On Comonotony Approximation in Quasi Normed Space

By

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Abstract: In this paper we are introduce the relationship between the best approximation by the polynomial $g_n \in \prod_n$ and modules of smoothness $\omega_k^{\varphi}(f, n^{-1})$, and the modules of smoothness τ_k , in quasi normed $L_{\psi,p}(I)$ space $0 < p < 1$,and the polynomial g_n which change its comonotone approximation with the function f at every point in an interval I.

Key word: Comonotone, degree of best approximation, Modulus of smoothens.

حول التقريب المحافظ على الرتابة في الفضاء شبه المعياري

بواسطة

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المستخلص : في هذا البحث اوجدنا العلاقة بين افضل تقريب بواسطة متعددة الحدود

, $g_n \in \prod_n$ ومقياس النعومة $\omega_k^{\varphi}(f, n^{-1})_{\psi,p}$, وبين مقياس النعومة τ_k في الفضاء المعياري $L_{\psi,p}(I)$ $0 < p < 1,$ ومتعددة الحدود g_n والتي تتغير رتابتها مع الدالة f عند كل نقطة من نقاط الفترة I .

الكلمات المفتاحية : التقريب المحافظ على الرتابة , درجة افضل تقريب , مقياس النعومة .

1.Introductions and definitions:

The approximation will be carried out by a polynomials $g_n \in \prod_n$, the space of polynomials of degree not exceeding n,which have the same shape in which we are interested as f, namely, have the same sign as f does in various parts of I, or change their monotonicity exactly where f does in I.

Interest in the subject began in the 1960s with work on monotone approximation by Shisha([3]), Lorentz and Zeller. It gained momentum in the 1970s and early 1980s with the work on monotone approximation of Devore,and the work on comonotone approximation of Shvedov, of Newman and of Beatson and Leviatan([5]).The last 15 years have seen extensive research and many new results, the most advanced of which are being summarized here([4]). We are not going to give an elaborate historical account and we direct the interested reader to an earlier survey by the author ([1]).

To be specific, let $s \geq 0$ and let Y_s be the set of all collections $Y_s = \{y_1, \dots, y_s | y_0 = -b < y_s < \cdots < y_1 = b\}$, where for $s = 0, y_0 = \phi$. For $y_s \in Y_s$ we set $\prod(x, y_s) = \prod_{i=1}^{s}(x - y_i)$, we let $\Delta^1(y_s)$ be the set of all functions f which change monotone at the points $y_i \in Y_s$, . In particular if$= 0$, then f is non-decreasing in $I = [-b, b]$, and we will write $f \in \Delta^1$. Moreover if f is differentiable in $(-b, b)$,then $f \in \Delta^1(y_s)$ iff $f'(x)\prod(x, y_s) \geq 0, -b < x < b$ .Now we are introduced some of definitions which are important in this paper .

The weighted quasi normed space $L_{\psi,p}(I), 0 < p < 1$ ([2]) is:

$$L_{\psi,p}(I) = \left\{ f \ni f: I \subset R \longrightarrow R: \left(\int_I \left| \frac{f(x)}{\psi(x)} \right|^p dx \right)^{\frac{1}{p}} < \infty, 0 < p < 1 \right\}$$

and the quasi normed $\|f\|_{L_{\psi,p}(I)} < \infty$.

Now for $\in \Delta^1(y_s) \cap L_{\psi,p}(I)$, the degree of best comonotone approximation ([2])we are denote by:

$$E_n^{(1)}(f, y_s)_{\psi,p} = \inf_{p_n \in \prod_n \cap \Delta^1(y_s)} \|f - p_n\|_{L_{\psi,p}(I)}.$$

Again if $y_0 = \phi$,then $E_n^{(1)}(f)_{\psi,p} = E_n^{(1)}(f, \phi)_{\psi,p}$,which is usually referred to as the degree of monotone approximation.

