Flow Analysis of Third Order Fluid in a Helical Pipe with Circular Cross- Section

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Abstract

In this paper, we studied viscous non –Newtonian fluid of third order flowing in a helical pipe with circular cross-section under action of the pressure gradient. Particular consideration is given to fluid flow which can be represented by the equation of state of the form:

$$T=\mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (trA_1^2) A_1.$$

where $\alpha_i(i = 1,2), \beta_i(i = 1,2,3)$ are material moduli and $A_i(i = 1\text{-}3)$ are the first three Rivlin-Ericksen tenser. The cylindrical coordinates have been used to describe the fluid motion. It is found that motion equations are controlled by the dimensionless numbers namely Dean number L, non-Newtonian parameter β, and material moduli (γ_1, γ_3). The motion equations are solved analytically. The analytic solutions of the secondary velocity and the axial velocity are obtained. The effects of each of the dimensionless numbers upon the components of the secondary and the axial velocity are analyzed.

المستخلص

في هذا البحث درس جريان مائع لانيوتيني من الرتبة الثالثة في انبوب حلزوني ذو مقطع عرضي دائري تحت تأثيرالضغط. وبصورة خاصة يمكن ان يمثل ذلك المائع بمعادلة حالة من النوع

$$T=\mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (trA_1^2) A_1$$

استخدمت الاحداثيات المتعامدة لوصف حركة المائع. وجد أن معادلات الحركة مسيطرعليها باعداد لا بعدية وهي رقم دين معلمة لا نيوتينية وثوابت المائع. ان معادلات الحركة قد حلت تحليليا. حصلنا على التحليلية للجريان الثنائي والسرعة المحورية. وبالاضافة لذلك قمنا بدراسة تأثير الاعداد اللابعدية وتحليل مركبات الجريان الثنائي والسرعة المحوريه.

1. Introduction

The science of hydrodynamic is that branch of applied mathematics which deals with the behavior of fluids in motion. Fluid is that state of matter which capable of changing shape and is capable of flowing. Fluids may be classified as “Viscous” and “Perfect” according to whether the fluid capable of exerting shearing stress or not. Viscous fluid is called Newtonian if the relation between stress and rate of strain (state of equation) is linear, otherwise is called non-Newtonian fluid. The flow of Newtonian and non-Newtonian fluids has been the subject extensive theoretical studies till date. Dean [6] in 1927 was the first researcher who worked in flow analysis of Newtonian fluids in curved pipes. He introduced a toroidal coordinate system to show that the relation between pressure gradient and the rate of flow through a curved pipe with a circular cross-section of in compressible Newtonian is dependent on the curvature. In that paper he couldn’t show this dependence but he did in his second paper [7]. He modified his analysis by including the higher order and he was able to show the rate of flow is straightly reduced by curvature. Jones [12] in 1969 made a theoretical analysis of the flow of an incompressible non-Newtonian viscous liquid in a curved by with circular cross-section. Keeping only the first order terms. He showed that the secondary motion consists of two symmetrical vortices and the distance of the stream line form the central plane decreases as the non-Newtonian parameter increases. Wang [17] in (1981) studied the flow of incompressible Newtonian fluid in a helical pipe with circular cross-section introduced non-orthogonal coordinates system to study the effect of torsion and the curvature. Employing a perturbation method and he found that the torsion has the first order effects on the secondary flow. In 1982 Germano [9] studied the same problem of Wang’s but his solutions were obtained in an orthogonal coordinate system and he found the effects of torsion to be the second order. This results confirmed in his second paper in (1989) [10] in which he studied the effect of torsion in a helical pipe with an elliptical cross-section showing that there is unexpected form of the secondary where the walls act as sources and sinks. In 1990, Tuttle [15] solved the motion of the flow in pipes of elliptical cross-section and circular cross-section successively. Then he qualitatively stated that the order of torsion effect on the secondary flow dependent the frame of references of the observer. Without any approximation in the governing equations. Chen and Jan in (1992) [5] studied the flow of Newtonian fluid in a helical pipe with circular cross-section in a non-orthogonal coordinates system. They obtained the solution by double series expansion method. But considering the series forms of dimensionless axial velocity and stream function used in their article return the method to have the same draw back as perturbation technique. Bolinder in (1996) [4] studied the first and higher order of effects of torsion on the flow in a helical duct with rectangular cross-section numerically and also introduced a method to obtain the Navier-Stocke equations in a helical coordinates system employing physical velocity components. In 2000. Hadi [1] studied the analysis of the flow of non-Newtonian fluid of a second order in helical pipes with ellipse cross-section and circular cross-section. In circular cross-section he showed that the secondary motion depended on two dimensionless parameters namely Dean and non-Newtonian parameter (β) also he studied the effects of torsion ($\lambda/\Re e$) β and Dean number on the secondary flow and axial velocity. Also, Zhang, Zhang, and Chen in 2000 [11] studied the viscous flow in annular pipes by a perturbation method . They found the secondary flow and the axial velocity are controlled by torsion, Dean number, and the radius of the cross section. Xue in 2002 [13] analyzed the laminar flow in helical circular pipes by using Galerkin method. His results indicate that Galerkin technique can effectively overcome the limitation of a small parameters for perturbation method finally this paper studies the flow of third order fluid in a helical pipe with circular cross-section founds the governing equations are controlled by dimensionless numbers namely Dean number(L), Reynoleds number($\Re e$), non-Newtonian parameter(β)and the material moduli (γ_1, γ_3) and studies the effects of (L,$\Re e, \beta, \gamma_1, \gamma_3$)on the secondary flow and the axial velocity.

2.Coordinates System

Let the position vector described by (Fig. 1)

$$R(s) = X(s)\,\mathbf{i} + Y(s)\,\mathbf{j} + Z(s)\,\mathbf{k} \qquad (1)$$

Where s is arc length along the pipe and i, j, k are units vector in the Cartesian direction. The **TNB**frame and Frenet formulas defined by:

$$\mathbf{T}=\frac{d\boldsymbol{R}}{ds}, \quad \boldsymbol{N}=\frac{1}{k}\frac{d\boldsymbol{T}}{ds}, \qquad \boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$$

$$\frac{d\boldsymbol{N}}{ds}=\tau\,\boldsymbol{B}-\kappa\,\boldsymbol{T}, \qquad \frac{d\boldsymbol{B}}{ds}=-\tau\,\boldsymbol{N}(2)$$

