Using differential transform method to solve fractional non-linear integro-differential equations

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L. Noor Ali Hussein
Department of Mathematics, Education of College , Al-Qadisiyah University

Abstract

In this paper we'll to solve the fractional integro-differential equations by employment differential transform method and compare with integro-differential equations by graph.

1- **Introduction**

In this paper we'll find solution the fractional non-linear integro-differential equations which the form

$$v^q(t) = (t, v(t), v'(t), \int_{t_0}^{t} G\big(s, v(s), v'(s)\big)ds , \quad (1)$$

with conditians

$$v(t_0) = v_0 , v'(t_0) = v_1 . \quad (2)$$

Where $t \in [t_0, T]$ and $m - 1 \leq q \leq m$, $m \in N$,

by using differential transform method .

There are sundry definitions of a fractional derivativ of order $q > 0$, here we depended on Caputo definition.

$$D_{t_0}^{q} f(t) = J^{m-q}\left[\frac{d^m}{dt^m} f(t)\right] \quad (3)$$

Where $m - 1 < q \leq m$ and $m \in N$.

The Caputo fractional drivative first calculates an ordnary drivative followed by a fractional integral to ascertain the wanted order of fractional derivative .

2- **Differential Transform**

Definition 2.1. Let $z(t)$, is anatomy function of one inconstant which is defined on $L = [0, t] \subseteq \Re$ and $t_0 \in L$. $Z(k)$, is Differential transform of $z(t)$ and is predefined on N union $\{0\}$ as the following:

$Z(k)=\frac{1}{k!}\left[\frac{d^k z(t)}{dt^k}\right]_{t=t_0}$ (4) ,

where $z(t)$ is the fundamental function and $Z(k)$ is called the transformed function .Inverse differential transform of $Z(k)$ in the is predefined as follows

$z(t)=\sum_{k=0}^{\infty} Z(k)(t-t_0)^k$. (5)

Then from the above two equations (4)and (5), with $t_0=0$, the function $z(t)$ can be written as:

$z(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k z(t)}{dt^k}\right]_{t=0} t^k$ (6)

the principal mathematical specifications of differential transform can be summarized in the following theorems .

3. Theorems [1],[2]

Theorem 3.1

If $Z(k), F(k)$ and $G(k)$ are differential transforms of the functions $z(t), f(t)$ and $g(t)$ consecutive , then :

1. If $z(t)=f(t)\pm g(t)$ then $Z(k)=F(k)\pm G(k)$.

2. If $z(t)=af(t)$ then $Z(k)=aF(k)$.

3. If $z(t)=f(t)g(t)$ then $Z(k)=\sum_{l=1}^{k} F(l)G(k-l)$.

4. If $z(t)=\frac{df(t)}{dt}$ then $Z(k)=(k+1)F(k+1)$.

5. If $z(t)=\frac{d^m f(t)}{dt^m}$ then $Z(k)=(k+1)(k+2)\cdots(k+m)F(k+m)$.

6. If $z(t)=\int_0^t f(s)ds$ then $Z(k)=\frac{F(k-1)}{k},\ K\geq 1, Z(0)=0$.

7. If $z(t)=t^m$ then $Z(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & O.W.\end{cases}$

8. If $z(t)=\sin(\omega t+a)$ then $Z(k)=\frac{\omega^k}{k!}sin\left(\frac{k\pi}{2}+a\right)$.

9. If $z(t)=\cos(\omega t+a)$ then $Z(k)=\frac{\omega^k}{k!}cos\left(\frac{k\pi}{2}+a\right)$.

10. If $z(t)=e^{\omega t}$ then $Z(k)=\frac{\omega^k}{k!}$.

Theorem 3.2. Assume that $Z(k), W(k), J_1(k)$ and $J_2(k)$, are the differential transforms of the functions $z(t), w(t), j_1(t)$ and $j_2(t)$, consecutive, then for $k=1,2,\cdots,N$,

1. If $z(t)=\int_{t_0}^{t} j_1(s)j_2(s)ds$ then $Z(k)=\frac{1}{k}\sum_{\ell=0}^{k-1} J_1(\ell)J_2(k-\ell-1)$

2. If $z(t)=w(t)\int_{t_0}^{t} j_1(s)j_2(s)ds$ then $Z(k)=\sum_{\ell=0}^{k}\sum_{s=0}^{k-\ell-1}\frac{1}{k-\ell}W(\ell)J_1(s)J_2(k-\ell-s-1)$.

