Chebyshev Polynomials in Collocation Methods for Solving Singular Perturbation Problems

Dr.Khalid. Mindeel. M. Al-Abrahemee
Department of Mathematics, College of Education / University of AL-Qadisiyha
(Email: Khalid.mohammed@qu.ed)

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Abstract

This paper discourse the utility Momentmethod on second, third and fourth kind Chebyshev polynomials as attempt functions in solution singular perturbation problems with boundary condition. The process is computational very unmingled and magnetic. Applications are justly demonstrated through numerical examples to illustrated the effectiveness and easiness of the approximate, all the calculation outcome are obtain using Matlab.

Keywords: Singular perturbation problems; Chebyshev Polynomials ;Ordinary differential equation; Two point boundary value problem; Boundary layer.

Physical Classification QA299.6- 433

1. Introduction

Singulare perturbation problems include a small perturbation parameter ε , rising very repeatedly in manybough of applied mathematics such as , fluide dynamiecs, quantume mechaniecs, chemicale reactor theory, elasticity, aerodynamicse, and the other dominion of the big mankind offluid motion. A few noticeable exemplify are Boundary stratum problems, the drift-diffusion equation of semiconductor invention modeling . These problems have allow a important amount of care in elapsed and recall years . It is well given actuality that the resolution of singular perturbation problems display a multi scaler charcter, that is, there are thin shift layer where the disruption modify fast, while hence from the layers the resolution carry ragularly and modify tardly. Therefore, the numerical dealing of singularly perturbed problems immediate some mayor compuetational difficulty. If we referthe existent canonical numerical method for solution these problems, huge oscillationse may proceed and taint the resolution in the whole period for of the boundary layer behavior. Thus, more effective and simplist computtional technique are request to resolve singulaerly perturbad two-boundary value problems. The examine paper by Kadalbajeoo and Reddy [1,2]. Natesan and Bawa [3], have study singularly perturbed reaction diffusion Robin boundary-value problems and say an almost second-order (up to a logarithmic factor) uniformly converging plan, which is a proper alliance of the canonical limited distinction plan and the cubic spline scheme. The intend plan has been appropriate on a piece-wise uniform Shishkin mess. Andargie and Reddy[4] have immediate a numerical integration way for the disruption of general singularly perturbed two-point boundary value problems with mixed conditions of left or right end boundary layer. In this way the primary secondorder differential equation has been refund by an approaching first order differentail eqaution with a small digress evidence and then worn the trapezoid formula, a three expression return relationship has been holdand solution by Thomas algorithmic program. For a elaborate disputation on singalur perturbatione problems one may relate to the books and hie level monographse : O'Maley [5], Nayfeeh [6], Keovorkian and Cole [7], Bendar and Orszeag [8], Farrel et. al. [9], and Roose et. al. [10]. The numerical management ofsingularly perturbed problems instant some adult computational difficulties and in late years a huge number of peculiar discourse methods have been intend to supply correct numereical solution [11-15] by Kadalebajoo. This typify of problem has been intensiveely learned analyticaly and itis understood that its resolution commonly has a multi scale character; i.e. It faetures regieons called "boundary layers" where the breachmodify fast. And these equations as well as numericale methodes have been learned by a number of author [16-20]. In this paper, produce modern Method to Approximate Solution of the singularly perturbed problems with boundary condition on Chebyshev polynomial .

2.Definitions of chebyshev polynomials

In this part, we give an preliminary to the Chebyshev polynomials and their fundamental properties. See the regard [21- 24] for more details.

2.1 The first-kind Chebyshev polynomial τ_n [25]

The first kindof Chebyshev polynomial Tn(×) of degree n, give details by:

$T_n = \cos n\theta$ when $\times = \cos\theta$ …..(1).

If the array of the varieable × is the period [−1,1], then the range of the confortmable ariable θ can be in use as [0, π]. These range are transversal in contrary directiones, since × = −1 agree to θ = π and × = 1 agree to θ = 0.