Here T, N, and B are the tangent normal and binormal vectors respectively, τ is the torsion and κ is the curvature[2]. To construct the orthogonal coordinate system(s, r ,θ) , let polar angle θ refers to a relation of the unit vector N* by the amount of ∅ + ∅∘ and is given by:

$$\emptyset=-\int_{s_\circ}^{s}\tau(s)ds \qquad (3)$$

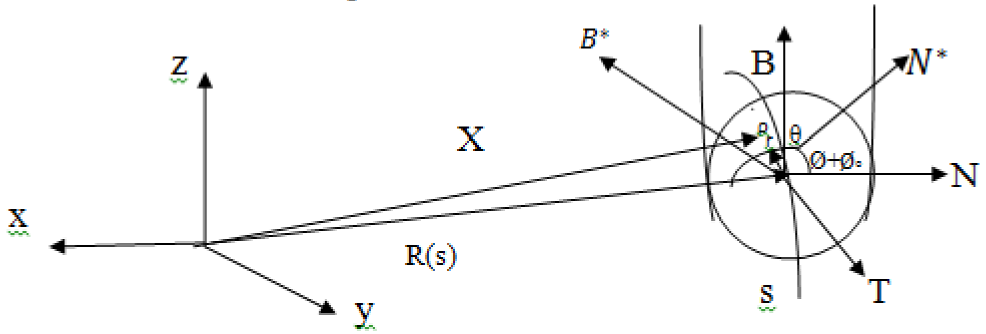


fig.(1) the coordinates system

3.Basic Equation

Consideration is given to a fluid characterized by a state equation of the form:

$$\mathbf{T}=\boldsymbol{\mu}\boldsymbol{A_1}+\boldsymbol{\alpha_1}\boldsymbol{A_2}+\boldsymbol{\alpha_2}\boldsymbol{A_1}^2+\boldsymbol{\beta_1}\boldsymbol{A_3}+\boldsymbol{\beta_2}(\boldsymbol{A_1}\boldsymbol{A_2}+\boldsymbol{A_2}\boldsymbol{A_1})+\boldsymbol{\beta_3}(\boldsymbol{tr}\boldsymbol{A_1}^2)\boldsymbol{A_1}$$

(4)

$$A_1=(\text{grad V})+(\text{grad V})^T \qquad (5a)$$

$$A_n=\frac{dA_{n-1}}{dt}+A_{n-1}(\text{grad V})+(\text{grad V})^T A_{n-1}, \qquad n>1, \qquad (5b)$$

Where **V**the is the velocity vector grad the gradient operater μ is the viscosity,α_i(i = 1,2) , βi (i = 1,2,3)are material moduli, d/dt is the material derivative and A_i (i= 1,2,3) are the first Rivlin-Eriksen tensors.[8], thermodynamic of third grade fluid requires that

$$\mu\geq 0, \qquad \alpha_1\geq 0, \qquad |\alpha_1+\alpha_2|\leq\sqrt{24\mu\beta_3}, \qquad (6) \qquad \beta_1=\beta_2=0, \qquad \beta_3\geq 0$$

4.Governing Equations

We write down the motion and continuity equations in curvilinear coordinate for unsteady viscous fluid flow in helical pipe without imposing any of our restrictions,[3],[14].

The motion equations are in curvilinear coordinates are:-

$$\frac{\partial u}{\partial t}+\omega\, u\,\frac{\partial u}{\partial s}+v\,\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}+\omega\, k\, u(v\sin(\theta+\emptyset)+w\cos(\theta+\emptyset))=-\omega\,\frac{\partial p}{\partial s}+\frac{1}{\rho}[\omega\,\frac{\partial T_{ss}}{\partial s}+\left(\frac{\partial}{\partial r}+\frac{1}{r}+2\omega\, k\sin(\theta+\emptyset)\right)T_{sr}+\left(\frac{1}{r}\frac{\partial}{\partial \theta}+2\omega\, k\cos(\theta+\emptyset)\right)T_{s\theta}\ (7)$$

$$\frac{\partial v}{\partial t}+\omega\, u\,\frac{\partial v}{\partial s}+v\,\frac{\partial v}{\partial r}+\omega\,\frac{\partial v}{\partial \theta}-\frac{w^2}{r^2}-\omega\, k\, u^2\sin(\theta+\emptyset)=-\frac{\partial p}{\partial r}+\frac{1}{\rho}\,[\left(\frac{\partial}{\partial r}+\frac{1}{r}+\omega\, k\sin(\theta+\emptyset)\right)T_{rr}+\left(\frac{1}{r}\frac{\partial}{\partial \theta}+2\omega\, k\cos(\theta+\emptyset)\right)T_{r\theta}+\omega\,\frac{\partial}{\partial s}T_{sr}-\frac{1}{r}T_{\theta\theta}-\omega\, k\sin(\theta+\emptyset)\,T_{ss}(8)$$

$$\frac{\partial w}{\partial t}+\omega u\frac{\partial w}{\partial s}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \theta}-\frac{vw}{r}-\omega k u^2\cos(\theta+\emptyset)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{\rho}\left[\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\omega k\cos(\theta+\emptyset)\right)T_{\theta\theta}+\omega\frac{\partial T_{\theta s}}{\partial s}+\left(\frac{\partial}{\partial \theta}+\frac{2}{r}+\omega k\cos(\theta+\emptyset)\right)T_{r\theta}-\omega k\cos(\theta+\emptyset)\,T_{ss}\right](9)$$

And $\omega\frac{\partial u}{\partial s}+\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial \theta}+\omega k(v\sin(\theta+\emptyset)+w\cos(\theta+\emptyset))=0(10)$

where p is the kinematic pressure, u, v and w represent the velocity components in s, r, θ respectively, k is the curvature of the pipe, ρis the density and ωisdefinedas:

$$\omega=\frac{1}{1+krsin(\theta+\emptyset)}$$

Introduce the following new dimensionless variables to obtain the dimensionless equations.

$$u=U_\circ u_1,\quad v=\frac{\nu}{a}v_1,\quad w=\frac{\nu}{a}w_1,\quad s=as_1,\quad r=ar_1\,,$$

$$\lambda={}^{\tau}/_{k}\,,\quad \varepsilon=ka\,,\quad p=U_\circ{}^2p_1\,,\quad \Re e=\frac{U_\circ a}{\nu}$$

Where a is the radius of the pipe,$U_\circ$isthemaximumvelocity in a straight pipe under the pressure gradient, ν is the viscosity and $p_1$is the pressure defined by:

$$p_1=p_\circ(s_1)+\varepsilon p_{11}(s_1,r_1)=-\frac{G}{\Re e}s_1+\varepsilon\, p_{11}(s_1,r_1)$$

Where G is the constant given by $[(\Re ea/\rho U_\circ{}^2)p^{**}],p^{**}$is the pressure gradient. For mathematical convenience, consideration is given to a helical pipe with constant curvature k and torsion τ .In this case it is possible to search helically symmetric solutions of the general equations, which is physically corresponding to a fully developed flow in a helical pipe and can be operated and setting all the resulting derivatives with respect to s equal zero except the pressure derivative the resulting of continuity and motion equations under these assumptions are:-

$$\frac{\partial (r_1 v_1)}{\partial r_1} + \frac{\partial}{\partial \theta_1}(w_1 - \varepsilon\lambda\Re e r_1 u_1) = 0 \qquad (11)$$

