On characterization of connected graphs without induced subgraphs -3- paws

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Abstract:

For the vertices a of the graph G, $G(a)$ denoted the subgraph on the set of all vertices adjeceent with a will be write as $[a]$ this subgraph is called a neighborhood of the vertex a in the graph G. In this work we investigates several properties of these connected graphs without induced subgraphs -3-paws.

Keywords: undireceted graphs , connected graphs, induced subgraphs , graphs without 3-paws.

1. Introduction:

In this work we consider only finite undireceted graphs without loops and multiple edges. An undireceted graph[9] is a graph in which edges have no orientation(i.e., the edge (x, y) is identical to the edge (y, x)) . The maximum number of edges in an undireceted graph without a loop is $\frac{n(n-1)}{2}$. Formally, let $G(V, E)$ be any graph. Tow vertices x, y of a graph G are said to be "adjacent" (to each other) if (x, y) is an edge of the graph G . A graph G is conncted when there is a path between every pair of vertices in G. An induced subgraph[5] of a graph G is another graph, formed from a subset of the vertices of the graph and all of the edges onnecting pairs of vertices in that subset . A clique[6], C, in an undireceted $G = (V, E)$ is a subset of the vertices, $C \subseteq V$, such that every two distinct vertices are adjaceent. This is equivalent to the condition that the induced subgraph of G induced by C is a complete graph. A coclique in a graph G is a clique in its complementary graph . If we fix the graph G then $G(a)$ denoted the subgraph on the set of all vertices adjeceent with a will be write as $[a]$. $a^{\perp}$ the subgraph on the set $[a] \cup \{a\}$. For the subgraph Δ of the graph G, $\Delta^{\perp}$ will be denoted the subgraph on the set $\bigcap_{a \in \Delta} a^{\perp}$. For a vertex a of a graph G, will be denote the i – neighborhood of a, i.e., the subgraph induced by G on the set of all vertices lying at a distance of i from a by $G_i(a)$. The core of the subgraph Δ containing more than one vertex, we will said to be the subgraph $K(\Delta) = \Delta^{\perp} \cap \Delta$. The core of the vertex a is called a subgraph $K(a)=\{x\in G \mid x^{\perp}=a^{\perp}\}$. The subgraph $[a]\bigcap[b]$ it is called « – subgraph [8] if the distance between the vertices a,b equal to 2 in the graph G . The set $\{a,b_1,...,b_m\}$ will be denoted m – paw in which the vertex a adjaceent with the vertices $b_1,...,b_m$. The graph with four edges or (sides) and four vertices or (corners) is called a quadrangle graph G. In our work [1],[2],[3] we introduced the new concepts of an induced subgraphs of the graph of binary relations of adjaceency and determined the algebraic system consisting of all binary relations of set and of all unordered pairs various adjeceent binary relations and we investigated some its subgraphs. In [7] obtained a classification of edge-regular graphs without 3-paws. Graphs without 3-paws with disconnected « – subgraphs (not necessarily finite) were studied by [4] . The conneceted locally *GQ*(*s*, *t*) graphs in [8] are studied in which every *μ*-subgraph is a known strongly regular graph . In this work we continue the study of class of conneceted graphs without induced subgraphs –3 -paws and investigates some its properties.

2.Conneceted graphs without 3-paws.

The purpose of this section we obtaining some general properties of graphs without 3-paws and without additional restrictions and G in this work denotes a connected graph without 3-paws.

Proposition 2.1: Let $a\in G$. Then:

1.If b, c – non-adjaceent vertices of $[a]$, then $[a]\subseteq b^{\perp}\bigcup c^{\perp}$;

2.For any edge ac of G then the subgraph $[a]-c^{\perp}$ is a clique;

3.If b,c – non-adjaceent vertices of $G-a^{\perp}$, then « – subgraph $[b]\bigcap[c]$ is contained in $G-a^{\perp}$ (i.e, $[b]\bigcap[c]\bigcap[a]=\varnothing$).

