathcal{P}_{\mathrm{i}} \geq \mathbb{f} \geq Q_{\mathrm{i}}, \forall x \in J_c$, which is satisfies :

$$\|\mathcal{P}_{\mathrm{i}} - Q_{\mathrm{i}}\|_{L_{\psi,p}(J_c)} \leq C\tau^{k-1}(\mathbb{f}, |J_c|J_c)_{\psi,p} .$$

Let p_i, q_i, on J_c, are two polynomials, define form $p_i = \mathcal{P}_{\mathrm{i}}, q_i = Q_{\mathrm{i}}$, if $(-1)^{\mathfrak{s}-\mathrm{i}} > 0$, and $p_{\mathrm{i}} = Q_{\mathrm{i}}, q_{\mathrm{i}} = \mathcal{P}_{\mathrm{i}}$, if $(-1)^{\mathfrak{s}-\mathrm{i}} < 0$, hence

$(-1)^{\mathfrak{s}-\mathrm{i}}(p_{\mathrm{i}}(x) - \mathbb{f}(x)) \geq 0, (-1)^{\mathfrak{s}-\mathrm{i}}(q_{\mathrm{i}}(x) - \mathbb{f}(x)) \leq 0, x \in J_c$.

And

$$\|p_{\mathrm{i}} - q_{\mathrm{i}}\|_{L_{\psi,p}(J_c)} = \|\mathcal{P}_{\mathrm{i}} - Q_{\mathrm{i}}\|_{L_{\psi,p}(J_c)} \leq C\tau^{k-1}(\mathbb{f}, |J_c|J_c)_{\psi,p} .$$

By ([3]), we have the contrast : $\tau^k(f, t)_p \leq C\, t\, \tau^{k-1}(\grave{f}, t)_p$, hence we get

$$\|p_{\mathrm{i}} - q_{\mathrm{i}}\|_{L_{\psi,p}(J_c)} \leq C|J_c|\tau^{k-1}(\mathbb{f}, |J_c|J_c)_{\psi,p} \quad \dots (1)$$

After we find a polynomials which are approximation with the function $\mathbb{f}$, now we are blend them for smoothness spline approximation on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$. Let $\mathrm{j}_{\mathrm{i}} \in \mathbb{J}_{\mathfrak{s}}$, and J_c, J_k, are non-contaminated, then $p_{\mathrm{i}}, p_{\mathrm{i}-1}$, overlap on J_k, which contains (d), interior knots from $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$.

By " Beaton's Lemma [3] , $\exists$ a spline $\overline{\mathbb{S}}_{\mathrm{i}}$, of order (r), at J_k , on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$, that relate with $p_{\mathrm{i}-1}$ and p_{i} . Furthermore the outline of $\overline{\mathbb{S}}_{\mathrm{i}}$, located between those of p_{i} , $p_{\mathrm{i}-1}$, hence

" $sgn\left(p_{\mathrm{i}-1}(x)-\mathbb{f}(x)\right)= sgn\left(p_{\mathrm{i}}(x)-\mathbb{f}(x)\right)= sgn\left(\overline{\mathbb{S}}_{\mathrm{i}}(x)-\mathbb{f}(x)\right)$ " , $x\in J_k$.

By the same method $\exists$ a spline $\mathbb{S}_{\mathrm{i}}$, of order (r), at J_k , on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$, that relate with $q_{\mathrm{i}-1}$ and q_{i} , and the outline of $\mathbb{S}_{\mathrm{i}}$, located between those of q_{i} , $q_{\mathrm{i}-1}$, hence

" $sgn\left(q_{\mathrm{i}-1}(x)-\mathbb{f}(x)\right)= sgn\left(q_{\mathrm{i}}(x)-\mathbb{f}(x)\right)= sgn\left(\mathbb{S}_{\mathrm{i}}(x)-\mathbb{f}(x)\right)$ " , $x\in J_k$.