And the motion equations are:

$$\frac{\partial u}{\partial t} + v_1 \frac{\partial u_1}{\partial r_1} + \frac{w_1}{r_1}\frac{\partial u_1}{\partial \theta_1} - \varepsilon\lambda\Re e u_1 \frac{\partial u_1}{\partial \theta_1} = -G + [\left(\frac{\partial}{\partial r_1} + \frac{1}{r_1}\right)\left(\frac{\partial u_1}{\partial r_1}\right) + (\frac{1}{r_1}\frac{\partial}{\partial \theta_1})\left(\frac{1}{r_1}\frac{\partial u_1}{\partial \theta_1}\right)] +$$
$$\beta[\left(\frac{\partial}{\partial r_1} + \frac{1}{r_1}\right)\left(2\frac{\partial u_1}{\partial r_1}\frac{\partial v_1}{\partial r_1} + \frac{1}{r_1}\frac{\partial u_1}{\partial \theta_1}\frac{\partial w_1}{\partial r_1} - \frac{w_1}{{r_1}^2}\frac{\partial u_1}{\theta_1} + \frac{1}{{r_1}^2}\frac{u_1}{\partial \theta_1}\frac{\partial v_1}{\partial \theta_1}\right)\left(\frac{1}{r_1}\frac{\partial}{\partial \theta_1}\right)\left(\frac{\partial u_1}{\partial r_1}\frac{\partial w_1}{\partial r_1} - \frac{w_1}{r_1}\frac{\partial u_1}{\partial r_1} +\right.$$
$$\left.\frac{1}{r_1}\frac{\partial u_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} + \frac{2}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1} + 2\frac{v_1}{r_1}\frac{\partial u_1}{\partial \theta_1}\right)] + \gamma_1[\left(\frac{\partial}{\partial r_1} + \frac{1}{r_1}\right)\left(\frac{\partial^2 u_1}{\partial r_1 t_1} + \frac{\partial u_1}{\partial r_1}\frac{\partial v_1}{\partial r_1} + \frac{1}{r_1}\frac{\partial u_1}{\partial \theta_1}\frac{\partial w_1}{\partial r_1}\right) +$$
$$\left(\frac{1}{r_1}\frac{\partial}{\partial \theta_1}\right)\left(\frac{1}{r_1}\frac{\partial^2 u_1}{\partial \theta_1 t_1} + \frac{2}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1} + 2\frac{v_1}{r_1}\frac{\partial u_1}{\partial \theta_1}\right)] + \gamma_2[(\frac{\partial}{\partial r_1} + \frac{1}{r_1})(4\frac{\partial u_1}{\partial r_1}(\frac{\partial v_1}{\partial r_1})^2 + 2\frac{\partial u_1}{\partial r_1}(\frac{\partial w_1}{\partial r_1})^2 +$$
$$\frac{2}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}(\frac{\partial v_1}{\partial \theta_1})^2 + 2\frac{{w_1}^2}{{r_1}^2}\frac{\partial u_1}{\partial r_1} - \frac{4}{r_1}\frac{\partial u_1}{\partial r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} - 4\frac{w_1}{r_1}\frac{\partial u_1}{\partial r_1} - 4\frac{w_1}{{r_1}^2}\frac{\partial u_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} + \frac{4}{{r_1}^2}\frac{\partial u_1}{\partial r_1}(\frac{\partial w_1}{\partial \theta_1})^2 +$$
$$8\frac{v_1}{{r_1}^2}\frac{\partial u_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1} + 4\frac{{v_1}^2}{{r_1}^2}\frac{\partial u_1}{\partial r_1} + (\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(4\frac{\partial u_1}{\partial \theta_1}(\frac{\partial v_1}{\partial r_1})^2 + \frac{2}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}(\frac{\partial w_1}{\partial r_1})^2 + \frac{2}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}(\frac{\partial v_1}{\partial \theta_1})^2 + 2\frac{{w_1}^2}{{r_1}^3}\frac{\partial u_1}{\partial \theta_1} -$$
$$\frac{4}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial r_1} - 4\frac{w_1}{{r_1}^2}\frac{\partial u_1}{\partial \theta_1}\frac{\partial w_1}{\partial r_1} - \frac{4}{{r_1}^3}\frac{\partial u_1}{\partial \theta_1}\frac{\partial v_1}{\partial \theta_1} + \frac{4}{{r_1}^3}\frac{\partial u_1}{\partial \theta_1}(\frac{\partial w_1}{\partial \theta_1})^2 + 8\frac{v_1}{{r_1}^3}\frac{\partial u_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1} + 4\frac{{v_1}^2}{{r_1}^3}\frac{\partial u_1}{\partial \theta_1}] +$$
$$\gamma_3[(\frac{\partial}{\partial r_1} + \frac{1}{r_1})(2(\frac{\partial u_1}{\partial r_1})^3 + \frac{2}{{r_1}^2}\frac{\partial u_1}{\partial r_1}(\frac{\partial u_1}{\partial \theta_1})^2) + (\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{2}{r_1}\frac{\partial u_1}{\partial \theta_1}(\frac{\partial u_1}{\partial r_1})^2 + \frac{2}{r^3}(\frac{\partial u_1}{\partial \theta_1})^3)]$$

(12a)