Proof: 1. Let b, c – non-adjaceent vertices of $[a]$, $x\in[a]$ and $x\notin b^{\perp}\bigcup c^{\perp}$,then the subgraph on $\{a;x,b,c\}$ is 3-paw. Therefore, $x\in b^{\perp}\bigcup c^{\perp}$.

2.Let $[a]-c^{\perp}$ is not clique. Then x,y non-adjaceent vertices of $[a]-c^{\perp}$ we get the 3-paw $\{a;x,y,c\}$ that contradicts with the hypothesis. Therefore $[a]-c^{\perp}$ is a clique.

3.Let $[b]\bigcap[c]\bigcap[a]\neq\varnothing$. Then x – the overall adjaceent to the b and c and hence we get 3-paw $\{x;a,b,c\}$. Therefore $[b]\bigcap[c]\bigcap[a]=\varnothing$.

Proposition 2.2 : If $acbd$ – quadranagle graph G. Then:

1. $a^{\perp}\bigcup b^{\perp}=c^{\perp}\bigcup d^{\perp}$;

2.If ce – edge of the graph $([a]\bigcap[b])-d^{\perp}$,then $e^{\perp}-d^{\perp}=c^{\perp}-d^{\perp}$.

Proof: 1. we prove that $a^{\perp}\bigcup b^{\perp}\subseteq c^{\perp}\bigcup d^{\perp}$ by propoisition(2.1)(1) $a^{\perp}\subseteq c^{\perp}\bigcup d^{\perp}$ and $b^{\perp}\subseteq c^{\perp}\bigcup d^{\perp}$ therefore $a^{\perp}\bigcup b^{\perp}\subseteq c^{\perp}\bigcup d^{\perp}$. Return (for c and d) similarly.

2. from (1) above $a^{\perp} \bigcup b^{\perp} = c^{\perp} \bigcup d^{\perp}$ and $a^{\perp} \bigcup b^{\perp} = e^{\perp} \bigcup d^{\perp}$, then $e \square d$ and hence $e^{\perp} \bigcup d^{\perp} = c^{\perp} \bigcup d^{\perp}$ and $e^{\perp} - d^{\perp} = c^{\perp} - d^{\perp}$.

"A pair of vertices a, b is called a strong pair of graph G[8] if « $-$subgraph $[a] \bigcap [b]$ is not a clique. We note that if $acbd-$ quadranagle graph G then pairs a, b and c, d are strong". Conversely, if a pair of vertices is strong, then it is contained in some quadranagle.

Proposition 2.3: Let a, b are strong pairs. Then:

1.If the neighborhood of some vertex x contains «-subgraph $[a] \bigcap [b]$, then $x^{\perp}$ contains $a^{\perp}$ or $b^{\perp}$.

2.if $x \in [a] - [b]$, then either $[x] \bigcap [b]$ contains not adjaceent vertex with the vertex a or $a^{\perp}$ contains $x^{\perp}$ or $[x] \bigcap [b]$ is clique of $[a] \bigcap [b]$.

Proof: 1. Consider the three possible cases for x.

$(*)$If $x \square a$ and $x \square b$ then by propoisition (2.2)(1) $a^{\perp} \bigcup x^{\perp} = c^{\perp} \bigcup d^{\perp}$ and $a^{\perp} \bigcup b^{\perp} = c^{\perp} \bigcup d^{\perp}$. Therefore, $a^{\perp} \bigcup b^{\perp} = a^{\perp} \bigcup x^{\perp}$ and $x^{\perp} - a^{\perp} = b^{\perp} - a^{\perp}$. By hypothesis, $x^{\perp}$ contains $[a] \bigcap [b]$, then $x^{\perp}$ contains $b^{\perp}$.

$(**)$ If $x \square a$ and $x \square b$,then, similarly , we can obtain: $x^{\perp}$ contains $a^{\perp}$.

$(***)$ Since $[x]$ contains «-subgraph $[a] \bigcap [b]$ then the case, where the vertex $x \square a$ and $x \square b$ impossible.