$$\|\overline{\mathbb{S}}_{\mathrm{i}}-\mathbb{S}_{\mathrm{i}}\|_{L_{\psi,p}(J_k)} \le C\,\|p_{\mathrm{i}-1}-q_{\mathrm{i}-1}\|_{L_{\psi,p}(J_k)} + C\,\|p_{\mathrm{i}}-q_{\mathrm{i}}\|_{L_{\psi,p}(J_k)},$$

by (1) for $J_c\subset J_k$, we get:

$$\|\overline{\mathbb{S}}_{\mathrm{i}}-\mathbb{S}_{\mathrm{i}}\|_{L_{\psi,p}(J_k)} \le C|J_c|\tau^{k-1}(\acute{\mathbb{f}},|J_k|,J_k)_{\psi,p} \quad \text{….(2)}$$

Now , if there is only one of a polynomial p_{i} , over I , set $\overline{\mathbb{S}}$, to this polynomial . If there are two polynomials overlapping on I , then there must be a combination spline $\overline{\mathbb{S}}_{\mathrm{i}}$, set $\overline{\mathbb{S}}$, to $\overline{\mathbb{S}}_{\mathrm{i}}$, it is clear $\overline{\mathbb{S}}-\mathbb{f}\in\Delta^0(\mathbb{J}_{\mathfrak{s}})$, on I , by the same manner $\exists$ $\mathbb{S}-\mathbb{f}\in\Delta^0(\mathbb{J}_{\mathfrak{s}})$, on I .

Hence by (1) and (2) , we get on I :

$$\|\overline{\mathbb{S}}-\mathbb{S}\|_{L_{\psi,p}(I)} \le C|J_c|\tau^{k-1}(\acute{\mathbb{f}},|I|,I)_{\psi,p} .$$

$J_c\subset I=[-b,b]$.

Theorem (2): Let $\mathbb{f}\in L_{\psi,p}(I)\cap\Delta^0(\mathbb{J}_{\mathfrak{s}}), 0<p<1$, $\mathbb{J}_{\mathfrak{s}}=\{\mathrm{j}_1,\dots,\mathrm{j}_{\mathfrak{s}}:-b<\mathrm{j}_{\mathfrak{s}}<\cdots<\mathrm{j}_1<b=\mathrm{j}_0\}, \mathfrak{s}\ge 0$. And $\{\bar{S}, S\}$, of order (r), on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$, is the same as in theorem (1) . Then :

$$\|\bar{S}-S\|_{L_{\psi,p}(I)} \le Cn^{-1}|J_c|\omega_{\varphi}^{k-1}(\acute{\mathbb{f}},n^{-1})_{\psi,p} .$$

Proof:

After found the polynomials that intertwining with $\mathbb{f}$, on $\mathbb{I}_{\mathbb{i}}$, and which have a approximate regulation, these multipliers will be concerted for smooth spline approximants $\bar{S}$ and S , on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$.

Using the same manner of the proof in theorem (1) we get on $\mathbb{I}_{\mathbb{i}}$, that :

$$\|\bar{S}-S\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})} \le C|J_c|\tau_{k-1}(\acute{\mathbb{f}},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})_{\psi,p} .$$

Where $\cup\,\mathbb{I}_{\mathbb{i}}=I=[-b,b], \mathbb{i}=1,\dots,\mathfrak{s}$, and every $x\in I$, is contained in at most (k), sub interval of $\mathbb{I}_{\mathbb{i}}$.

Then the assessment can be on the other hand afflicted for on $\mathbb{I}_{\mathbb{i}}$:

$$\sum_{\mathbb{i}=1}^{\mathfrak{s}} \tau_k\,(\mathbb{f},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})^p{}_{\psi,p} \leq C\sum_{\mathbb{i}=1}^{\mathfrak{s}} |\mathbb{I}_{\mathbb{i}}|^p\,\omega_\varphi^{k-1}(\acute{\mathbb{f}},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})^p{}_{\psi,p} \ .$$

