

Analytical Solution for the Fractional Partial Differential Equations by Adomian Decomposition and Modified Decomposition Method

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ABSTRACT

In this paper, analytical solution of the fractional partial differential equation has been presented. The algorithm for the analytical solution for this equation is based on Adomian's decomposition and modified decomposition method. The fractional derivative is described in Caputo's sense. The analytical method has been applied to solve a practical example and its results have been compared with exact solution.

Key words: Adomian's Decomposition Method Modified Decomposition Method, Fractional Derivative, Fractional Partial Differential Equation.

الحل التحليلي للمعادلات التفاضلية الجزئية الكسرية بواسطة تحليل ادوميان وطريقة التحلل المعدلة

الخلاصة

في هذا البحث ، تم تقديم حلا تحليليا للمعادلة التفاضلية الجزئية الكسرية . وان خوارزمية الحل التحليلي قائمة على اساس تحليل ادوميان وطريقة التحلل المعدلة . المشتقة الكسرية هو موصوف بتعريف كابوتو وقد تم تطبيق الطريقة التحليلية في حل مثال عملي وتمت مقارنة نتائج مع الحل المضبوط .

INTRODUCTION

The fractional calculus is used in many fields of science and engineering [1, 2, 3, 4]. The solution of differential equation containing fractional derivatives is much involved and its classical analytic methods are mainly integral transforms, such as Laplace transform, Fourier transform, Mellin transform, etc. [1, 2, 5]

In recent years Adomian decomposition method is applied to solving fractional differential equations. This method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations, and even for

stochastic systems [6] as well. By using this method Saha Ray [7, 8, 9, 10] solved linear differential equations containing fractional derivative of order 1/2 or 3/2, and nonlinear differential equation containing fractional derivative of order 1/2.

In this paper, we consider the fractional partial differential equations of the form: [11]

$$\frac{\partial u(x,t)}{\partial t} = a(x) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + q(x,t) \quad \dots (1)$$

on a finite domain $x_L < x < x_R$ with $1 < \alpha < 2$. We assume that the fractional partial differential coefficient $a(x) > 0$. We also assume an initial condition $u(x,0) = f(x)$ for $x_L < x < x_R$ and boundary conditions of the form $u(x_L,t) = 0$ and $u(x_R,t) = g_R(t)$ eq.(1) uses a caputo fractional derivative of order α .

When $\alpha=2$, eq.(1) becomes the following classical parabolic partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = a(x,t) \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t)$$

Similarly, when $\alpha=1$, eq.(1) reduces to the following classical hyperbolic partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = a(x,t) \frac{\partial u(x,t)}{\partial x} + q(x,t)$$

In the present work, we apply the Adomian's Decomposition and modified Decomposition method for solving eq.(1) and compare the results with exact solution. The paper is organized as follows: In section 2, mathematical aspects are presented. In section 3, basic idea of Adomian's decomposition and modified decomposition method is presented. In section 4 the fractional partial differential equations and its solution by Adomian's decomposition and modified decomposition method is presented. In section 5 An example is solved numerically using the Adomian's decomposition and modified decomposition method. Finally, we present conclusion about solution of the fractional partial differential equation.

MATHEMATICAL ASPECTS

The mathematical definition of fractional calculus has been the subject of several different approaches [12, 13]. The Caputo fractional derivative operator D^α of order α is defined in the following form:

$$D^\alpha f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{f^{(m)}(t)}{(x - t)^{\alpha - m + 1}} dt, \quad \alpha > 0,$$

Where $m - 1 < \alpha < m, m \in \mathbb{N}, x > 0$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^\alpha (\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x),$$

where λ and μ are constants.

For the Caputo's derivative we have:

$$D_{L+}^\alpha (x - L)^n = \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \alpha)} (x - L)^{n - \alpha}$$

and

$$D_{R-}^\alpha (R - x)^n = \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \alpha)} (R - x)^{n - \alpha}$$

Basic Idea of the Adomian's Decomposition and Modified Decomposition Method

Consider a general nonlinear equation, [14]

$$Lu + R(u) + F(u) = g(t) \quad \dots (2)$$

Where L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. The nonlinear term is represented by F (u). Thus we get

$$Lu = g(t) - R(u) - F(u) \quad \dots (3)$$

The inverse L^{-1} is assumed an integral operator given by

$$L^{-1} = \int_0^t (\cdot) dt ,$$

The operating with the operator L^{-1} on both sides of Equation (3) we have

$$u = f + L^{-1}(g(t) - R(u) - F(u)) \quad \dots (4)$$

where f is the solution of homogeneous equation

$$Lu = 0 \quad \dots (5)$$

Involving the constants of integration. The integration constants involved in the solution of homogeneous Equation (5) are to be determined by the initial or boundary conditions according to the problem.