2. Suppose that, $x \square a$ and $x \square b$. If $[a] \bigcap [b] \not\subset [a]$ is performed (1) above. Now let $[a] \bigcap [b] \subseteq [a]$ and we assume that there are two non-adjaceent vertices in $[x] \bigcap [b]$. Therefore, x, b strong pair then from (1) above, $a^{\perp}$ contains $x^{\perp}$ or $b^{\perp}$ and since $a \square b$, then $a^{\perp}$ contains $x^{\perp}$.

Fix a quadranagle $acbd$ of the graph G. And proved that the following propositions:

Proposition 2.4: If $e \notin a^{\perp} \bigcup b^{\perp}$ and the subgraph $[b] \bigcap [e]$ is not clique, then $x^{\perp} = b^{\perp}$ for any non-adjaceent with a the vertices $x \in [c] \bigcap [d]$.

Proof: We take the vertex x of $([c] \bigcap [d]) - a^{\perp}$.Then from propoisition (2.2)(1) and since $acbd$ quadrangle graph then we get: $a^{\perp} \bigcup x^{\perp} = c^{\perp} \bigcup d^{\perp}$ and $a^{\perp} \bigcup b^{\perp} = c^{\perp} \bigcup d^{\perp}$.Hence $a^{\perp} \bigcup b^{\perp} = a^{\perp} \bigcup x^{\perp}$ and $b^{\perp} - a^{\perp} = x^{\perp} - a^{\perp}$. Since the subgraph $[b] \bigcap [e]$ is not clique then there are non-adjaceent vertices y and z for which we can write the equality $x^{\perp} \bigcup e^{\perp} = y^{\perp} \bigcup z^{\perp}$ and $b^{\perp} \bigcup e^{\perp} = y^{\perp} \bigcup z^{\perp}$ and from this side $b^{\perp} - e^{\perp} = x^{\perp} - e^{\perp}$ hence $b^{\perp} = x^{\perp}$.

Suppose that:

$\Delta = a^{\perp} \bigcup b^{\perp}$ and $X(\Delta) = \{x \in \Delta \mid x^{\perp} \subseteq \Delta\}$....................(*).

And we obtain the following proposition.

Proposition 2.5: Let $b \in G - \Delta$ and $[b]$ intersects Δ. Then:

1. $[w] \cap X(\Delta)$ is a clique for $w \in [b] \cap \Delta$;

2. If $w \in [a] \cap [c] \cap [b]$, then $[w] \cap ([a] - c^{\perp}) \subset [b]$.

Proof: 1. Let there exist non-adjaceent vertices x, y in $X(\Delta)$, therefore from $(*)$ $[w] \subseteq x^{\perp} \cup y^{\perp}$ and hence $\{w; b, x, y\} - 3$-paws and this a contradiction.

2.Suppose that there exist a vertex x in $[w] \cap ([a] - c^{\perp})$ and $x \notin [b]$ then $\{w; b, x, c\} - 3$-paws and this a contradiction .

In propoisitions (2.6) and (2.7), we assumed that the subgraph $([a] \cap [b]) - d^{\perp}$ contains an edge cx.

Proposition 2.6: If $G_2(c) - \Delta$ contains a vertex f, then:

1.The graphs $[f] \cap [c]$ and $[f] \cap [x]$ coincide;

2.If $[x] - c^{\perp}$ contains vertex from $[a] - [b]$, then $[f] \cap [x] \subseteq [b]$;

3.If $[f] \cap [c]$ intersects $[a]$ and $[b]$, then $c^{\perp}$ and $x^{\perp}$ coincide out $[f] \cap [b] \cap [d]$.

Proof: 1. Graph $[f] \cap [d]$ does not intersect $c^{\perp}$ and $x^{\perp}$ then from $(*)$ $\Delta = c^{\perp} \cup d^{\perp}$ and $\Delta = x^{\perp} \cup d^{\perp}$ and hence $c^{\perp} \cup d^{\perp} = x^{\perp} \cup d^{\perp}$. Therefore $c^{\perp} - d^{\perp} = x^{\perp} - d^{\perp}$ and hence $[f] \cap [c] = [f] \cap [x]$.