By (Theorem (1.6.3) [5]) , we get

$$\sum_{\mathbb{i}=1}^{\mathfrak{s}} \tau_k\,(\mathbb{f},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})^p{}_{\psi,p} \leq C\sum_{\mathbb{i}=1}^{\mathfrak{s}} |\mathbb{I}_{\mathbb{i}}|^p\,\omega_\varphi^{k-1}(\acute{\mathbb{f}},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})^p{}_{\psi,p} \ .$$

$$\leq Cn^{-p}\omega_\varphi^{k-1}(\acute{\mathbb{f}},n^{-1})^p{}_{\psi,p} \ .$$

That is :

$$\|\bar{\mathbb{S}}-\mathbb{S}\|^p{}_{L_{\psi,p}(I)} \leq C|J_c|^p\sum_{\mathbb{i}=1}^{\mathfrak{s}} \tau_{k-1}\,(\acute{\mathbb{f}},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})^p{}_{\psi,p}$$

$$\leq C|J_c|^p\sum_{\mathbb{i}=1}^{\mathfrak{s}} |\mathbb{I}_{\mathbb{i}}|^p\,\omega_\varphi^{k-2}(\acute{\acute{\mathbb{f}}},|\mathbb{I}_{\mathbb{i}}|,\mathbb{I}_{\mathbb{i}})^p{}_{\psi,p} \ .$$

$$\leq Cn^{-p}|J_c|^p\omega_\varphi^{k-2}(\acute{\acute{\mathbb{f}}},n^{-1})^p{}_{\psi,p} \ .$$

Hence

$$\|\bar{S}-S\|_{L_{\psi,p}(I)} \leq Cn^{-1}|J_c|\omega_\varphi^{k-2}(\acute{\acute{\mathbb{f}}},n^{-1})_{\psi,p} \ .$$

Theorem (3): Let $\mathbb{J}_{\mathfrak{s}} = \{\mathrm{j}_1,\dots,\mathrm{j}_{\mathfrak{s}} : -b < \mathrm{j}_{\mathfrak{s}} < \cdots < \mathrm{j}_1 < b = \mathrm{j}_0\}, \mathfrak{s} \geq 0$. And $S(x)$, be a spline of an order (r), on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n}$. Then there exist "an intertwining of a polynomials" $\{\mathcal{P}_1,\mathcal{P}_2\}$, for $S(x)$, with respect to $\mathbb{J}_{\mathfrak{s}}$ ∋

$$\|\mathcal{P}_1-\mathcal{P}_2\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}}\cup\mathbb{I}_{\mathbb{i}+1})} \leq C\omega_\varphi^k(\mathbb{f},|,\mathbb{I}_{\mathbb{i}}\cup\mathbb{I}_{\mathbb{i}+1}|,\mathbb{I}_{\mathbb{i}}\cup\mathbb{I}_{\mathbb{i}+1})_{\psi,p} \ .$$

Proof:

Firstly ,we show that the polynomial:

$$\mathcal{P}_{\mathrm{i}}(x) = P_n(x) + \sum_{j=0}^{\mathfrak{s}} \gamma_j\ \mathcal{T}_{i,j}(x) \ ,\quad i = 1,2 ,$$

Where $P_n(x)$, be a polynomial of degree $\leq r$, and $\gamma_j = \frac{\mathbb{S}_{j+1}^{(2m)}(x)-\mathbb{S}_j^{(2m)}(x)}{(2m)!}$, $j = 0,\dots,\mathfrak{s}$, also

$$\mathcal{T}_{i,j}(x) = \begin{cases}\mathfrak{J}_{a,j}(x) & ; \quad a = i = 1\\ \chi_{a,j}(x) & ; \quad a = i = 2\end{cases} \ , \quad \ni$$

$\mathcal{T}_{i,j}(x)$, be a polynomial of degree $\leq r$, that pleasure the confirmation (p_{i}), of theorem (1) , and which is pleasure the conclusion in (Lemma (5.22) [1]) , and

$$\chi_{a,j}(x) = \begin{cases}1 & ; \quad x \geq x_a\\ 0 & ; \quad o.w\end{cases} \ , x \in I \ ,$$

∋

$$\left|\mathfrak{J}_{a,j}(x) - \chi_{a,j}(x)\right| \leq C(\mathcal{M})\psi_{a,j}^{\mathcal{M}}(x) \ ,$$

where $\mathcal{M} \in N$, be a fixed .It is comparatively directly transmit to confirm , then:

$\|\mathcal{P}_1 - \mathcal{P}_2\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})}$

$$= \left\| \sum_{j=0}^{n} \gamma_j (\mathcal{T}_{1,j} - \mathcal{T}_{2,j}) \right\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})}$$

$\leq C \sum_{j=0}^{n} \|\gamma_j\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})} |\mathcal{T}_{1,j}(x) - \mathcal{T}_{2,j}(x)|$

