Differential Subordinations for New Classes of ω-Qusi Convex Functions with Respect to Symmetric Points

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Abstract . This paper introduce new classes of meromorphic multivalent ω-quasi convex functions with respect to symmetric points and discuss its differential subordination properties in the punctured unit disk U with applications in fractional calculus .

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1 .Introduction

Let $\sum_{\omega,p,\gamma}^{+}$ denote the family of all functions f,of the form:

$$f(z)=\frac{1}{(z-\omega)^p}+\sum_{k=2}^{\infty}a_k(z-\omega)^{k-\frac{k}{\gamma}}\quad ,\gamma\in N\backslash\{1\},p=1,2,\ldots \qquad (1)$$

which are analytic in the punctured unit disk $U=\{z\in C:0<|z|<1\}$,and ω is a fixed point in U.

Let $\sum_{\omega,p,\gamma}^{-}$ denote the family of all functions f,of the form:

$$f(z)=\frac{1}{(z-\omega)^p}-\sum_{k=2}^{\infty}a_k(z-\omega)^{k-\frac{k}{\gamma}}\quad ,\gamma\in N\backslash\{1\},p=1,2,\ldots \qquad (2)$$

which are analytic in the punctured unit disk $U=\{z\in C:0<|z|<1\}$.and ω is a fixed point in U.

For two functions f and g analytic in $\Delta=\{z\in C:|z|<1\}$,we say that the function f is subordinate to g in Δ and write $f(z)\prec g(z)$ or simply $f\prec g$ if there exists a schwarz function w which is analytic in Δ with $w(0)=0$ and $|w|<1$ such that

$$f(z)=g(w(z)),z\in\Delta.$$

Let $\psi:C^3\times\Delta\to C$ and let h analytic in Δ. Assume that $p.\psi$ are analytic and univalent in Δ and p satisfies the differential superordination

$$h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z). \qquad (3)$$

Then p is a solution of the differential superordination .An analytic function q is called a subordinat if $q\prec p$, for all p satisfying equation(3).A univalent function q such that $p\prec q$ for all subordinates p of equation(3) is said to be the best subordinate.

A functions $f \in \sum_{\omega,p,\gamma}^{+} \left(\sum_{\omega,p,\gamma}^{-}\right)$ is meromorphic multivalent ω-starlike with respect to symmetric points if $f(z) \neq 0$ and

$$-Re\left\{\frac{(z-\omega)f'(z)}{f(z)-f(-z)}\right\} > 0, (z-\omega) \in U. \quad (4)$$

A functions $f \in \sum_{\omega,p,\gamma}^{+} \left(\sum_{\omega,p,\gamma}^{-}\right)$ is meromorphic multivalent ω-convex with respect to symmetric points if $f(z) \neq 0$ and

$$-Re\left\{1+\frac{\left((z-\omega)f'(z)\right)'}{\left(f(z)-f(-z)\right)'}\right\} > 0, (z-\omega) \in U. \quad (5)$$

A functions $f \in \sum_{\omega,p,\gamma}^{+} \left(\sum_{\omega,p,\gamma}^{-}\right)$ is meromorphic multivalent ω-Quasi convex function if there exist a meromorphic multivalent ω-convex function g such that $g'(z) \neq 0$ and

$$-Re\left\{\frac{\left((z-\omega)f'(z)\right)'}{(g(z)-g(-z))'}\right\} > 0, (z-\omega) \in U. \quad (6)$$

This study establish some sufficient conditions for the functions belong to the classes $\sum_{\omega,p,\gamma}^{+}$ and $\sum_{\omega,p,\gamma}^{-}$ to satisfy

$$-Re\left\{\frac{2\left((z-\omega)^p f'(z)\right)'}{(g(z)-g(-z))'}\right\} \prec q(z), (z-\omega) \in U. \quad (7)$$

and q is the given univalent function in U. Moreover , we give application for these results in fractional calculs.

We need to the following lammas.

Lemma 1.[8] Let q be convex univalent in the unit disk Δ and $\phi, \alpha \in C$ with

$$Re\left(1+\frac{zq''(z)}{q'(z)}+\frac{\phi}{\alpha}\right) > 0.$$

If p is analytic in Δ and

$$\phi p(z) + \alpha z p' \prec \phi q(z) + \alpha z q'(z),$$

then $p \prec q$ is the best dominant

.