$$\frac{\partial v_1}{\partial t_1} + v_1\frac{\partial v_1}{\partial r_1} + \frac{w_1}{r_1}\frac{\partial v_1}{\partial \theta_1} - \frac{{w_1}^2}{r_1} - \varepsilon\Re e^2 {u_1}^2 \sin\theta_1 - \varepsilon\Re e u_1\frac{\partial v_1}{\partial \theta_1} = -\varepsilon\Re e^2\frac{\partial P_{11}}{\partial r_1} + [\left(\frac{\partial}{\partial r_1} + \frac{1}{r_1}\right)\left(2\frac{\partial v_1}{\partial r_1}\right)$$
$$+ \left(\frac{1}{r_1}\frac{\partial}{\partial \theta_1}\right)\left(\frac{\partial w_1}{\partial r_1} - \frac{w_1}{r_1} + \frac{1}{r_1}\frac{\partial v_1}{\partial \theta_1}\right) - \frac{1}{r_1}\left(\frac{2}{r_1}\frac{\partial w_1}{\partial \theta_1} + 2\frac{v_1}{r_1}\right)] + \beta[(\frac{\partial}{\partial r_1} + \frac{1}{r_1})(4(\frac{\partial v_1}{\partial r_1})^2 + (\frac{\partial w_1}{\partial {r_1}^2})^2 + \frac{w_1}{{r_1}^2} +$$
$$(\frac{\partial v_1}{\partial \theta_1})^2 - 2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1} + \frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} - 2\frac{w_1}{{r_1}^2}\frac{\partial v_1}{\partial \theta_1}) + \frac{1}{r_1}\frac{\partial}{\theta_1}(2\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1} - 2\frac{w_1}{r_1}\frac{\partial v_1}{\partial r_1} + \frac{2}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} + \frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1} +$$
$$2\frac{v_1}{r_1}\frac{\partial w_1}{\partial r_1} - 2\frac{w_1}{{r_1}^2}\frac{\partial w_1}{\partial \theta_1} - 2\frac{v_1 w_1}{{r_1}^2} + \frac{2}{{r_1}^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1} + 2\frac{v_1}{{r_1}^2}\frac{\partial v_1}{\partial \theta_1}) - \frac{1}{r_1}((\frac{\partial w_1}{\partial r_1})^2 + \frac{{w_1}^2}{{r_1}^2} + \frac{1}{{r_1}^2}(\frac{\partial v_1}{\partial \theta_1})^2 -$$
$$2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1} + \frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} - \frac{w_1}{{r_1}^2}\frac{\partial v_1}{\partial \theta_1} + \frac{4}{{r_1}^2}(\frac{\partial w_1}{\partial \theta_1})^2 + 4\frac{{v_1}^2}{{r_1}^2} + 8\frac{v_1}{{r_1}^2}\frac{\partial w_1}{\partial \theta_1})] + \gamma_1[(\frac{\partial}{\partial r_1} + \frac{1}{r_1})(2\frac{\partial^2 v_1}{\partial r_1 \partial t_1} +$$
$$4(\frac{\partial v_1}{\partial r_1})^2 + 2(\frac{\partial w_1}{\partial r_1})^2 + \frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} - 2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1}) + (\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{\partial^2 w_1}{\partial r_1 \partial t_1} + \frac{1}{r_1}\frac{\partial^2 v_1}{\partial \theta_1 \partial t_1} - \frac{1}{r_1}\frac{\partial w_1}{\partial t_1} + \frac{3}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1} -$$
$$3\frac{w_1}{r_1}\frac{\partial v_1}{\partial r_1} + \frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1} + \frac{3}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1} + 3\frac{v_1}{r_1}\frac{\partial w_1}{\partial r_1} + \frac{2}{{r_1}^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1} + 2\frac{v_1}{{r_1}^2}\frac{\partial v_1}{\partial \theta_1} + 2\frac{w_1}{{r_1}^2}\frac{\partial w_1}{\partial \theta_1} - 2\frac{v_1 w_1}{{r_1}^2}) -$$

$$2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-2\frac{w_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}+\frac{4}{r_1^2}(\frac{\partial w_1}{\partial \theta_1})^2+4\frac{v_1^2}{r_1^2}+8\frac{v_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1})]+\gamma_1[(\frac{\partial}{\partial r_1}+\frac{1}{r_1})(2\frac{\partial^2 v_1}{\partial r_1 \partial t_1}+$$

$$4(\frac{\partial v_1}{\partial r_1})^2+2(\frac{\partial w_1}{\partial r_1})^2+\frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1})+(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{\partial^2 w_1}{\partial r_1 \partial t_1}+\frac{1}{r_1}\frac{\partial^2 v_1}{\partial \theta_1 \partial t_1}-\frac{1}{r_1}\frac{\partial w_1}{\partial t_1}+\frac{3}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-$$

$$3\frac{w_1}{r_1}\frac{\partial v_1}{\partial r_1}+\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1}+\frac{3}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}+3\frac{v_1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{2}{r_1^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}+2\frac{v_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}+2\frac{w_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1}-2\frac{v_1 w_1}{r_1^2})-$$

$$\frac{1}{r_1}(\frac{2}{r_1}\frac{\partial^2 w_1}{\partial \theta_1 \partial t_1}+\frac{2}{r_1}\frac{\partial v_1}{\partial t_1}+\frac{2}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2+\frac{2}{r_1}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial r_1}-4\frac{w_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}+2\frac{w_1^2}{r_1^2}-2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{4}{r_1^2}(\frac{\partial w_1}{\partial \theta_1})^2+$$

$$8\frac{v_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1}+4\frac{v_1^2}{r_1^2})]\ +\gamma_2[(\frac{\partial}{\partial r_1}+\frac{1}{r_1})(4\frac{\partial v_1}{\partial r_1}(\frac{\partial w_1}{\partial \theta_1})^2+\frac{4}{r_1^2}\frac{\partial v_1}{\partial r_1}(\frac{\partial v_1}{\partial \theta_1})^2+4\frac{w_1^2}{r_1^2}\frac{\partial v_1}{\partial r_1}+\frac{8}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial \gamma_2 v_1}{\partial \theta_1}-$$

$$8\frac{w_1}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1}-8\frac{w_1}{r_1^2}\frac{\partial v_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}+\frac{8}{r_1^2}\frac{\partial v_1}{\partial r_1}(\frac{\partial w_1}{\partial \theta_1})^2+16\frac{v_1}{r_1^2}\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}+8(\frac{\partial v_1}{\partial r_1})^3+8\frac{v_1^2}{r_1^2}\frac{\partial v_1}{\partial r_1})$$

$$+(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{4}{r_1}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial v_1}{\partial r_1})^2+4\frac{\partial w_1}{\partial r_1}(\frac{\partial v_1}{\partial r_1})^2-\frac{w_1}{r_1}(\frac{\partial v_1}{\partial r_1})^2+\frac{2}{r_1^3}(\frac{\partial v_1}{\partial \theta_1})^3+\frac{6}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2\frac{\partial w_1}{\partial r_1}-6\frac{w_1}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2+$$

$$\frac{6}{r_1}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial w_1}{\partial r_1})^2\text{-}12\frac{w_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial r_1}-6\frac{w_1}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2+4\frac{w_1^2}{r_1^2}\frac{\partial w_1}{\partial r_1}-4\frac{w_1^2}{r_1^2}\frac{\partial v_1}{\partial \theta_1}+2(\frac{\partial w_1}{\partial r_1})^3+\frac{4}{r_1^3}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial w_1}{\partial \theta_1})^2-$$

$$2\frac{w_1^3}{r_1^3}+\frac{4}{r_1^2}\frac{\partial w_1}{\partial r_1}(\frac{\partial w_1}{\partial \theta_1})^2-4\frac{w_1}{r_1^3}(\frac{\partial w_1}{\partial \theta_1})^2+8\frac{v_1}{r_1^3}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}+8\frac{v_1}{r_1^2}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}-8\frac{v_1 w_1}{r_1^3}\frac{\partial w_1}{\partial \theta_1}+4\frac{v_1^2}{r_1^2}\frac{\partial v_1}{\partial \theta_1}+$$