Let $f \in L_{\psi,p}(I)$,and the symmetric kth difference ([2]) :

$$\Delta_h^k(f,x,I)_\psi = \Delta_h^k(f,x)_\psi$$
$$= \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \dfrac{f(x-\frac{kh}{2}-ih)}{\psi(x+\frac{kh}{2})} & , x \pm \frac{kh}{2} \in I \\ 0 & o.w \end{cases}$$

The Ditizian-Totik modulus of smoothness of f which is defined for such an f as follows ([2]):

$$\omega_\varphi^k(f,\delta,I)_{\psi,p} = \sup_{0<h\le\delta} \left\|\Delta_h^k(f,.)\right\|_{L_{\psi,p}(I)}$$

Where φ be a function of , and the so called τ-modulus (or sendove-popov modulus), an averaged modulus of smoothness, defined for a function f on an interval I by :

$\tau_k(f,\delta,I)_{\psi,p} = \|\omega_k(f,.,\delta)\|_{L_{\psi,p}(I)}$,([2]) where

$$\omega_k(f,x,\delta)_{\psi,p}$$
$$= sup\left\{\left|\frac{\Delta_h^k(f,y)}{\psi(y,\frac{kh}{2})}\right| : (y \pm \frac{kh}{2}) \right.$$
$$\left. \square \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right] \cap [-b,b]\right\}$$

Is the kth local modulus of smoothness of f. We take new chepyshev partition $X_j = acos\frac{j\pi}{n}$, $1 \le a < \infty$, $0 \le j \le n$ ([3]).

2.Auxilary and main results:

our aim in auxiliary results are the following Lemmas are the main components of the proof:

Theorem (2.1): There exists a polynomial $g_{k-1} \in \prod_{k-1} \cap \bar{\Delta}^1(I)$, $k > 1$, interpolate $f \in L_{\psi,p}(I) \cap \Delta^1(I), 0 < p < 1$, at $k > 1$ points in side an interval of $J_{\mathcal{A}} = [y_0 + \mathcal{S}|I|, y_1 - \mathcal{S}|I|]$, where $\mathcal{S} < \frac{1}{2}$ is a strictly positive constant then:

$$\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} \le \qquad c(p,k)\omega_k^\varphi(f,|I|,I)_{\psi,p} .$$

Theorem (2.2): For a function $f \in L_{\psi,p}(I) \cap \Delta^1(I)$, there exists a polynomial $g_{k-1} \in \prod_{k-1} \cap \square^1(Y_s)$, $k > 1$, satisfies:

$$\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} \leq \qquad c(p,k)\tau_k(f,|I|,I)_{\psi,p}.$$

Lemma (2.3): There exists a polynomial $\mathfrak{K}_{k-1}$, interpolate $f \in L_{\psi,p}(\ell_i) \cap \Delta^1(\ell_i), 0 < p < 1$ at $k-1$, points in side $Y_i \subset \ell_i = \left[y_i^{(v)}, y_i^{(k-1)}\right]$, $v = 1, \dots, k-2$ such that :

$$\|\mathfrak{N}_{k-1}(f)\|_{L_{\psi,p}(\mathfrak{I})} \leq c(p,\mathfrak{m})\|f\|_{L_{\psi,p}(\mathfrak{I})}.$$

Where $\square = [\frac{y_i + y_i^{(v)}}{k-1}, \breve{a}]$ and $\breve{a} = \frac{y_i + y_i^{(k-1)}}{k-1} + \mathfrak{m}|Y_i| < y_i^{(k-1)}, \mathfrak{m} > o$.

Proof: Suppose that $\mathfrak{N}_{k-1}(f)$interpolate f at the points y_i inside Y_i, $i = 1, \dots, s$ then $\mathfrak{N}_{k-1}(f) = \left\{f(y_1, \dots, y_s) \prod^s_{\substack{i=1 \\ 0 \leq y_i \leq n}} (x_j - y_i)\right\}$, since $f(y_i, \dots, y_s) \geq 0$ and $\mathfrak{N}_{k-1}(f)$ is non decreasing for $x_j \geq y_s$, hence $\mathfrak{N}_{k-1}(f) \geq 0$ for $x_j \geq y_s$ (since $f(y_s) \geq 0$) thus $f \geq \mathfrak{N}_{k-1}(f)$ for $\frac{y_i + y^{(v)}}{k-1} < \frac{y_i + y_i^{(k-1)}}{k-1} + \mathfrak{m}|Y_i| < y_i^{(k-1)}$ there fore $\|\mathfrak{N}_{k-1}(f)\|_{L_{\psi,p}[\frac{y_i + y_i^{(k-1)}}{k-1}, \breve{a}]} \leq c(p,\mathfrak{m})\|f\|_{L_{\psi,p}[\frac{y_i + y_i^{(k-1)}}{k-1}, \breve{a}]}$