It is well implicit the cos n θ is a polynamial of degre n in cos θ, and inded we are easy with the introductory formula

$Cos0\theta = 1, Cos1\theta = Cos\theta, Cos2\theta = 2\,Cos2\theta - 1, Cos3\theta = 4\,Cos3\theta -$

$3\, Cos\theta, Cos4\theta = 8\, Cos4\theta - 8\, Cos2\theta + 1, \dots$

The first kind of Chebyshev polynomials $T_n(\times)$ definitely e×plain by the vocation return detail. Set

$\tau_0(\times) = 1$ and $\tau_1(\times) = \times$,then $\tau_n(\times) = 2 \times \tau_{n-1}(\times) - \tau_{n-2}(\times)$ $n = 1,2, \dots$ (2).

The properties of Chebyshev polynomialscan be formulate as:-[25]
1-They are rectangular with moment duty
$w(\times) = \frac{1}{\sqrt{1-\times^2}}$ on the period [-1,1] that is:-

$$\int_{-1}^{1} \frac{1}{\sqrt{1-\times^2}} \tau_n(\times)\, \tau_m(\times) d\times = \begin{cases} 0 & n = m \\ \pi & n \neq m \\ \frac{\pi}{2} & n = m = 1,2,.. \end{cases}$$

And

$$\tau_0 x = 1, \tau_1 x = x, \tau_{m+1} x = 2x\, \tau_m(x) - \tau_{m-1}(x) \quad \dots.. \ (3).$$

2.2 The second- kind Chebyshev Polynomials Un [25]

The second kind of Chebyshev polynomial Ʊn(×) of degre n in × definite by

$Un(\times) = sin(n + 1)\theta / sin\theta \ when \ x = Cos\theta. \dots$ (4)

The runges of × and θ are the same as for $\tau_n(\times)$. The basic formula give

sin 1 θ= sin θ , sin 2 θ = 2sin θ cos θ ,sin θ 3 θ =sin θ (4cos 2 θ-1) , sin4 θ=sin θ (8cos 3 θ-4cos4 θ) , ….

, so the functions (4) is in fact a polynomial in Cos θ, and we may at once take that

$Ʊ_0(x) = 1,\ Ʊ_1(x) = 2x,\ Ʊ_2(\times) = 4 \times 2 - 1,\ Ʊ_3(\times) = 8 \times 3 - 4 \times .$

By confederate the trigonometric sameness sin(n + 1)θ + sin(n − 1)θ = 2 cos θ sin nθ

We get that Un(×) satiate the return detail

$$Ʊ_n(\times) = 2xƱ_n - 1(\times) - Ʊ_n - 2(\times), n = 2, 3, ..\ with\ U_0(\times) = 1, Ʊ_1(\times) = 2 \times \dots\dots..(5).$$

2.3 The Third and Fourth kind Chebyshev Polynomials V_n and W_n :-[25]

Therese other familiesof polynomails $V_n(\times)$ and $W_n(\times)$ may be create, which are told to $\tau_n(\times)$ and $Ʊ_n(\times)$ butwhich have trigonomtric definition twist the half angle θ⁄2 (where × = cosθ as before. The Chebyshev polynomails $V_n(\times)$ and $F_n(\times)$ of the third and fourth kinds are polynomails of degree n in ×:-

$$\mathbf{V}_n(\times) = 2x\mathbf{V}_{n-1}(\times) - \mathcal{H}_{n-2}(x) with\ \mathbf{V}_0(\times) = 1, \mathbf{V}_1(\times) = 2\times - 1. \ \dots\dots\dots(6)$$

$$\mathbf{W}(\times) = 2 \times \mathbf{W}_{n-1}(x) - \mathbf{W}_{n-2}(\times) with\ W_0(x) = 1, \mathbf{W}_1(\times) = 2 \times +1. \dots\dots.(7) \quad where\ n = 2,3,4,..$$

then

$$\mathbf{V}_0(\times) = 1, \mathbf{V}_1(\times) = 2\times - 1, \mathbf{V}_2 \times = 4\times^2 - 2\times - 1, V_3(\times) = 8\times^3 - 4\times^2 - 4\times + 1 \dots\dots (8)$$

and

$$\mathbf{V}_0(\times) = 1, \mathbf{V}_1(\times) = 2\times + 1, \mathbf{V}_2 \times = 4\times^2 + 2\times - 1, V_3(\times) = 8\times^3 + 4\times^2 - 4\times - 1 \dots (9)$$

Thus $V_n(\times)$ and $W_n(\times)$plowshare exactly the plowshare accurately the $\tau_n(\times)$, Ʊn(×) and their production contend only in the rescriptions of the initial condition n=1.