2. Assume that the vertex g from $[a] - [b]$ lies in $[x] - c^{\perp}$ and there exist a vertex y of $[f] \cap [x]$, non-adjaceent with b then $x^{\perp} \cup d^{\perp} = g^{\perp} \cup b^{\perp}$, in the other means $\Delta = g^{\perp} \cup b^{\perp}$. $([x] \cap [f]) \cap [g] = \varnothing$ and $g^{\perp} \cup b^{\perp} = a^{\perp} \cup b^{\perp}$, that's means $g^{\perp}$ and $a^{\perp}$ coincide out $b^{\perp}$ we get 3-paws $\{y; f, g, c\}$ and this a contradiction therefore $[f] \cap [x] \subseteq [b]$.

3. Its easy we can obtain from (2).

Proposition 2.7: If $e \in G_3(a)$ and $acbe - 3$-path, then « – subgraphs $[a] \cap [b]$, $[c] \cap [e]$ are cliques.

Proof: Assume that, $[a] \cap [b]$ contains a non-adjaceent vertices c and d then from propoisition (2.1) (1) we obtain $e \in c^{\perp} \cup d^{\perp}$ and this a contradiction with the fact that e is at a distance 3 from a.

3. Strong pairs in connected graphs without 3-paws.

In this section we investigates some properties of the concept of the strong pairs in graphs without 3-paws.

Proposition 3.1: Let $a \in G$, $b, e \in G_2(a)$ and the vertex c from $[a] \cap ([b] - e^{\perp})$ adjaceent with the vertex f from $[a] \cap ([e] - b^{\perp})$, then $[c]$ and $[f]$ coincide on $a^{\perp} - ([b] \cap [e])$.

Proof: from propoisition (2.1) $c^{\perp} \subseteq b^{\perp} \cup f^{\perp}$, $f^{\perp} \subseteq c^{\perp} \cup e^{\perp}$ This implies $c^{\perp} - b^{\perp} \subseteq f^{\perp}$ and $f^{\perp} - c^{\perp} \subseteq e^{\perp}$ and hence:

$[c] \cap (a^{\perp} - ([b] \cup [e])) = [f] \cap (a^{\perp} - ([b] \cup [e]))$, then $[c]$ and $[f]$ coincide on $a^{\perp} - ([b] \cap [e])$. In propoisitions (3.2 and 3.3) assumed that $\{a,b,c\} -$ 3 –coclique of G and $[a] \cap [b]$ contains a non-adjaceent vertices c, d.

Proposition 3.2: Let $[c] \cap [e]$ contains a vertex w from $[a]$ and w' from $[b]$. If $([a] \cap [b]) - d^{\perp}$ contains the vertex z does not lie in the $K(c)$, then for any vertex x of $([a] \cap [b] \cap [c]) - z^{\perp}$ her neighborhood $[x]$ contains $[b] - ([a] \cup [e])$ and $[a] - ([b] \cup [e])$.

Proof: Since $z \notin K(c)$, that's mean $z^{\perp} \neq c^{\perp}$, then $z \notin d^{\perp}, z \in [a] \cap [b]$. By proposition(2.6) $e \in G_2(c)$ and $([c] \cap [e]) \cap [a]$ contains the vertex w, $([c] \cap [e]) \cap [b]$ contains the vertex w' therefore $[c]$ and $[z]$ coincide out $[a] \cap [b] \cap [d]$. And since $a^{\perp} \cup b^{\perp} = z^{\perp} \cup x^{\perp} = c^{\perp} \cup d^{\perp} = z^{\perp} \cup d^{\perp}$ therefore $z^{\perp} - d^{\perp} = c^{\perp} - d^{\perp}$ and $x^{\perp} - z^{\perp} = d^{\perp} - z^{\perp}$. We take the element $y \in [b] - ([a] \cup [e])$, and let it non-adjaceent with x .Then, since $z, x -$ non-adjaceent from $[b]$ and $y, x -$ non-adjaceent then $y, z -$ are adjaceent vertices. Since $[z]$ and $[c]$ coincide out $[a] \cap [b] \cap [d]$, then $y, c -$ are adjaceent vertices. Similarly, we can prove that $z, w -$ are adjaceent vertices. $w, y -$ are non-adjaceent vertices, because otherwise $\{w; a, y, e\} -$ this 3-paw. Then $x, w -$ adjaceent vertices, because otherwise $\{c; x, y, w\} -$ this 3-paw now we get a contradiction with the fact that $\{w; z, x, e\} -$ 3-paws and hence $y, x -$ are adjaceent vertices that's mean $y \in [x]$ and propoisition is proved.