3. If $z(t)=\int_{t_0}^{t}\frac{d^{n_1}}{dt^{n_1}}j_1(s)\frac{d^{n_2}}{dt^{n_2}}j_2(s)ds$, then $Z(k)=\frac{1}{k}\sum_{\ell=0}^{k-1}\frac{(n_1+\ell)!(n_2+k-\ell-1)!}{l!(k-\ell-1)!}\times J_1(n_1+l)J_2(n_2+k-\ell-1)$.

4. If $z(t) = \frac{d^m}{dt^m} w(t) \int_{t_0}^{t} \frac{d^{n_1}}{dt^{n_1}} j_1(s) \frac{d^{n_2}}{dt^{n_2}} j_2(s) ds$ the $Z(k) = \sum_{\ell=0}^{k} \sum_{s=0}^{k-\ell-1} \frac{(m+\ell)!(n_1+s)!(n_2+k-\ell-s-1)!}{(k-\ell)!\ell!s!(k-\ell-s-1)!} \times J_1(n_1+s) J_2(n_2+k-\ell-s-1)\, W(m+\ell)$

4. Fractional differential transform

Let the anatomy and continuous function $z(t)$ in terms of a fractional reinforce series as follows:

$$z(t) = \sum_{k=0}^{\infty} Z(k)(t-t_0)^{k/\alpha}, \quad (7)$$

where α is the order frction and $Z(k)$ is the frctional differential transform of $z(t)$.

The fractional derivative in Caputo is

$$D_{t_0}^{q} z(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \left\{ \int_{t_0}^{t} \left[\frac{z(s) - \sum_{k=0}^{m-1} \left(\frac{1}{k!}\right)(s-t_0)^k z^{(k)}(t_0)}{(t-s)^{1+q-m}} \right] ds \right\} \quad (8)$$

The transformation of the initial conditions are defined as follows:

$$Z(k) = \begin{cases} \text{If } k/\alpha \in Z^+, & \frac{1}{(k/\alpha)!} \left[\frac{d^{k/\alpha} z(t)}{dt^{k/\alpha}} \right]_{t=t_0} \\ \text{If } k/\alpha \notin Z^+, & 0 \end{cases} ,(9)$$

where, q is the order of fractional differential equation considered .

we succinct the fractional differential transform method with some theorems

Theorem 4.1.

1. If $h(t) = g(t) \pm f(t)$, then $H(k) = G(k) \pm F(k)$.

2. If $h(t) = g(t)f(t)$, then $H(k) = \sum_{l=0}^{k} G(l)F(k-l)$.

3. If $h(t) = f_1(t) f_2(t) \cdots f_{n-1}(t) f_n(t)$, then

$$H(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} F_1(k_1) F_2(k_2-k_1) \cdots F_{n-1}(k_{n-1}-k_{n-2}) F_n(k-k_{n-1})$$

4. If $h(t) = (t-t_0)^p$, then $H(k) = \delta(k-\alpha p)$ where ,

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

5. If $h(t) = D_{t_0}^{q}[f(t)]$, then $H(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} F(k+\alpha q)$.

6. If $h(t) = \frac{d^{q_1}}{dt^{q_1}}[f_1(t)] \frac{d^{q_2}}{dt^{q_2}}[f_2(t)] \cdots \frac{d^{q_{n-1}}}{dt^{q_{n-1}}}[f_{n-1}(t)] \frac{d^{q_n}}{dt^{q_n}}[f_n(t)]$, then $H(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)} \frac{\Gamma[q_2+1+(k_2-k_1)/\alpha]}{\Gamma[1+(k_2-k_1)/\alpha]} \cdots \frac{\Gamma[q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha]}{\Gamma[1+(k_{n-1}-k_{n-2})/\alpha]} \Gamma[q$ ... $\alpha q_1) \times F_2(k_2-k_1+\alpha q_2) \cdots F_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1}) \times F_n(k-k_{n-1}+\alpha q_n)$, where $\alpha q_i \in Z^+$ for $i = 1,2,\cdots,n$.