Since $\left|\left[\frac{y_i + y_i^{(k-1)}}{k-1}, \breve{a}\right]\right| \approx |Y_i|$, we get

$$\|\mathfrak{N}_{k-1}(f)\|_{L_{\psi,p}(\mathfrak{I})} \leq c(p,m)\|f\|_{L_{\psi,p}(\mathfrak{I})}.$$

Corollary 2.4: Let $Y_i \subset \ell_i$ and $f \in L_{\psi,p}(\ell_i) \cap \Delta^1(\ell_i), 0 < p < 1$ .Let $\mathfrak{N}_{k-1}(f) \in \prod_{k-1} \cap \Delta^1(\ell_i)$, interpolate f at $k-1$ points in side Y_i then for any constant $m > 0$ such that :

$$\|\mathfrak{N}_{k-1}(f)\|_{L_{\psi,p}(\wp)} \leq c(p,m)\|f\|_{L_{\psi,p}(\wp)}.$$

Where $\square = [\breve{b}, \frac{y_i + y_i^{(k-1)}}{k-1}]$ and $\breve{b} = \frac{y_i + y_i^{(v)}}{k-1} - n\,|Y_i| < y_i^{(v)}$.

Proof: by the same method in Lemma (2.3) where □ = $[\breve{b}, \frac{y_i + y_i^{(k-1)}}{k-1}]$ and $\breve{b} = \frac{y_i + y_i^{(v)}}{k-1} - n\ |Y_i| < y_i^{(v)}$ we get the result. If $n = 0$, then the result is no true.

Lemma 2.5: Let $f \in \Delta^1(I)$, $k \geq 1$ and let ℓ_i be an interval of length $\leq y_i^{(k-1)} - y_i^{(v)}/(4k+1)$ in the center of I (i.e; $dis(\ell_i, -y_i^{(k-1)}) = dis(\ell_i, y_i^{(k-1)})$, let $q_{k-1} \in \prod_{k-1}$, interpolate f at $k-1$ points in ℓ_i. If $f \in L_{\psi,p}(I), 0 < p < 1$ then:

$$\|f - q_{k-1}\|_{L_{\psi,p}(I)} \leq c(p,k)\omega_k^{\varphi}(f, |I|, I)_{\psi,p} .$$

Proof: For an interval $\ell_i = \left[y_i^{(v)}, y_i^{(k-1)}\right]$ we denote that $|\ell_i| = y_i^{(k-1)} - y_i^{(v)}$ and $[(2n+1)\ell_i] = \left[y_i^{(v)} - n.|\ell_i|, y_i^{(k-1)} + m|\ell_i|\right]$, it is sufficient to prove (2.5) for the interval $\ell_i \ni |\ell_i| = (k-1)|Y_i|$ where $Y_i = \left[\frac{y_i + y_i^{(v)}}{k-1}, \frac{y_i + y_i^{(k-1)}}{k-1}\right]$ and $|Y_i| = \frac{y_i^{(k-1)} - y_i^{(v)}}{k-1}$, $\ell_i = (k-1)Y_i$, which is means ℓ_i consists of $k-1$ of interval Y_i with $(k-1)|Y_i| = |\ell_i|$, $k \geq 4$ assume that $Y_i \subset (k-1)Y_i = \ell_i$, $k \geq 4$ now let $f \in \Delta^1(I)$ and q_{k-1} interpolate f at $k-1$ points inside ℓ_i

$$\|f - q_{k-1}\|_{L_{\psi,p}[(k-1)Y_i]} \leq c(p,k)\omega_k^{\varphi}(f, |\ell_i|, \ell_i)_{\psi,p} .$$

Take $|\ell_i| \approx |Y_i|$ and since $k \geq 1$,$0 < p < 1$, $Y_i \subset I$ and $f \in L_{\psi,p}(I), \ell_i \subset I \ni Y_i \subset \ell_i$, then

$$\|f - q_{k-1}\|_{L_{\psi,p}(\ell_i)} \leq c\|f - q_{k-1}\|_{L_{\psi,p}[(k-1)Y_i]} + c\omega_k^{\varphi}(f, |\ell_i|, \ell_i)_{\psi,p} , \ell_i \sqcup I$$