$$4\frac{v_1^2}{r_1^2}\frac{\partial w_1}{\partial r_1}-4\frac{v_1^2 w_1}{r_1^3})-\frac{1}{r_1}(\frac{8}{r_1}(\frac{\partial v_1}{\partial r_1})^2\frac{\partial w_1}{\partial \theta_1}+8\frac{v_1}{r_1}(\frac{\partial v_1}{\partial r_1})^2+\frac{4}{r_1}(\frac{\partial w_1}{\partial r_1})^2\frac{\partial w_1}{\partial \theta_1}+4\frac{v_1}{r_1}(\frac{\partial w_1}{\partial r_1})^2+$$

$$\frac{4}{r_1^3}(\frac{\partial v_1}{\partial \theta_1})^2\frac{\partial w_1}{\partial \theta_1}+4\frac{v_1}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2+2\frac{w_1^2}{r_1^3}\frac{\partial w_1}{\partial \theta_1}+2\frac{v_1 w_1^2}{r_1^3}-8\frac{w_1}{r_1^2}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}-8\frac{v_1 w_1}{r_1^2}\frac{\partial w_1}{\partial r_1}+$$

$$\frac{8}{r_1^2}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}+8\frac{v_1}{r_1^2}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-8\frac{w_1}{r_1^3}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}-8\frac{v_1 w_1}{r_1^3}\frac{\partial v_1}{\partial \theta_1}+\frac{8}{r_1^3}(\frac{\partial w_1}{\partial \theta_1})^3+24\frac{v_1^2}{r_1^3}\frac{\partial w_1}{\partial \theta_1}+$$

$$16\frac{v_1}{r_1^3}(\frac{\partial w_1}{\partial \theta_1})^2+8\frac{v_1^3}{r_1^3})]+\gamma_3[(\frac{\partial}{\partial r_1}+\frac{1}{r_1})(4\frac{\partial v_1}{\partial r_1}(\frac{\partial u_1}{\partial r_1})^2+\frac{4}{r_1^2}\frac{\partial v_1}{\partial r_1}(\frac{\partial u_1}{\partial \theta_1})^2)+(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{2}{r_1}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial u_1}{\partial r_1})^2+$$

$$2\frac{\partial w_1}{\partial r_1}(\frac{\partial u_1}{\partial r_1})^2+\frac{2}{r_1^3}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial u_1}{\partial \theta_1})^2+\frac{2}{r_1^2}\frac{\partial w_1}{\partial r_1}(\frac{\partial u_1}{\partial \theta_1})^2-2\frac{w_1}{r_1}(\frac{\partial u_1}{\partial r_1})^2-2\frac{w_1}{r_1}(\frac{\partial u_1}{\partial \theta_1})^2)-\frac{1}{r_1}(\frac{4}{r_1}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial u_1}{\partial r_1})^2+$$

$$4\frac{v_1}{r_1}(\frac{\partial u_1}{\partial r_1})^2+\frac{4}{r_1^3}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial u_1}{\partial \theta_1})^2+4\frac{v_1}{r_1^3}(\frac{\partial u_1}{\partial \theta_1})^2]+\gamma_4[(\frac{\partial}{\partial r_1}+\frac{1}{r_1})(2(\frac{\partial u_1}{\partial r_1})^2+\frac{2}{r_1}\frac{\partial u_1}{\partial r_1}\frac{\partial u_1}{\partial \theta_1})+$$

$$(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{1}{r_1^2}(\frac{\partial u_1}{\partial \theta_1})^2+\frac{1}{r_1}\frac{\partial u_1}{\partial r_1}\frac{\partial u_1}{\partial \theta_1})-\frac{1}{r_1^3}(\frac{\partial u_1}{\partial \theta_1})^2] \qquad (12b)$$

$$\frac{\partial w_1}{\partial t_1}+v_1\frac{\partial w_1}{\partial r_1}+\frac{w_1}{r_1}\frac{\partial w_1}{\partial \theta_1}+\frac{v_1 w_1}{r_1}-\varepsilon \Re e^2 u_1^2 Cos\theta_1-\varepsilon\lambda \Re e u_1\frac{\partial w_1}{\partial \theta_1}=$$

$$-\frac{1}{r_1}\varepsilon \Re e^2\frac{\partial P_{11}}{\partial \theta_1}+\left[\left(\frac{1}{r_1}\frac{\partial}{\partial \theta_1}\right)\left(\frac{2}{r_1}\frac{\partial w_1}{\partial \theta_1}+2\frac{v_1}{r_1}\right)+\left(\frac{\partial}{\partial r_1}+\frac{2}{r_1}\right)\left(\frac{\partial w_1}{\partial r_1}-\frac{w_1}{r_1}+\frac{1}{r_1}\frac{\partial v_1}{\partial \theta_1}\right)\right]+$$

$$\beta[(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})((\frac{\partial w_1}{\partial r_1})^2+\frac{w_1^2}{r_1^2}+\frac{1}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2-2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-2\frac{w_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}+\frac{4}{r_1^2}(\frac{\partial w_1}{\partial \theta_1})^2+4\frac{v_1^2}{r_1^2}+$$

$$8\frac{v_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1})+\left(\frac{\partial}{\partial r_1}+\frac{2}{r_1}\right)\left(2\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1}-2\frac{w_1}{r_1}\frac{\partial v_1}{\partial r_1}+\frac{2}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}+\frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}+2\frac{v_1}{r_1}\frac{\partial w_1}{\partial r_1}-2\frac{w_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1}-\right.$$

$$\left.2\frac{v_1 w_1}{r_1^2}+\frac{2}{r_1^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}+2\frac{v_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}\right)]+\gamma_1[(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{2}{r_1}\frac{\partial^2 w_1}{\partial \theta_1 \partial t_1}+\frac{2}{r_1}\frac{\partial v_1}{\partial t_1}+\frac{2}{r_1^2}(\frac{\partial v_1}{\partial \theta_1})^2+2\frac{w_1^2}{r_1^2}+$$

$$\frac{2}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-4\frac{w_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}-2\frac{w_1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{4}{r_1^2}(\frac{\partial w_1}{\partial \theta_1})^2+4\frac{v_1^2}{r_1^2}+8\frac{v_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1})+\left(\frac{\partial}{\partial r_1}+\frac{2}{r_1}\right)\left(\frac{\partial^2 w_1}{\partial r_1 \partial t_1}+\right.$$

$$\frac{1}{r_1}\frac{\partial^2 v_1}{\partial \theta_1 \partial t_1}-\frac{1}{r_1}\frac{\partial w_1}{\partial t_1}+\frac{3}{r_1}\frac{\partial v_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-3\frac{w_1}{r_1}\frac{\partial v_1}{\partial r_1}+\frac{\partial v_1}{\partial r_1}\frac{\partial w_1}{\partial r_1}+\frac{2}{r_1^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}+2\frac{v_1}{r_1^2}\frac{\partial v_1}{\partial \theta_1}-2\frac{w_1}{r_1^2}\frac{\partial w_1}{\partial \theta_1}-2\frac{v_1 w_1}{r_1^2}+$$