for $k = 0,1,\ldots,q\alpha$

4. Numerical examples

Example 1 To solve the equation

$$z^q(t) = \frac{1}{2} z'(t)u(t) - u(t) - \int_0^t [z'(s)]^2 ds + \frac{1}{2} + t , \quad t \geq 0 \quad (10)$$

with conditions

$z(0) = -1, \quad z'(0) = 1, \quad z''(0) = \frac{1}{2}$ (11)

By using differential transformation method on Equ.(10),for $k = 1,2,\dots$, we acquire

$Z(k+\alpha q) = \frac{\Gamma(1+k/\alpha)}{\Gamma(q+1+k/\alpha)}\left[\frac{1}{2}\sum_{\ell=0}^{k}(\ell+1)Z(\ell+1)Z(k-\ell) - Z(k) - \frac{1}{k}\sum_{\ell=0}^{k-1}(\ell+1)(k-\ell)Z(\ell+1)Z(k-\ell) + \frac{1}{2}\delta(k) + \delta(k-1)\right]$,
(12)

where α is the unknown value of the fraction of q .

By using Eq.(9) the initial conditions is

$Z(0) = -1$

$Z(1) = 1$

$Z(2) = \frac{1}{2}$

$Z(3) = 0$,
$for\ k = 3,\dots,9,11,12,\dots,19,21$ (13)

$Z(10) = 1$

$Z(20) = \frac{1}{4}$

Now, in [1], when $q = 2$, the exact solution of Eq.(10) is $(z(t) = \sin(t) - \cos(t))$ and it's got from the serise solution

$$z(t) = -1 + t + \frac{1}{2}t^2 - \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{5040}t^7 + \cdots$$

Here, we take $q = 2.1$ then the approximate solution for Eq.(10) is

$$z(t) = -1 + t^{\frac{1}{10}} + \frac{1}{2}t^{\frac{2}{10}} + t + \frac{1}{4}t^2 - 2\frac{\Gamma(\frac{11}{10})}{\Gamma(\frac{32}{10})}t^{\frac{11}{5}} + \frac{1}{4}\frac{\Gamma(\frac{12}{10})}{\Gamma(\frac{33}{10})}t^{\frac{23}{10}} - \frac{1}{12}\frac{\Gamma(\frac{13}{10})}{\Gamma(\frac{34}{10})}t^{\frac{24}{10}} + \dots$$

Fig.1 shows the complete solution for Eq.(10), when $q = 2$, Fig.2 shows the Sacrificial solution for Eq.(10), when $q = 2.1$

Fig.1 plot eq.(4.1) when q=2

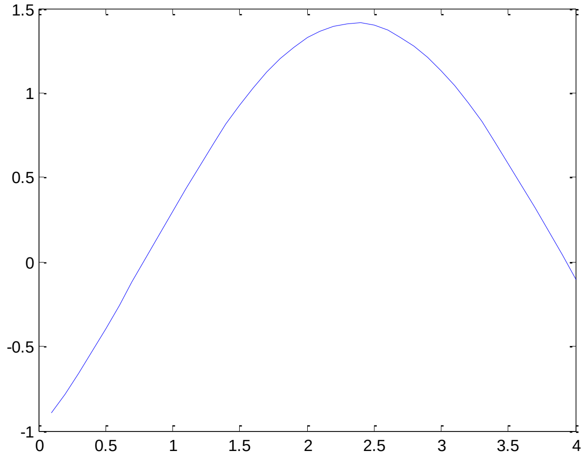
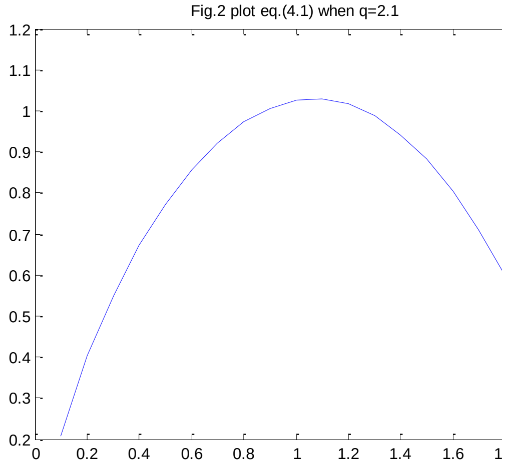