The ADM assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

and the nonlinear operator $F(u)$ can be decomposed by an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomials of u_0, u_1, \dots, u_n defined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0,1,2,\dots,$$

But the modified decomposition method introduced by Wazwaz [15], is based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x)$$

We apply this decomposition when the function f consists of several parts and can be decomposed into two different parts. In this case, f is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part f_1 is important.

For the method to be more efficient, we select f_1 as one term of f or at least a number of terms if possible and f_2 consists of the remaining terms of f .

The Adomian’s Decomposition and Modified Decomposition Method for Solving the Fractional Partial Differential Equations

We adopt Adomian decomposition method for solving Equation (1). In the light of this method we assume that

$$u = \sum_{n=0}^{\infty} u_n$$

Now, Equation (1) can be rewritten as

$$L_t u(x, t) = a(x, t) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + q(x, t)$$

where $L_t = \frac{\partial}{\partial t}$ which is an easily invertible linear operator, $\frac{\partial^\alpha}{\partial x^\alpha}$ is the Caputo derivative of order α .

Therefore, by Adomian decomposition method, we can write,

$$u(x, t) = u(x, 0) + L_t^{-1} \left(a(x, t) \frac{\partial^\alpha \left(\sum_{n=0}^{\infty} u_n \right)}{\partial x^\alpha} \right) + L_t^{-1}(q(x, t)) \quad \dots (6)$$

Each term of series (6) is given by the standard Adomian decomposition method recurrence relation

$$u_0 = f$$

$$u_{n+1} = L_t^{-1} \left(a(x, t) \frac{\partial^\alpha \left(\sum_{n=0}^{\infty} u_n \right)}{\partial x^\alpha} \right), n \geq 0$$

Where $f = u(x, 0) + L^{-1}(q(x, t))$

Then the modified decomposition method (MDM) recursive scheme is as follows

$$u_0 = f_1$$

$$u_1 = f_2 + L_t^{-1} \left(a(x, t) \frac{\partial^\alpha u_0}{\partial x^\alpha} \right)$$

$$u_{n+1} = L_t^{-1} \left(a(x,t) \frac{\partial^\alpha \left(\sum_{n=0}^{\infty} u_n \right)}{\partial x^\alpha} \right), n \geq 0$$

NUMERICAL APPLICATION

In this section, we apply Adomian decomposition and modified decomposition method for finding the analytical solution of fractional partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = a(x,t) \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t) \dots (7)$$

with the coefficient function: $a(x,t)=0.183634x^{2.8}$, and the source function: $q(x,t) = -e^{-t}x^3 - e^{-t}x^4$, subject to the initial condition $u(x,0) = x^3$, $0 < x < 1$, and the boundary conditions $u(0,t) = 0, t > 0$, $u(1,t) = e^{-t}, t > 0$.

Implementation of Adomian’s Decomposition and Modified Decomposition Method Equation (7) can be rewritten in operator form as

$$L_t u(x,t) = a(x) \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t)$$

where $L_t = \frac{\partial}{\partial t}$ which is an easily invertible linear operator, $\frac{\partial^{1.8}}{\partial x^{1.8}}$ is the Caputo derivative of order 1.8.

By Adomian decomposition method, we can write,

$$u(x,t) = u(x,0) + L_t^{-1} \left(a(x) \frac{\partial^{1.8} \left(\sum_{n=0}^{\infty} u_n \right)}{\partial x^{1.8}} \right) + L_t^{-1}(q(x,t)) \dots (8)$$

Each term of series (8) is given by the standard Adomian decomposition method recurrence relation

$$u(x,t) = u_0 + u_1 + u_2$$

$$= e^{-t}x^3 + e^{-t}x^4 + .13697e^{-5} \cdot x^4 - .5e^{-19} \cdot x^5 [-36364.-.10330e21 x + 51648. x^{t^2} e^t - 36364e^t - .1033e21xte^t]e^{-t} - 5.16478x^6$$

And the modified decomposition method (MDM) recursive scheme is as follows

$$u(x,t) = u_0 + u_1 + u_2 = e^{-t}x^3 + 0 + 0$$

Note that the exact solution to this problem is: $u(x,t) = x^3e^{-t}$. Table (1) shows the analytical solutions for fractional partial differential equation obtained for different values and comparison between exact solution and analytical solution. But, in the Table (2), comparison between the absolute errors of approximate solution using [4] and the approximate solution using our proposed method.