$\leq C \sum_{j=0}^{n} \left\|\mathbb{S}_{j+1}^{(2m)} - \mathbb{S}_j^{(2m)}\right\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})} |\Im_{a,j}(x) - \chi_{a,j}(x)|$

$\leq C \sum_{j=0}^{n} \|\mathbb{S}_{j+1} - \mathbb{S}_j\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})} \psi_{a,j}^{\mathcal{M}-2m}(x)$,

Where $\mathbb{S}_{j+1}(x)$, denotes for "smooth spline approximation" on $\{\mathbb{X}_{\mathrm{j}}\}_{\mathrm{j}=0}^{n} \ni$ $\mathbb{S}_{j+1}(x) \equiv \overline{\mathbb{S}}\ \mathrm{I}_{\mathbb{I}_{\mathbb{i}}}$, and $\mathbb{S}_j(x) \equiv \mathbb{S}\ \mathrm{I}_{\mathbb{I}_{\mathbb{i}}}$, $x \in \mathbb{I}_{\mathbb{i}}$.

That is sense $\mathbb{S}_{j+1}$, $\mathbb{S}_j$, are pleasure the confirmation of (Theorem (1)), hence

$\overline{\mathbb{S}} - \mathbb{f}$, $\mathbb{S}_j - f \in \Delta^0(\mathbb{J}_{\mathfrak{s}}^*)$, where $\mathbb{f} \in L_{\psi,p}(\mathbb{I}_{\mathbb{i}}) \cap \triangle^0 (\mathbb{J}_{\mathfrak{s}}^*)$,

$\|\mathbb{S}_{j+1} - \mathbb{S}_j\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})} = \|\overline{\mathbb{S}} - \mathbb{S}\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})}$

$\|\mathcal{P}_1 - \mathcal{P}_2\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})}$

$$\leq \sum_{j=0}^{n} C \ \|\overline{\mathbb{S}} - \mathbb{S}\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}})} \ \psi_{a,j}^{\mathcal{M}-2m}(x) .$$

Since $\mathbb{I}_{\mathbb{i}} \subset \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1}$,

$\|\mathcal{P}_1 - \mathcal{P}_2\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})}$

$$\leq \|\overline{\mathbb{S}} - \mathbb{S}\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} \left(\sum_{j=0}^{n} C \ \psi_{a,j}^{\mathcal{M}-2m}(x) \right),$$

Not that $\sum_{j=0}^{n} C \ \psi_{a,j}^{\mathcal{M}-2m}(x) \leq C(p,k,\mathcal{M})$, $\mathcal{M} - 2m \geq 2$,

$\|\mathcal{P}_1 - \mathcal{P}_2\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})}$

$$\leq C\|\overline{\mathbb{S}} - \mathbb{S}\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})},$$

$$\leq C\|\overline{\mathbb{S}} - P\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} + C\|\mathbb{S} - P\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})}$$

$$= \mathbb{L}_1 + \mathbb{L}_2 .$$

$$\mathbb{L}_1 \leq C\|\overline{\mathbb{S}} - \mathbb{f}\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} + C\|\mathbb{f} - P\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} .$$

And

$$\mathbb{L}_2 \leq C\|\mathbb{S} - \mathbb{f}\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} + C\|\mathbb{f} - P\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} .$$

By [(Lemma (5.3.4) , Theorem (1.6.1) and Theorem (2.1.2)) ([5])], we get

$$\mathbb{L}_1 \leq C\omega_\varphi^k(\mathbb{f}, |, \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1}|, \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})_{\psi,p} \ ,$$

also

$$\mathbb{L}_2 \leq C\omega_\varphi^k(\mathbb{f}, |, \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1}|, \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})_{\psi,p} \ .$$

Hence

$$\|\mathcal{P}_1 - \mathcal{P}_2\|_{L_{\psi,p}(\mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})} \leq C\omega_\varphi^k(\mathbb{f}, |, \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1}|, \mathbb{I}_{\mathbb{i}} \cup \mathbb{I}_{\mathbb{i}+1})_{\psi,p} \ .$$

Where C , be a constant which is dependent on p , k , $\mathcal{M}$.

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