$$\left.\frac{3}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}+3\frac{v_1}{r_1}\frac{\partial w_1}{\partial r_1}\right)]\ +\gamma_2[(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{8}{r_1}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial v_1}{\partial r_1})^2+8\frac{v_1}{r_1}(\frac{\partial w_1}{\partial r_1})^2+\frac{4}{r_1}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial w_1}{\partial r_1})^2+4\frac{v_1}{r_1}(\frac{\partial w_1}{\partial r_1})^2+$$

$$\frac{4}{r_1{}^3}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial v_1}{\partial \theta_1})^2+4\frac{v_1}{r_1{}^3}(\frac{\partial v_1}{\partial \theta_1})^2+4\frac{w_1{}^2}{r_1{}^3}\frac{\partial w_1}{\partial \theta_1}+4\frac{v_1 w_1{}^2}{r_1{}^3}+\frac{8}{r_1{}^2}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}+8\frac{v_1}{r_1{}^2}\frac{\partial w_1}{\partial r_1}\frac{\partial v_1}{\partial \theta_1}-$$

$$8\frac{w_1}{r_1{}^2}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}-8\frac{v_1 w_1}{r_1{}^2}\frac{\partial v_1}{\partial \theta_1}-8\frac{w_1}{r_1{}^2}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}-8\frac{v_1 w_1}{r_1{}^2}\frac{\partial w_1}{\partial r_1}+\frac{8}{r_1{}^3}(\frac{\partial w_1}{\partial \theta_1})^2+24\frac{v_1}{r_1{}^3}(\frac{\partial w_1}{\partial \theta_1})^2+24\frac{v_1{}^2}{r_1{}^3}\frac{\partial w_1}{\partial \theta_1}+$$

$$8\frac{v_1{}^3}{r_1{}^3})+(\frac{\partial}{r_1}+\frac{2}{r_1})(4\frac{\partial w_1}{\partial r_1}(\frac{\partial v_1}{\partial r_1})^2+\frac{4}{r_1}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial v_1}{\partial r_1})^2-4\frac{w_1}{r_1}(\frac{\partial v_1}{\partial r_1})^2+\frac{6}{r_1{}^2}\frac{\partial w_1}{\partial r_1}(\frac{\partial v_1}{\partial \theta_1})^2-\frac{2}{r_1{}^3}(\frac{\partial v_1}{\partial \theta_1})^2-$$

$$2(\frac{\partial w_1}{\partial r_1})^3-2\frac{w_1{}^3}{r_1{}^3}-6\frac{w_1}{r_1{}^3}(\frac{\partial v_1}{\partial \theta_1})^2+\frac{6}{r_1}(\frac{\partial w_1}{\partial r_1})^2\frac{\partial v_1}{\partial \theta_1}-6\frac{w_1}{r_1}(\frac{\partial w_1}{\partial r_1})^2-12\frac{w_1}{r_1{}^2}\frac{\partial w_1}{\partial \theta_1}\frac{\partial v_1}{\partial \theta_1}-4\frac{w_1{}^2}{r_1{}^3}\frac{\partial v_1}{\partial \theta_1}-$$

$$4\frac{w_1{}^2}{r_1{}^2}\frac{\partial w_1}{\partial r_1}+\frac{4}{r_1{}^2}\frac{\partial w_1}{\partial r_1}(\frac{\partial w_1}{\partial \theta_1})^2+\frac{4}{r_1{}^3}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial w_1}{\partial \theta_1})^2-4\frac{w_1}{r_1{}^2}(\frac{\partial w_1}{\partial \theta_1})^2+8\frac{v_1}{r_1{}^2}\frac{\partial w_1}{\partial r_1}\frac{\partial w_1}{\partial \theta_1}+8\frac{v_1}{r_1{}^3}\frac{\partial v_1}{\partial \theta_1}\frac{\partial w_1}{\partial \theta_1}-$$

$$8\frac{v_1 w_1}{r_1{}^2}+4\frac{v_1{}^2}{r_1{}^2}\frac{\partial w_1}{\partial r_1}+4\frac{v_1{}^2}{r_1{}^2}\frac{\partial v_1}{\partial \theta_1}-4\frac{v_1{}^2 w_1}{r_1{}^3})]+\gamma_3[(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{4}{r_1}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial u_1}{\partial r_1})^2+4\frac{v_1}{r_1}(\frac{\partial u_1}{\partial r_1})^2+$$

$$\frac{4}{r_1{}^2}\frac{\partial w_1}{\partial \theta_1}(\frac{\partial u_1}{\partial \theta_1})^2+4\frac{v_1}{r_1{}^2}(\frac{\partial u_1}{\partial \theta_1})^2+(\frac{\partial}{\partial r_1}+\frac{2}{r_1})(2\frac{4}{r_1}\frac{\partial w_1}{\partial r_1}(\frac{\partial u_1}{\partial r_1})^2+\frac{2}{r_1}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial u_1}{\partial r_1})^2-2\frac{w_1}{r_1}(\frac{\partial u_1}{\partial r_1})^2+$$

$$\frac{2}{r_1{}^2}\frac{\partial w_1}{\partial r_1}(\frac{\partial u_1}{\partial \theta_1})^2+\frac{2}{r_1{}^3}\frac{\partial v_1}{\partial \theta_1}(\frac{\partial u_1}{\partial \theta_1})^2-2\frac{w_1}{r_1{}^3}(\frac{\partial u_1}{\partial \theta_1})^2)]+\gamma_4[(\frac{1}{r_1}\frac{\partial}{\partial \theta_1})(\frac{2}{r_1{}^2}(\frac{\partial u_1}{\partial \theta_1})^2+\left(\frac{\partial}{\partial r_1}+\frac{2}{r_1}\right)\left(\frac{\partial u_1}{\partial r_1}\frac{\partial u_1}{\partial \theta_1}\right)]$$

(12c)

The above equations are controlled by the following dimensionless numbers: $\beta=\alpha_2/\rho\, a^2$, $\gamma_1=\alpha_1/\rho a^2$, $\gamma_2=\beta_3\nu/\rho a^4$, $\gamma_3=\beta_3 u_\circ{}^2/\rho a^2\nu$,

and $\gamma_4=\alpha_1 u_\circ{}^2/\rho\,\nu^2$.

5.The Flow of fluid in Circular Cross-Section

In equations (12a),(12b)and (12c) we set

$$v=\frac{-1}{r}\frac{\partial \Psi}{\partial \theta},\ w=\frac{\partial \Psi}{\partial r}+\frac{\lambda}{2\Re e}Lru \quad (13a)$$

$$\frac{\partial u}{\partial t}=0,\frac{\partial u}{\partial t}=0\,\text{and}\,\frac{\partial \Psi}{\partial t}=0 \quad (13b)$$

to gives the steady flow of third order fluid in a helical pipe with circular cross-section.