Proposition 3.3: Let $[a] \cap [e]$ contains a non-adjaceent vertices f, g then:

1.For all vertex from $[a] - ([b] \cup K(a))$ adjaceent exactly with one vertex of the sets $\{c, d\}, \{f, g\}$ (In particular, we can assume that $cf, dg -$ edges).

2.If « – subgraphs $[c] \cap [g], [f] \cap [d]$ has a non-empty intersection with $[b] \cap [e]$, then for any vertex from $[c] \cap [g] \cap [b] \cap [e]$ is not adjaceent with the vertex from $[f] \cap [d] \cap [b] \cap [e]$.

Proof: 1. Take the vertex x of $[a] - [b]$ Then x necessarily adjaceent with c or d (otherwise $\{a; x, c, d\} -$ 3-paws) .The vertex x can not be adjaceent with c and d in the same time otherwise, by proposition (2.5) $x^{\perp} = a^{\perp}$ and this contradicts the condition $x \notin K(a)$.

2. If w is the vertex of $[c] \cap [g] \cap [b] \cap [e]$, adjaceent with $w' \in [f] \cap [d] \cap [b] \cap [e]$, then $\{w', d, f\}$ is a 3-coclique of $[w]$ and this a contradiction then propoisition is proved. Assume that $[a] \cap [e]$ contains a non-

adjaceent vertices f, g wherein cf, dg are edges in G then we can proof that the following proposition.

Proposition 3.4: Let $[c]\bigcap[d]$ contains the vertex x of $[a]\bigcap[b]$ Without loss of generality $x\in[g]-[f]$. Then:

1.The graphs $[c],[x]([d],[g])$ coincide on $[b]-([a]\bigcup[e])$ (respectively on $[a]-([b]\bigcup[e])$);

2.If x is not adjaceent with some vertex g' of $[d]\bigcap[a]\bigcap[e]$, then $[b]-([a]\bigcup[e])$ contained in $[c]\bigcap[x]$ and $[a]-([b]\bigcup[e])$ contained in $[d]\bigcap[x]\bigcap[g']$;

Proof: 1. We prove that :

$[c]\bigcap([b]-([a]\bigcup[e]))=[x]\bigcap([b]-([a]\bigcup[e]))$ by two inclusions:

$(\subseteq)$: let $w\in[c]\bigcap([b]-([a]\bigcup[e]))$ and w is not a adjaceent with x then $\{c;w,x,f\}-3$-paws at $x\in[g]-[f]$. At $x\in[f]-[g]$ assuming that wx – is not edge, we get $\{a;c,d,f\}-3$-paws.

$(\supseteq)$: let $w'\in[x]\bigcap([b]-([a]\bigcup[e]))$ and $w'\notin[c]$ then $w'\in[g]$ since otherwise $\{x;c,w',g\}-3$-paws and we get $\{g;a,e,w'\}$ and this a contradiction then $w'\in[c]$. Now we prove that $[d],[g],[x]$ coincide on $[b]-([a]\bigcup[e])$. By propoisition (3.1) $a\in G$, $b,e\in G_2(a)$, and consider that $\{d,g\}$ – they satisfy the propoisition and $\{x,g\}$ – they satisfy the propoisition that’s mean :

$[d]\bigcap([a]-([b]\bigcup[e]))=[g]\bigcap([a]-([b]\bigcup[e]))=$
$[x]\bigcap([a]-([b]\bigcup[e]))$.