3.Descript method

The method(Moment) is one of theefficacy remaining methods, There is a previous study to discuss type the first kind and the second kind by scientists respectively by Sarker and Su usefulness in [26] and Abdolerza, Farhad and Jafar amended in [27]. And to illustrate the new method of research we think the following sacond order linear singalur perturebation probleme with two-point boundary conditions:-

$M[y(\times)] = \varepsilon y'' + p(\times)y' + q(\times)y = f(\times)10)$

with b.c $y(a) = g_0 \, , y(b) = g_1$ (11)
where $\times\in [a,b]$ andthep,q and f are continuos function on [a,b] and $0 < \varepsilon \ll 1$. The problm of findeng an approch resolution to the singular perturbation problem (10) is often possess by pretentious the solution y(×) as :-

$y_N(\times) = \sum_{i=0}^{N} d_i C_i = \frac{1}{2} d_0 C_0(\times) + d_1 C_1(\times) + \cdots + d_N C_N(\times)$ (12)

for all $\times\in [a,b]$ [25] .

Where Ci ‘s are kind Chebyshv polynomail defined in (5), (8) and (9). (second , third and fourth) This approach must satiate the baundary

conditiones (11), exchang (12) in (10) we get the relics in the differential equation:-

$E(\times) = |M(y)(\times) - f(\times)|$

4. The Solution of Second Order SPBVP Using Moment Method

If we have a second order of singular perturbation problem with boundary conditions (**SPBVP**)

$\varepsilon y'' + p(\times)y' + q(\times)y = f(\times)$ (14)
with bc $y(a) = g_0 \, , y(b) = g_1$
and $0 < \varepsilon \ll 1$

Can be approximated the unknown function u(×) by:- $y_N(\times) = \sum_{i=0}^{N} d_i C_i(\times)$ (15)

where Ci ‘s are the second , third and fourth kind Chebyshev polynomiales. since these approximation mustsatisfy the b .c , we get:-

$$y(a) = \frac{1}{2} d_0 C_0(a) + d_1 C_1(a) + \cdots + d_N C_N(a) = g_0$$

hence $d_0 = 2\left(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}\right)$ (16)

$y(b) = \frac{1}{2} d_0 C_0(b) + d_1 C_1(b) + \cdots + d_N C_N(b) = g_1$

hence

$d_1 = \frac{1}{C_1(b)} [\left(\left(g_1 - 2\left(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}\right)\right)\frac{C_0(b)}{2}\right) - \sum_{i=2}^{N} d_i C_i(b)]$ …………… (17).

By substitute equation (16) and (17) into (15) we get

$y_N(\times) = 2\left(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}\right) C_0(\times) + \frac{1}{C_1(b)} [\left(\left(g_1 - 2\left(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}\right)\right)\frac{C_0(b)}{2}\right) - \sum_{i=2}^{N} d_i C_i(b)] C_1(b) + \sum_{i=2}^{N} d_i C_i(\times)$,

Now we use oparator form to get:- M[y] = f(×) and we defined M is $M[y(\times)] = \varepsilon \frac{d^2}{d\times^2} y_N + p(\times)\frac{d}{d\times} y_N + q(\times) y_N$ We can conclude E(×) = M[y] - f(×) such that t w_0 , w_1 ,..., w_N linearly independen on period [a,b] that is:-

$\int_a^b W_j(\times) E(\times) dx = 0 \; j = 1,2, \dots \dots, N$
where $w_j = \times^j$

Now we solve the system of N-1 equations by Gaussian elimination process to find exchange in eq.(14) to hold the approaching solutionof z(×).

5. Numerical Examples

Consider the following linear two-point singular perturbation problem:-

Example 5.1[28]

Consider the following linear two-point singular perturbation problem:-

$\boldsymbol{\varepsilon y''(\times) - \times y'(\times) - y(\times) = f(\times) = \left(\frac{\times+1}{\varepsilon} - 1\right) e^{\frac{\times+1}{\varepsilon}} - 2\left(\frac{\times-1}{\varepsilon} + 1\right) e^{\frac{\times-1}{\epsilon}}}$ ‘
boundary condition y(-1)=1, y(1)=2,and exact solution is $y = e^{-(\times+1)/\varepsilon} + 2e^{(\times-1)/\varepsilon}$, $\varepsilon = 10^{-3}$.