Where$\Psi$is the pseudo − stream functionand $L=2\varepsilon\Re e$is Dean number.

$$\nabla^2 u+G=\frac{1}{r}\left(-\frac{\partial u}{\partial r}\frac{\partial \Psi}{\partial \theta}\right)-\beta\left[\frac{2}{r^3}\frac{\partial^2 u}{\partial r^2}\frac{\partial \Psi}{\partial \theta}-\frac{2}{r}\frac{\partial^2 u}{\partial r^2}\frac{\partial^2 \Psi}{\partial r\,\partial \theta}-\frac{2}{r^3}\frac{\partial u}{\partial r}\frac{\partial \Psi}{\partial \theta}+\frac{1}{r^2}\frac{\partial u}{\partial r}\frac{\partial^2 \Psi}{\partial r\,\partial \theta}-\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial^3 \Psi}{\partial r^2\,\partial \theta}-\frac{1}{r^3}\frac{\partial u}{\partial r}\frac{\partial^3 \Psi}{\partial \theta^3}\right]+\gamma_1\left[\frac{-1}{r^3}\frac{\partial u}{\partial r}\frac{\partial \Psi}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial r^2}\frac{\partial \Psi}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial r^2}\frac{\partial^2 \Psi}{\partial r\,\partial \theta}-\frac{1}{r}\frac{\partial^2 u}{\partial r^2}\frac{\partial^2 \Psi}{\partial r\,\partial \theta}-\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial^3 \Psi}{\partial r^2\partial \theta}\right]+2\gamma_3[\frac{1}{r^2}(\frac{\partial u}{\partial r})^2\left(r\frac{\partial u}{\partial r}+3r^2\frac{\partial^2 u}{\partial r^2}\right)] \quad (14)$$