Fig.2 plot eq.(4.1) when q=2.1



Example 2. To solve the equation

$$z^q(t) = \frac{1}{2}z'(t) - z(t)\int_0^t z'(s)z'(s)ds + \frac{1}{2}e^{3t} \quad (14)$$

with conditions

$$z(0) = z'(0) = 1 \quad (15)$$

By using differential transformation method on Eq.(14),for $k = 1,2,\dots$, we acquire

$$Z(k+19) = \frac{\Gamma(1+k/10)}{\Gamma(q+1+k/10)}\left[\frac{k+1}{2}Z(k+1) - \sum_{\ell=0}^{k-1}\sum_{s=0}^{k-\ell-1}\frac{(k-\ell-s)(s+1)}{k-\ell}Z(\ell)Z(s+1)Z(k-\ell-s) + \frac{3^k}{2k!}\right], \quad (16)$$

where α is the unknown value of the fraction of q .

By using Eq.(9) the initil conditions is

$Z(0) = 1$

$Z(1) = 1$

$Z(3) = 0, \ for\ k = 2,3,\dots,9,11,12,\dots,19$

$$Z(10) = 1 \quad (17)$$

Now, in [1], when $q = 2$, the the exact solution of Eq.(14) is $(z(t) = e^t)$ and it's got from the series solution

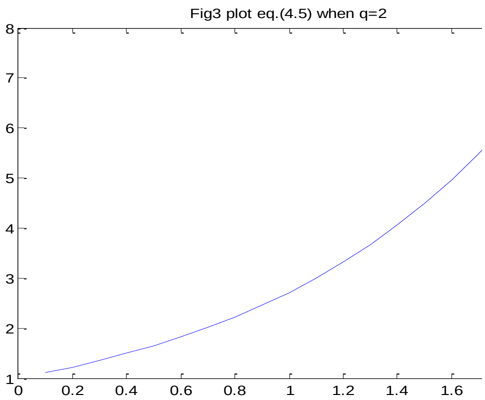
$$z(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{5040}t^7 + \cdots$$

If we continues for $k > 5$ the solution is $z(t) = e^t$

Here, we take $q = 2.1$ then the approximate solution for Eq.(14) is

$$z(t) = 1 + t^{\frac{1}{10}} + t + \frac{1}{2}\frac{\Gamma\left(\frac{11}{10}\right)}{\Gamma\left(\frac{30}{10}\right)}t^{\frac{20}{10}} + \frac{5}{4}\frac{\Gamma\left(\frac{12}{10}\right)}{\Gamma\left(\frac{31}{10}\right)}t^{\frac{21}{10}} + \frac{9}{4}\frac{\Gamma\left(\frac{13}{10}\right)}{\Gamma\left(\frac{32}{10}\right)}t^{\frac{22}{10}} + \cdots$$

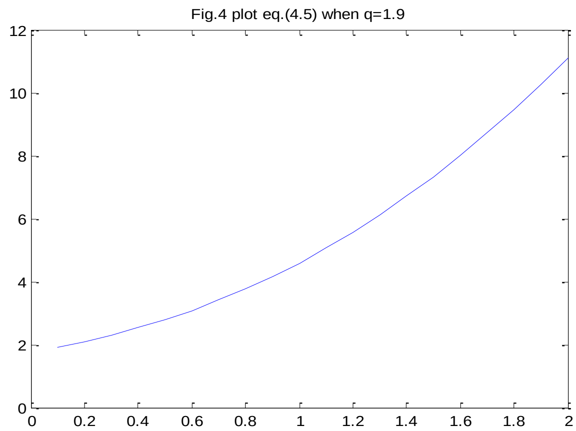
Fig.3 shows the complete solution for Eq.(14) acquired for the value of $q = 2$, i.e $(z(t) = e^t)$. Fig.4 shows the Sacrificial solution for eq.(14) acquired for the value of $q = 1.9$.