By use moment method for this problem with N = 6 we get the approximat solution

$$y_N(\times) = \sum_{i=0}^{6} d_i C_i\,(\times)$$

To illustration of this example then the compare betwen the exactand approxemated solution and find the error by M.S.E from table (1). Figure (1) shows a illustration betweenthe analytic solution and the approximat solution of theproblem which is immediate in example (5.1) using The proposed methodof Chebeyshev polynomails in all its kinds .

Example 5.2 [29]

Suppose we have the following equation of linear two-point singular perturbation problem:-

$$\varepsilon y''(\times) + y'(\times) = f(\times) = 0\ ,\ y(0)=0,\ (1)=1$$

And analytic solution is $y(x) = \frac{1-e^{-\times/\varepsilon}}{1-e^{-1/\varepsilon}}$, $\varepsilon = 10^{-6}$

By use moment method for this problem with N = 7 we get the appro×imat solution

$$y_N(\times) = \sum_{i=0}^{7} d_i C_i\,(\times)$$

To illustration of this example then the compare betwen the exactand approxemated solution and find the error by M.S.E from table (2). Figure (2) shows a illustration betweenthe analytic solution and the approximat solution of theproblem which is immediate in example (5.2) using The proposed methodof Chebeyshev polynomails in all its kinds .

6. Conclusions

In this study, we used a new method to solve the problems of linear and nonlinear singular perturbation problems , which is one of the most important issues in life applications. The softness used was superior in accuracy and results through some examples used linear and nonlinear by numerical comparison between the analytical solution and the approximate solutions of Ghabayshiv Error through least square error.

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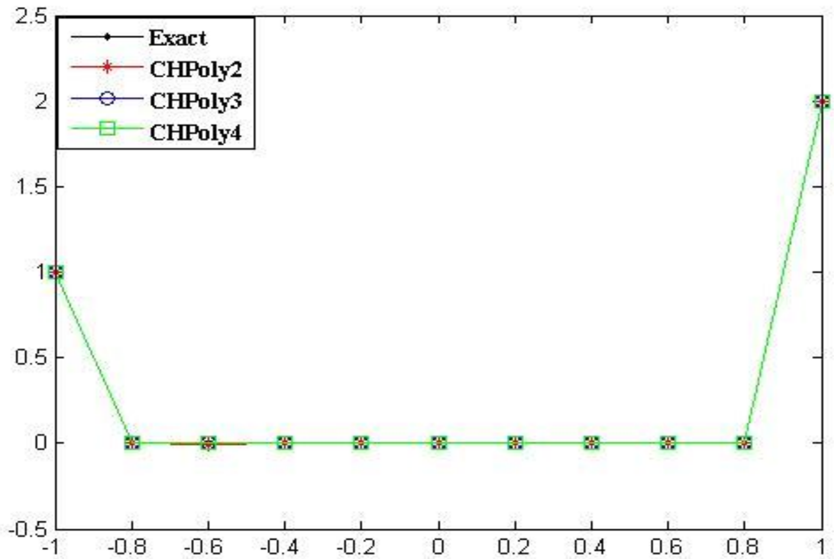

Figure(1) : shows a comparison between the exact solution and the approximate solution of the problem which is presented in e×ample 5.1 .

Table(1): presents a comparison between the e×act and approximated solution which depends on the least square error for e×ample 5.1 .

x	"Analytic solution"	"Suggested solutions"		
	$ya(x)$	"second − kind Chebyshev polyn	"third − kind Chebyshev p	"fourth − kind Chebyshev pol
−1.0	1.0000000000	1.0000003937	1.0000000034	1.0000000000
−0.8	0.0000000000	0.0000000298	0.0073764358	0.0000000000
−0.6	0.0000000000	−0.0027170337	0.0000001497	0.0011930953
−0.4	0.0000000000	−0.0002514149	−0.0000010480	0.0000000000
−0.2	0.0000000000	−0.0000059206	0.0000033015	−0.0001719885
0	0.0000000000	0.0000168176	0.0000017426	−0.0000959423
0.2	0.0000000000	0.0000225346	−0.0000007325	0.0000000000
0.4	0.0000000000	0.0000246020	−0.0000008292	0.0000000000
0.6	0.0000000000	−0.0000195530	−0.0000012017	−0.0013701290
0.8	0.0000000000	0.0000008998	0.0000002269	0.0000000000
1.0	2.0000000000	2.0000000063	2.0000000639	2.0000000000
$M.S.E$		0.0000006770	0.0000049465	0.0000003036