Lemma 2.[3] Let q be univalent in the unit disk Δ and Θ be analytic in adomain D containing $q(\Delta)$. If $zq'(z)\Theta(z)$ is starlike in Δ and

$$zp'(z)\Theta(p(z)) \prec zq'(z)\Theta(q(z)),$$

then $p \prec q$ and q is the best dominat.

2 .Subordination Theorems

This section establish some sufficient conditions for subordination of analytic function in the classes $\sum_{\omega,p,\gamma}^{+}$ and $\sum_{\omega,p,\gamma}^{-}$ .The same properties have been found for other classes in [1,2].

Theorem 1. Let the function q be convex univalent in U such that $q'(z) \neq 0$ and

$$Re\left(1+\frac{(z-\omega)q''(z)}{q'(z)}+\frac{\phi}{\alpha}\right) > 0, \alpha \neq 0. \quad (8)$$

Suppose that $\frac{-2\left((z-\omega)^p f'(z)\right)'}{(g(z)-g(-z))'}$ is the analytic in U . If $f \in \sum_{\omega,p,\gamma}^{+}$ satisfies the subordination

$$\frac{-2\left((z-\omega)^p f'(z)\right)'}{(g(z)-g(-z))'} \times \left[\phi + \alpha\left(\frac{(z-\omega)((z-\omega)^p f'(z))''}{((z-\omega)^p f'(z)'} - \frac{(z-\omega)(g(z)-g(-z))''}{(g(z)-g(-z))'}\right)\right]$$

$$\prec \phi q(z) + \alpha(z-\omega)q'(z),$$

then

$$\frac{-2((z-\omega)^p f'(z))'}{(g(z)-g(-z))'} \prec q(z),$$

and q is the best dominant.

Proof.Let the function

$p(z) = \frac{-2\left((z-\omega)^p f'(z)\right)'}{(g(z)-g(-z))'}$. Differentiating this function ,with respect to $(z-\omega)$and perfoming calculations we get

$$p'(z) =$$

$$\frac{-2((z-\omega)^p f'(z))''(g(z)-g(-z))'+2((z-\omega)^p f'(z))'(g(z)-g(-z))''}{[((g(z)-g(-z))']^2}$$

.

It can be observed that
$\phi p(z)+\alpha(z-\omega)p'(z)$

$$= \frac{-2((z-\omega)^p f'(z))'}{(g(z)-g(-z))'}\left[\phi + \alpha\left(\frac{(z-\omega)((z-\omega)^p f'(z))''}{((z-\omega)^p f'(z))'} - \frac{(z-\omega)(g(z)-g(-z))''}{(g(z)-g(-z))'}\right)\right] \prec \phi q(z)+\alpha(z-\omega)q'(z).$$

The Using the assumption the theorem the assertion of the theorem follows by an application of Lemma 1.

Corollary 1. Let equation(8)holds.Let the function q be univalent in U. Let $k=1$,if q satisfies the subordination

$$\frac{-2((z-\omega)f'(z))'}{(g(z)-g(-z))'}\left[\phi + \alpha\left(\frac{(z-\omega)((z-\omega)f'(z))''}{((z-\omega)f'(z))'} - \frac{(z-\omega)(g(z)-g(-z))''}{(g(z)-g(-z))'}\right)\right] \prec \phi q(z)+\alpha(z-\omega)q'(z),$$

then

$$\frac{-2((z-\omega)f'(z))'}{(g(z)-g(-z))'} \prec q(z),$$

and q is the best dominant.

Theorem 2. Let the function q be univalent in U such that $q \neq 0$,$(z-\omega) \in U$ and $\frac{(z-\omega)q'(z)}{q(z)}$ is starlike univalent in U . If $f \in \sum^{-}_{\omega,p,\gamma}$ satisfies the subordination

$$\eta\left(\frac{(z-\omega)((z-\omega)^p f'(z))''}{((z-\omega)^p f'(z))'} - \frac{(z-\omega)(g(z)-g(-z))''}{(g(z)-g(-z))'}\right) \prec \eta(z-\omega)\frac{q'(z)}{q(z)},$$

then

$$\frac{-2((z-\omega)^p f'(z))'}{(g(z)-g(-z))'} \prec q(z),$$

and q is the best dominant.