Hence

$$\|f - q_{k-1}\|_{L_{\psi,p}(I)} \leq c(p,k)\omega_k^{\varphi}(f, |I|, I)_{\psi,p} .$$

Lemma 2.6 : Let ℓ_i be an interval in the center of I and q_{k-1} interpolate $f \in L_{\psi,p}(I), 0 < p < 1$ at $k-1$ points inside ℓ_i then :

$$\|f - q_{k-1}\|_{L_{\psi,p}(I)} \leq c(p,k)\tau_k(f, |I|, I)_{\psi,p} .$$

Proof : By lemma (2.5) there exist $q_{k-1} \in \prod_{k-1}$ comonotony approximation and interpolate with f inside in ℓ_i ,such that $\|f - q_{k-1}\|_{L_{\psi,p}(I)} \leq c(p,k)\omega_k^{\varphi}(f,|I|,I)_{\psi,p}$

Since $|f - q_{k-1}| \lesssim c(p)\|f - q_{k-1}\|_{L_{\psi,p}(I)}$

$\omega_k^{\varphi}(f,|I|,I)_{\psi,p} \leq \omega_k^{\varphi}(f,|I|,I)_{\psi,\infty} \leq \omega_k^{\varphi}(f,x,I)_{\psi,\infty}$

We get

$$|f - q_{k-1}| \leq c(p.k)\omega_k^{\varphi}(f,|x|,I)_{\psi,\infty}$$

By take $L_{\psi,p}(I)$ −quasi norm of both sides we get $\|f - q_{k-1}\|_{L_{\psi,p}(I)} \leq c(p,k)\left\|\omega_k^{\varphi}(f,.,I)_{\psi,\infty}\right\|_{L_{\psi,p}(I)}$

By take τ −modulus, we get the result.

Proof theorem (2.1): Let $\mathcal{A} > 0$, be fixed and let ℓ_i , $i = 1,\dots,s$ be an interval of length $|\ell_i| = y_i^{(k-1)} - y_i^{(v)}$, $k > 1$, $v = 1,\dots,k-2$ in the center of I , i.e, $dist\left(\ell_i, -y_i^{(k-1)}\right) = dist(\ell_i, y_i^{(k-1)})$,let $q_{k-1} \in \prod_{k-1}$ interpolate f at $k-1$ points inside $\ell_i \cap J_{\mathcal{A}}$, by lemma (2.5) implies that $\|f - q_{k-1}(f)\|_{L_{\psi,p}(I)} \leq c(p,k)\omega_k^{\varphi}(f,|I|,I)_{\psi,p}$.

Now, let $\mathcal{F}_{k-1} = \mathcal{F}_{k-1}(f) \in \prod_{k-1}$ interpolate f at $k-1$ points in $\left[y_i^{(k-1)} - \mathcal{A}|I|, y_i^{(k-1)} - \frac{1}{2}\mathcal{A}|I|\right]$, then

$$\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)} = \|f - q_{k-1}(f) + q_{k-1}(f) - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)}$$

$$\leq c(p)\|f - q_{k-1}(f)\|_{L_{\psi,p}(I)} + c(p)\|\mathcal{F}_{k-1}(f - q_{k-1})\|_{L_{\psi,p}(I)} \leq$$

$c(p,k)\|f - q_{k-1}(f)\|_{L_{\psi,p}(I)} +$
$c(p,k)\|\mathcal{F}_{k-1}(f - q_{k-1})\|_{L_{\psi,p}\left[y_i^{(k-1)} - \mathcal{A}|I|, y_i^{(k-1)}\right]}$, by lemma (2.3)

$$\leq c(p,k)\|f - q_{k-1}(f)\|_{L_{\psi,p}(I)} + c(p,k)\|f - q_{k-1})\|_{L_{\psi,p}\left[y_i^{(k-1)} - \mathcal{A}|I|, y_i^{(k-1)}\right]}$$