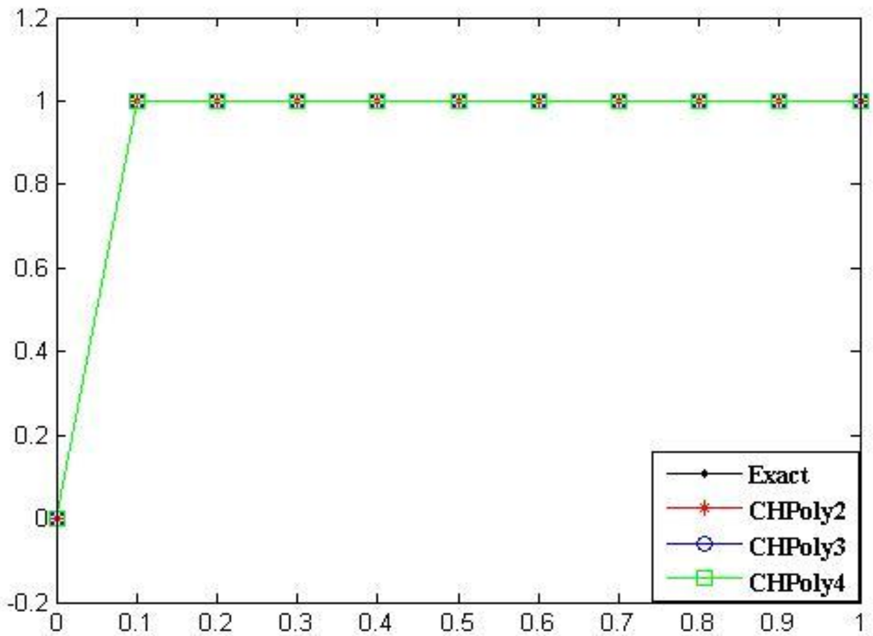

Figure (2): shows a comparison between the e×act solution and the approximate solution of the problem which is presented in e×ample (5.2) .

Table (2): presents a comparison between the exact and appro×imated solution which depends on the least square error for e×ample 5.2 .

x	Analytic solution	Suggested solutions		
	ya(x)	second − kind Chebyshev poly	third − kind Chebyshev polyno	fourth − kind Chebyshev poly
0.0	0	0.0000130832	0.0000000001	−0.0000000306
0.1	1	1.0000034922	0.9999999996	0.9999999958
0.2	1	1.0000554567	0.9999999998	1.0000000493
0.3	1	0.9998675023	0.9999999997	0.9999998127
0.4	1	1.0002554458	1.0002729360	1.0000210329
0.5	1	0.9997890236	1.0000361803	1.0000016391
0.6	1	1.0000808177	1.0000000116	0.9999966236
0.7	1	1.0006286158	0.9999992516	1.0000018832
0.8	1	1.0005880716	0.9999998711	1.0000086743
0.9	1	0.9995769010	1.0000001175	1.0000136983
1.0	1	0.9976329902	1.0000002701	1.0000167176
L.S.E		0.0000006054	0.0000000069	0.0000000001

متعددة حدود جبيشيف لحل مسائل الاضطراب الشاذة

خالد منديل محمد
جامعة القادسية- كلية التربيه -قسم الرياضيات
الاميل: Khalid_math98@yahoo.com

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الخلاصه :

في هذا البحث استخدمت الطريقة اللحظية على متعدد حدود تشيبيشيف من من الرتبة الثانية والثالثة والرابعة لحل مسائل الاضطراب المنفردة ذات الشروط الحدودية . الطريقة تكون سهلة حسابيا وفعالة وأظهرت التطبيقات على حد سواء من خلال أمثلة عددية لتوضيح كفاءة وبساطة الطريقة، ويتم الحصول على كل النتائج الحسابية باستخدام ماتلاب.

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