By practise differential transformation method on Eq.(18),for $k = 1,2,\ldots$, we acquired

$$Z(k+\alpha q) = \frac{\Gamma(1+k/\alpha)}{\Gamma(q+1+k/\alpha)}\left[(k+1)Z(k+1) - 2\sum_{\ell=0}^{k-1}\sum_{s=0}^{k-\ell-1} Z(\ell)Z(s)Z(k-\ell-s) + \frac{1}{k!}\sum_{\ell=0}^{k}\frac{\delta(\ell-3)3^{k-\ell}}{(k-\ell)!}\right] , \quad (20)$$

where α is the obscure value of the fraction of q .



Initial conditions in Eq.(19)are transformed by employment Eq.(9) as follows:

$$Z(0) = 0$$

$$Z(1) = -1$$

$$Z(2) = 0 \, , for\ k = 2,3,\ldots,9,11,12,\ldots,18$$
$$Z(10) = 1 \, ,$$

Now, in [1], when $q = 2$, the the exact solution of Eq.(18) is $(z(t) = te^t)$ and it's got from the series solution

Example 3 we take the D.T. for the following integro-differential equation

$$z^q(t) = z'(t) - 2z(t)\int_0^t z(s)z'(s)ds + e^t + t^3e^{3t} \qquad (18)$$

with initial conditions

$$z(0) = 0, \quad z'(0) = 1 \qquad (19)$$

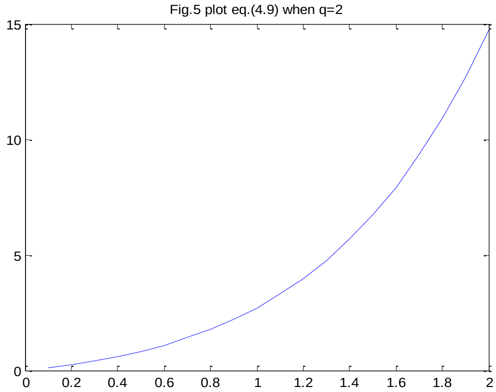
$$z(t) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 - \frac{1}{24}t^5 + \frac{1}{120}t^6 - \frac{1}{720}t^7 + \cdots$$

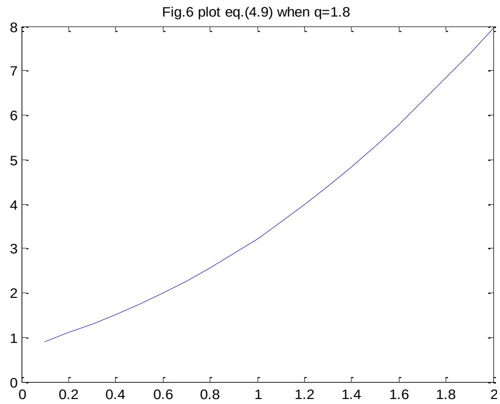
If we continues for $k > 5$ the solution is $z(t) = te^t$

when, $q = 1.8$ the approximate solution for eq.(18) is

$$z(t) = t^{\frac{1}{10}} + t + \frac{\Gamma(\frac{11}{10})}{\Gamma(\frac{29}{10})} t^{\frac{19}{10}} + \frac{1}{2}\frac{\Gamma(\frac{12}{10})}{\Gamma(\frac{30}{10})} t^{\frac{20}{10}} + \frac{7}{6}\frac{\Gamma(\frac{13}{10})}{\Gamma(\frac{31}{10})} t^{\frac{21}{10}} + \cdots$$

Fig.5 shows the complete solution for Eq.(18) when $q = 2$. Fig.6 shows the Sacrificial solution for eq.(18) acquired for the value of $q = 1.8$.





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