2. We show that $[d]-[g']\subseteq[x]$. We take an arbitrary vertex $y\in[d]-[g']$ if $y\notin[x]$ then we obtain $\{d;x,y,g'\}-3$-paws and therefore $y\in[x]$. We now prove that the inclusion $[d]\bigcap([b]-([a]\bigcup[e]))\subseteq[x]$.

Suppose that the vertex $w''\in[d]\bigcap([b]-([a]\bigcup[e]))$ and $w''\notin[x]$ then $\{d;x,g',w''\}-3$-paws therefore $[d]\bigcap([b]-([a]\bigcup[e]))\subseteq[x]$. Then by propoisition (2.1) $w''\notin[g']$ and hence $[b]-([a]\bigcup[e])$ lies in $[c]\bigcap[x]$ from propoisition (3.4) (1) above . Further,

$[d]\bigcap([a]-([b]\bigcup[e]))=[g']\bigcap([a]-([b]\bigcup[e]))=$
$[g]\bigcap([a]-([b]\bigcup[e]))=[x]\bigcap([a]-([b]\bigcup[e]))$.

Since $[x]$ and $[g']$ coincide on the difference $[a]-([b]\bigcup[e])$ and x, g' – non-adjaceent then: $([a]-([b]\bigcup[e]))\subseteq[d]\bigcap[x]\bigcap[g']$.

حول خصائص البيانات المتصلة بدون البيانات الجزئية المستحثة ذات 3− مخالب

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الملخص:

لرؤوس البيان a للبيان G , $G(a)$ يمثل البيان الجزئي لمجموعة كل الرؤوس(القمم) المتصلة مع الرأس a والتي سوف نرمز لها بـ $[a]$ هذا البيان الجزئي يسمى بالجوار للرأس a في البيان G. في هذا العمل قمنا ببرهنة بعض القضايا المتعددة لمفهوم البيان المتصل بدون البيانات الجزئية المستحثة ذات 3− مخالب.

الكلمات المفتاحية للبحث: البيانات الغير موجهة, البيانات المتصلة, البيانات الجزئية المستحثة, البيانات بدون 3− مخالب

References

[1] Al’ Dzhabri Kh.Sh., Rodionov V.I., The graph of partial orders. Vestn. Udmurt. Univ. Mat. Mekh. Komp’yut. Nauki. 4 (2013) 3-12.

[2] Al’ Dzhabri Kh.Sh., The graph of reflexive-transitive relations and the graph of finite topologies. Vestn. Udmurt. Univ. Mat. Mekh. Komp’yut. Nauki. 1 (2015) 3-11.

[3] Al’ Dzhabri Kh.Sh., On subgraphs of graph of binary relations // Groups and Graphs, Algorithms and Automata: Abstacat of the international conference and PhD summer School in honor of the 80th Birthday of professor Vitaly A.Baranasky /Ural Federal University, Institute of Mathematics and Mechanics UB RAS, Institute of Mathematics SB RAS. Yekaterinburg, 2015. P. 30.

[4] Brouwer A.E., Numata M. , A characterization of some graphs which do not contain 3-claws / / Discrete Math. 1994. Vol. 124. P.49-54.

[5]Diestel, Reinhard (2006),Graph Theory, Graduate texts in mathematics, 173, Springer-Verlag, pp. 3–4,ISBN9783540261834.

[6] Hamelink, Ronald C. (1968). ,A partial characterization of clique graphs. Journal of Combinatorial Theory. 5: 192–197.

[7]Numata M. , On a characterization of a class of regular graphs of diameter 2 // Osaka J. Math. 1974. Vol.11. P.389-400.

[8]V. I. Kazarina and A. A. Makhnev, On locally GQ(s, t) graphs with strongly regular

μ-subgraphs St. Petersburg Math. J. Vol. 17 (2006), No. 3, Pages 443–452.

[9] Ore. O. ,Theory of graphs, Providence: Amer. Math. Soc. Colloq. Publ. 1962, vol.18, 270p. Translated under the title Teoriya grafov, Moscow: Nauka, 198, 336p.