Table (1) Comparison between exact solution and analytical solution when $\alpha = 1.8$ for fractional partial differential equation.

x	t	Exact Solution	Adomian Decomposition Method	Modified Adomian Decomposition Method	$ u_{ex}-u_{ADM} $	$ u_{ex}-MADM $
0	1	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	1	0.0003679	0.0004289	0.0003679	0.0000610	0.0000000
0.2	1	0.0029430	0.0042840	0.0029430	0.0013410	0.0000000
0.3	1	0.0099330	0.0180000	0.0099330	0.0085260	0.0000000
0.4	1	0.0240000	0.0560000	0.0240000	0.0320000	0.0000000
0.5	1	0.0460000	0.1360000	0.0460000	0.0900000	0.0000000
0.6	1	0.0790000	0.2890000	0.0790000	0.2090000	0.0000000
0.7	1	0.1260000	0.5520000	0.1260000	0.4260000	0.0000000
0.8	1	0.1880000	0.9750000	0.1880000	0.7870000	0.0000000
0.9	1	0.2680000	1.6160000	0.2680000	1.3480000	0.0000000
1	1	0.3680000	2.5410000	0.3680000	2.1730000	0.0000000
0	2	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	2	0.3680000	0.0001832	0.3680000	0.0000479	0.0000000
0.2	2	0.0010830	0.0022560	0.0010830	0.0011730	0.0000000
0.3	2	0.0036540	0.0110000	0.0036540	0.0072760	0.0000000
0.4	2	0.0086610	0.0340000	0.0086610	0.0250000	0.0000000
0.5	2	0.0170000	0.0770000	0.0170000	0.0600000	0.0000000
0.6	2	0.0290000	0.1400000	0.0290000	0.1110000	0.0000000
0.7	2	0.0460000	0.2060000	0.0460000	0.1600000	0.0000000
0.8	2	0.0690000	0.2270000	0.0690000	0.1570000	0.0000000
0.9	2	0.0990000	0.1070000	0.0990000	0.0085810	0.0000000
1	2	0.1350000	-0.312000	0.1350000	0.4470000	0.0000000
0	3	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	3	0.0000498	0.0000976	0.0000498	0.0000478	0.0000000

0.2	3	0.0003983	0.0014430	0.0003983	0.0010440	0.0000000
0.3	3	0.0013440	0.0059980	0.0013440	0.0046540	0.0000000
0.4	3	0.0031860	0.0094150	0.0031860	0.0062290	0.0000000
0.5	3	0.0062230	-0.015000	0.0062230	0.0210000	0.0000000
0.6	3	0.0110000	-0.142000	0.0110000	0.1530000	0.0000000
0.7	3	0.0170000	-0.528000	0.0170000	0.5450000	0.0000000
0.8	3	0.0250000	-1.454000	0.0250000	1.4800000	0.0000000
0.9	3	0.0360000	-3.381000	0.0360000	3.4170000	0.0000000
1	3	0.0500000	-7.009000	0.0500000	7.0580000	0.0000000

Table (2) Comparison between absolute error of approximate solution in [4], and our proposed method when $\alpha = 1.8, t=2$.

x	$ u_{ex}-u_{Cheb} $ m= 2	$ u_{ex}-u_{ADM} $	$ u_{ex}-u_{MADM} $
0	0.170849 e-03	0.0000000	0.0000000
0.1	0.021094 e-03	0.0000479	0.0000000
0.2	0.176609 e-03	0.0011730	0.0000000
0.3	0.301420 e-03	0.0072760	0.0000000
0.4	0.404138 e-03	0.0250000	0.0000000
0.5	0.489044 e-03	0.0600000	0.0000000
0.6	0.563305 e-03	0.1110000	0.0000000
0.7	0.633367 e-03	0.1600000	0.0000000
0.8	0.705677 e-03	0.1570000	0.0000000
0.9	0.786679 e-03	0.0085810	0.0000000
1	0.882821 e-03	0.4470000	0.0000000

CONCLUSIONS

- 1- Analytical solutions for fractional partial differential equation obtained for different values of using the Adomian decomposition and modified decomposition method have been described and demonstrated.
- 2- It is clear that the Adomian decomposition method solution converges very slowly to the exact solution. On the other hand, the modified decomposition method is in high agreement with the exact solutions

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