Proof . Let the function ϕ be defined by

$$\phi(z) = \frac{-2((z-\omega)^p f'(z))'}{(g(z)-g(-z))'}. \quad (z-\omega) \in U$$

By setting

$$\tau(\varsigma) = \eta\backslash\varsigma, \varsigma \neq 0$$

it can be cleared that τ is analytic in $C-\{0\}$. Then by simply computation we have

$$\eta\frac{(z-\omega)\phi'(z)}{\phi(z)} = \eta\left(\frac{(z-\omega)((z-\omega)^p f'(z))''}{((z-\omega)^p f'(z))'} - \frac{(z-\omega)(g(z)-g(-z))''}{(g(z)-g(-z))'}\right) \prec \eta\frac{(z-\omega)q'(z)}{q(z)}.$$

Then Using the assumption the theorem the assertion of theorem follows by application of Lemma 2.

Corollary 2. Assume that q is convex univalent in U .Let $p=1$, if $f \in \sum^{-}_{\omega,p,\gamma}$ and

$$\eta\left(\frac{(z-\omega)((z-\omega)f'(z))''}{((z-\omega)f'(z))'} - \frac{(z-\omega)(g(z)-g(-z))''}{(g(z)-g(-z))'}\right) \prec \eta\frac{(z-\omega)q'(z)}{q(z)},$$

then

$$\frac{-2((z-\omega)f'(z))'}{(g(z)-g(-z))'} \prec q(z),$$

and q is the best dominant.

3 .Applications of Fractionl Integral Operator

In this section , we introduce some application of section (2) containing fractional integral operators. Let $f(z) = \sum_{k=0}^{\infty} \varphi_k (z-\omega)^k$ and let us begin with the following definition. Following the ealier work by Srivastava and Owa [7,8] and Miller and Ross [4] and recently Darus and Faisal [1].

Difinition 1. [6] .For afunction f ,the fractional integral of order λ is defined by :

$$I_z^{\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^z f(\xi)(z-\xi)^{\lambda-1} d\xi, \lambda > 0 \quad (9)$$

where the function f is analytic in simply connected region of the complex z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed be requiring $log(z-\xi)$ to be real when $(z-\xi) > 0$. Note that $I_z^{\lambda} f(z) = f(z) \times \frac{z^{\lambda-1}}{\Gamma(\lambda)}$, for $z \geq 0$.

Let $a, b, c \in C$ with $C \neq 0, -1, -2, \ldots$. The Gaussian hyper geometric function ${}_2F_1$ is defined by:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (10)$$

where (x_n) is the pochhammer symbol defined , in terms of the Gamma function by:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{Bmatrix} 1 & ,(n=0) \\ x(x+1)(x+2)\ldots(x+n-1), n \in N \end{Bmatrix}.$$

Now, we introduce a ω-fractional integral operators .

Difinition 2. For $\lambda > 0$ and $\beta, \mu \in R$, the ω -fractional integral operator $I_{0,z,\omega}^{\lambda,\beta,\mu}$ is defined by :

$$I_{0,z,\omega}^{\lambda,\beta,\mu} f(z) = \frac{(z-\omega)^{-\lambda-\beta}}{\Gamma(\lambda)} \int_0^{z-\omega} ((z-\omega) - \xi)^{\lambda-1} {}_2F_1\left(\lambda+\beta, -\mu; \lambda; 1 - \frac{\xi}{z-\omega}\right) f(\xi) d\xi, \quad (11)$$

where the function f is analytic in a simply -connected region of the z-plane containing hte origin with the order

$$f(z) = O(|z-\omega|)^{\varepsilon} \quad (z-\omega \to 0),$$
$$\varepsilon > max\{0, \beta-\mu\} - 1,$$

and the multiplicity of $((z-\omega)-\xi)^{\lambda-1}$ is removed by requiring $log(z-\omega)$ to be real when $(z-\omega)-\xi > 0$.