Since $y_i^{(k-1)} - \mathcal{J}|I| \le y_i^{(k-1)} - \frac{1}{2}\mathcal{A}|I|$,using lemma (2.3) with a constant $m = 1$ hence $\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)} \le c(p,k)\|f - q_{k-1})\|_{L_{\psi,p}(I)}$

$$\le c(p,k)\omega_k^{\varphi}(f,|I|,I)_{L_{\psi,p}(I)}$$

$$\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} = \|f - \mathcal{F}_{k-1} + \mathcal{F}_{k-1} - g_{k-1}(f)\|_{L_{\psi,p}(I)}$$

$$\le c(p,k)\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)} + \|g_{k-1}(f - \mathcal{F}_{k-1}(f))\|_{L_{\psi,p}}$$

$$\le c(p,k)\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)} + c(p,k)\|g_{k-1}(f - \mathcal{F}_{k-1}(f))\|_{L_{\psi,p}\left[-y_i^{(k-1)}, y_i^{(k-1)} - \mathcal{A}|I|\right]}$$

$$\le c(p,k)\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)} + c(p,k)\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}\left[-y_i^{(k-1)}, y_i^{(k-1)} - \mathcal{A}|I|\right]}$$

By corollary(2.4) with $[-y_i^{(k-1)} + \mathcal{A}|I|, y_i^{(k-1)} - \mathcal{J}|I|]$ and $n = \mathcal{A}/(2\mathcal{A} - \mathcal{A})$

thus $\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} \le c(p,k)\|f - \mathcal{F}_{k-1}(f)\|_{L_{\psi,p}(I)} \le c(p,k)\omega_k^{\varphi}(f,|I|,I)_{L_{\psi,p}(I)}$.

Proof of theorem (2.2): let $\mathcal{G}_{k-1} \in \prod_{k-1}, k > 1 \ni$

$\mathcal{G}_{k-1} = \frac{\mathcal{P}_{k-1}}{\psi(x)} + E_{k-1}(f,\ell_i)_{\psi,p}$, and $\mathcal{G}_{k-1} > \frac{f(x)}{\psi(x)}$, when $f(x) \ge 0 \Rightarrow \frac{f(x)}{\psi(x)} \ge 0 \Rightarrow \mathcal{G}_{k-1} \ge 0$, and when $f(x) < 0 \Rightarrow \frac{f(x)}{\psi(x)} < 0$ hence $\mathcal{G}_{k-1} < 0$ this implies that $\mathcal{G}_{k-1}(x) \in \Delta^1(\ell_i)$ and since $\mathcal{G}_{k-1} \in \prod_{k-1}$, we get $\mathcal{G}_{k-1}(x) \in \prod_{k-1} \cap \Delta^1(\ell_i)$, this meaning that $\mathcal{G}_{k-1}$ comonotony with f at every points in an interval I. Now

$$\mathcal{G}_{k-1} > \frac{\mathcal{G}_{k-1}}{\psi(x)} \Rightarrow \mathcal{G}_{k-1} - \frac{f(x)}{\psi(x)} > \frac{\mathcal{G}_{k-1}(x)}{\psi(x)} - \frac{f(x)}{\psi(x)}$$

$$\frac{\mathcal{P}_{k-1}(x) - f(x)}{\psi(x)} \le \frac{\mathcal{P}_{k-1}(x) - f(x)}{\psi(x)} + E_{k-1}(f,\ell_i)_{\psi,p}$$

Hence

$$\|\mathcal{G}_{k-1} - f\|^p_{L_{\psi,p}(I)} \le c(p)\|f - \mathcal{P}_{k-1}\|^p_{L_{\psi,p}(I)} + c(p)E_{k-1}(f,\ell_i)^p_{\psi,p}$$

Since $\mathcal{P}_{k-1}$ best approximation of f then

$$\|\mathcal{G}_{k-1} - f\|^p_{L_{\psi,p}(I)} \le c(p)\|f - \mathcal{P}_{k-1}\|^p_{L_{\psi,p}(I)}$$

By theorem (2.1) there exist $\mathcal{g}_{k-1}(f) \in \prod_{k-1} \cap \Delta^1(I)$ such that

$$\mathcal{P}_{k-1}|_{\ell_i} = \mathcal{g}_{k-1}(f,x)|_{\ell_i} = \mathcal{g}_{k-1}(f,x,y_1,\dots,y_s)|_{\ell_i} \;, k > 1$$

By using lemma (2.6) , we get

$$\|f - \mathcal{F}\|_{L_{\psi,p}(I)} \le c(p,k)\tau_k(f,|I|,I)_{\psi,p} \,.$$

Hence

$$\|f - \mathcal{g}_{k-1}\|_{L_{\psi,p}(I)} \le \qquad c(p,k)\tau_k(f,|I|,I)_{\psi,p} \,.$$

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