From Definition (2)with $\beta < 0$,we have

$$I_{0,z,\omega}^{\lambda,\beta,\mu} f(z) = \frac{(z-\omega)^{-\lambda-\beta}}{\Gamma(\lambda)} \int_0^{z-\omega} ((z-\omega) - \xi)^{\lambda-1} {}_2F_1\left(\lambda+\beta, -\mu; \lambda; 1 - \frac{\xi}{z-\omega}\right) f(\xi) d\xi$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda+\beta)_k (-\mu)_k}{(\lambda)_k (1)_k} \frac{(z-\omega)^{-\lambda-\beta}}{\Gamma(\lambda)} \times \int_0^{z-\omega} ((z-\omega)-\xi)^{\lambda-1} \left(1 - \frac{\xi}{z-\omega}\right)^k f(\xi) d\xi$$

$$= \sum_{k=0}^{\infty} B_k \frac{(z-\omega)^{-\lambda-\beta-k}}{\Gamma(\lambda)} \int_0^{z-\omega} ((z-\omega) - \xi)^{k+\lambda-1} f(\xi) d\xi$$

$$= \sum_{k=0}^{\infty} B_k \frac{(z-\omega)^{-\beta-1}}{\Gamma(\lambda)} f(\xi)$$

$$= \frac{H_k}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \varphi_k (z-\omega)^{k-\beta-1},$$

where $H_k = \sum_{k=0}^{\infty} B_k$. Denote $a_k = \frac{H_k \varphi_k}{\Gamma(\lambda)}$, for all $k = 2,3, \ldots$ and let $\beta = k \backslash \gamma + 1$, thus

$$\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\in\sum\nolimits_{\omega,p,\gamma}^{+}\quad and\quad \frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\in\sum\nolimits_{\omega,p,\gamma}^{-}\qquad(\varphi_k\geq 0).$$

Note that ,when $\omega=0$ the ω fractional integral operator $I_{0,z,0}^{\lambda,\beta,\mu}$ was studied by Raina and srivastava [5]

Teorem 3. Let the function q be convex univalent in U such that $q'\neq 0$ and

$$Re\left(1+\frac{(z-\omega)q''(z)}{q'(z)}+\frac{\phi}{\alpha}\right)>0,\alpha\neq 0.\qquad(12)$$

Suppose that

$$\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}$$

is analytic in$U$. If $f\in\sum_{\omega,p,\gamma}^{+}$ satisfies the subordination

$$\phi\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}$$

$$+\alpha\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}\times$$

$$\left(\frac{(z-\omega)\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)''}{\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}-\frac{(z-\omega)\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)''}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}\right)$$

$$\prec\phi q(z)+\alpha(z-\omega)q'(z),$$

then

$$\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}\prec q(z),$$

and q is the best dominant.

Proof.Let the function p be defined by

$$p(z)=\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'},$$

$(z-\omega)\in U$.

It can easily observed that

$$\phi p(z)+\alpha(z-\omega)p'(z)=$$

$$\phi\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}+\alpha\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}$$

$$\times\left(\frac{(z-\omega)\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)''}{\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}-\frac{(z-\omega)\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)''}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}\right)$$

$$\prec\phi q(z)+\alpha(z-\omega)q'(z).$$

Then using the assumption the theorem the assertion of the theorem follows by an application of Lemma 1.

Teorem 4. Let the assumptions of Theorem (2) hold ,then

$$\frac{-2\left((z-\omega)^p\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}f(z)\right)'\right)'}{\left(\left(\frac{1}{(z-\omega)^p}+I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)-\left(\frac{-1}{(z-\omega)^p}-I_{0,z,\omega}^{\lambda,\beta,\mu}g(z)\right)\right)'}$$

$\prec q(z), \quad (z-\omega)\in U$

and q is the best dominat.

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التبعيات التفاضلية لأصناف جديدة من دوال *w-Quasi* المحدبة بالنسبة لنقاط التناظر

تاريخ الاستلام 2016/1/14 **تاريخ القبول 2016/3/9**

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الخلاصة :

يقدم هذا البحث اصناف جديدة من دوال w -quasi المحدبة ميرومورفك متعددة التكافؤ مع نقاط التناظر ومناقشة خصائص التبعية التفاضلية في ثقب قرص الوحدة مع تطبيقات تفاضل والتكامل الكسوري.

الكلمات الافتتاحية:

التبعية التفاضلية ، حساب التفاضل والتكامل الكسوري، شبه محدب مع نقاط التناظر .