and

$$
\begin{aligned}
&\nabla^4\Psi = \frac{1}{r}\left(\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial\theta} - \frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial r}\right)\nabla^2\Psi + Lu\left(-Cos\theta\frac{\partial u}{\partial r}\right) + 2GL\frac{\lambda}{\Re e} + \gamma_1[\frac{30}{r^6}\frac{\partial\Psi}{\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} + \\
&\frac{8}{r^5}\frac{\partial\Psi}{\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{4}{r^4}\frac{\partial\Psi}{\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{13}{r^5}\frac{\partial^2\Psi}{\partial\theta^2}\frac{\partial^3\Psi}{\partial\theta^3} - \frac{1}{r^5}\frac{\partial\Psi}{\partial\theta}\frac{\partial^4\Psi}{\partial\theta^4} + \frac{20}{r^5}\frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial\theta} - \frac{3}{r^3}\frac{\partial\Psi}{\partial r}\frac{\partial\Psi}{\partial\theta} + \\
&\frac{3}{r^4}\frac{\partial\Psi}{\partial r}\frac{\partial^3\Psi}{\partial\theta^3} - \frac{30}{r^5}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{6}{r^4}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{4}{r^3}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} - \\
&\frac{20}{r^4}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial\Psi}{\partial\theta} + \frac{8}{r^3}\frac{\partial\Psi}{\partial r}\frac{\partial^2\Psi}{\partial r\partial\theta} + \frac{3}{r^2}\frac{\partial\Psi}{\partial r}\frac{\partial^2\Psi}{\partial r\partial\theta} + \frac{10}{r^5}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\frac{\partial\Psi}{\partial\theta} - \frac{9}{r^4}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\frac{\partial\Psi}{\partial\theta} + \\
&\frac{1}{r^3}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\frac{\partial\Psi}{\partial\theta} - \frac{2}{r^4}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\frac{\partial^3\Psi}{\partial\theta^3} + \frac{10}{r^4}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^3\Psi}{\partial r\partial\theta^2} + \frac{9}{r^3}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^3\Psi}{\partial r\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^3\Psi}{\partial r\partial\theta^2} - \\
&\frac{3}{r^4}\frac{\partial^4\Psi}{\partial r\partial\theta^3}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{1}{r^3}\frac{\partial\Psi}{\partial r}\frac{\partial^4\Psi}{\partial r\partial\theta^3} + \frac{16}{r^4}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial\Psi}{\partial\theta} - \frac{3}{r^2}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial\Psi}{\partial\theta} - \frac{6}{r^2}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^2\Psi}{\partial r\partial\theta} - \\
&\frac{16}{r^3}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^2\Psi}{\partial r\partial\theta} - \frac{6}{r^3}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^3\Psi}{\partial\theta^3} + \frac{3}{r}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^2\Psi}{\partial r\partial\theta} + \frac{2}{r^2}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^4\Psi}{\partial r\partial\theta^3} + \frac{9}{r^4}\frac{\partial^3\Psi}{\partial r^2\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} - \\
&\frac{8}{r^3}\frac{\partial^3\Psi}{\partial r^2\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{7}{r^3}\frac{\partial\Psi}{\partial r}\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \frac{6}{r}\frac{\partial\Psi}{\partial r}\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \\
&\frac{2}{r^3}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + \frac{4}{r^2}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + \frac{8}{r^2}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + \frac{4}{r}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + 2\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + \\
&\frac{1}{r^4}\frac{\partial^4\Psi}{\partial r^2\partial\theta^2}\frac{\partial\Psi}{\partial\theta} - \frac{2}{r^3}\frac{\partial^4\Psi}{\partial r^2\partial\theta^2}\frac{\partial\Psi}{\partial\theta} - \frac{1}{r^3}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^4\Psi}{\partial r^2\partial\theta^2} + \frac{2}{r^2}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^4\Psi}{\partial r^2\partial\theta^2} - \frac{11}{r^3}\frac{\partial^3\Psi}{\partial r^3}\frac{\partial\Psi}{\partial\theta} - \\
&\frac{2}{r}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^3\Psi}{\partial r^3} + \frac{11}{r^2}\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^3\Psi}{\partial r^3} - 2\frac{\partial^2\Psi}{\partial r\partial\theta}\frac{\partial^3\Psi}{\partial r^3} - \frac{4}{r}\frac{\partial^3\Psi}{\partial r^2\partial\theta}\frac{\partial^3\Psi}{\partial r^3} + 4\frac{\partial^3\Psi}{\partial r^2\partial\theta}\frac{\partial^3\Psi}{\partial r^3} - \\
&\frac{1}{r^3}\frac{\partial^2\Psi}{\partial\theta^2}\frac{\partial^4\Psi}{\partial r^3\partial\theta} - \frac{1}{r}\frac{\partial\Psi}{\partial r}\frac{\partial^4\Psi}{\partial r^3\partial\theta} - \frac{2}{r}\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^4\Psi}{\partial r^3\partial\theta} + 4\frac{\partial^2\Psi}{\partial r^2}\frac{\partial^4\Psi}{\partial r^3\partial\theta} + \frac{2}{r^2}\frac{\partial^4\Psi}{\partial r^4}\frac{\partial\Psi}{\partial\theta} - \\
&\frac{2}{r}\frac{\partial^4\Psi}{\partial r^4}\frac{\partial^2\Psi}{\partial r\partial\theta} + L\frac{\lambda}{\Re e}(-\frac{3}{r^3}\frac{\partial u}{\partial r}\frac{\partial\Psi}{\partial\theta} + \frac{1}{2r}\frac{\partial u}{\partial r}\frac{\partial\Psi}{\partial\theta} - \frac{3}{r^2}\frac{\partial u}{\partial r}\frac{\partial^3\Psi}{\partial\theta^3} - \frac{3}{r^2}\frac{\partial u}{\partial r}\frac{\partial^2\Psi}{\partial r\partial\theta} - \frac{1}{2}\frac{\partial u}{\partial r}\frac{\partial^2\Psi}{\partial r\partial\theta} + \\
&\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial^3\Psi}{\partial r\partial\theta^2} - \frac{5}{2r^2}\frac{\partial^2 u}{\partial r^2}\frac{\partial\Psi}{\partial\theta} + \frac{\partial^2 u}{\partial r^2}\frac{\partial\Psi}{\partial\theta} + \frac{5}{2r}\frac{\partial^2 u}{\partial r^2}\frac{\partial^2\Psi}{\partial r\partial\theta} - r\frac{\partial^2 u}{\partial r^2}\frac{\partial^2\Psi}{\partial r\partial\theta} + 6\frac{\partial u}{\partial r}\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \\
&r\frac{\partial u}{\partial r}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + 2\frac{\partial^2 u}{\partial r^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + 2r\frac{\partial^2 u}{\partial r^2}\frac{\partial^3\Psi}{\partial r^2\partial\theta} + \frac{1}{r}\frac{\partial^3 u}{\partial r^3}\frac{\partial\Psi}{\partial\theta} - \frac{\partial^3 u}{\partial r^3}\frac{\partial^2\Psi}{\partial r\partial\theta} - \frac{\partial u}{\partial r}\frac{\partial^4\Psi}{\partial r^3\partial\theta} + \\
&2r\frac{\partial u}{\partial r}\frac{\partial^4\Psi}{\partial r^3\partial\theta} - (r^3-6)\frac{3}{r^3}u\frac{\partial\Psi}{\partial\theta} - \frac{18}{r^3}u\frac{\partial^2\Psi}{\partial r\partial\theta} + \frac{1}{r^2}u\frac{\partial^2\Psi}{\partial r\partial\theta} + \frac{3}{r}u\frac{\partial^2\Psi}{\partial r\partial\theta} + \\
&\frac{15}{r^2}u\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \frac{3}{2r^5}u\frac{\partial^3\Psi}{\partial r^3} - \frac{3}{r}u\frac{\partial^2\Psi}{\partial r\partial\theta} + \frac{1}{2r^2}u\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \frac{1}{r}u\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \frac{3}{2}u\frac{\partial^3\Psi}{\partial r^2\partial\theta} - \\
&\frac{3}{2r}u\frac{\partial^4\Psi}{\partial r^2\partial\theta^2} - 2u\frac{\partial^4\Psi}{\partial r^2\partial\theta^2})] \quad + \gamma_3[\frac{-2}{r^4}(\frac{\partial u}{\partial r})^2\frac{\partial^4\Psi}{\partial\theta^4} - \frac{4}{r^3}(\frac{\partial u}{\partial r})^2\frac{\partial\Psi}{\partial r} - \frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\frac{\partial^3\Psi}{\partial r\partial\theta^2} + \\
&\frac{4}{r}(\frac{\partial^2 u}{\partial r^2})^2\frac{\partial\Psi}{\partial r} + \frac{4}{r^2}(\frac{\partial u}{\partial r})^2\frac{\partial^2\Psi}{\partial r^2} - 4(\frac{\partial^2 u}{\partial r^2})^2\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r^2}(\frac{\partial u}{\partial r})^2\frac{\partial^4\Psi}{\partial r^2\partial\theta^2} + \frac{4}{r}\frac{\partial u}{\partial r}\frac{\partial^3 u}{\partial r^3}\frac{\partial\Psi}{\partial r} - \\
&4\frac{\partial u}{\partial r}\frac{\partial^3 u}{\partial r^3}\frac{\partial^2\Psi}{\partial r^2} - \frac{6}{r^4}(\frac{\partial u}{\partial r})^2\frac{\partial^2\Psi}{\partial\theta^2} - \frac{2}{r^2}(\frac{\partial^2 u}{\partial r^2})^2\frac{\partial^2\Psi}{\partial\theta^2} + \frac{2}{r^3}\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial r}\frac{\partial^3 u}{\partial r^3}\frac{\partial^2\Psi}{\partial\theta^2} - \\
&\frac{2}{r}(\frac{\partial u}{\partial r})^2\frac{\partial^3\Psi}{\partial r^3} - 8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}\frac{\partial^3\Psi}{\partial r^3} - 2(\frac{\partial u}{\partial r})^2\frac{\partial^4\Psi}{\partial r^4} - L\frac{\lambda}{\Re e}(12(\frac{\partial u}{\partial r})^2\frac{\partial^2 u}{\partial r^2} + \\
&6r\frac{\partial u}{\partial r}(\frac{\partial^2 u}{\partial r^2})^2 + 3\frac{\partial u}{\partial r}\frac{\partial^3 u}{\partial r^3})]
\end{aligned}
\tag{15}
$$

6. The Solution

we are going to solve the equations (14) and (15). We start by the successive approximation for u and Ψ. This method equivalent to expand u and Ψ in secondary power of Dean number .In this way we obtain recursive relations. These equations are solved analytically.

This equation in polar coordinates is

$$r^2 = 1 \quad \text{or} \quad 1 - r^2 = 0 \tag{16}$$

Where r is the radius of the cross section and the non slip conditions are

$$u = \Psi = \frac{\partial \Psi}{\partial r} = 0 \quad at r = 1 \tag{17}$$

The solution of equations (14) and (15) subject to associate boundary conditions are $\Psi(\mathrm{r}, \theta, L, \lambda/\Re e, \gamma_1, \gamma_3), u(\mathrm{r}, \theta, L, \lambda/\Re e, \gamma_1, \gamma_3)$.The prime parameter Dean number L, and the successive approximation method is adopted. This method equivalent to expand Ψand u in a secondary power of Dean number L.

$$\Psi = L\Psi_1 + L^2\Psi_2$$

$$u = u_\circ + Lu_1 \tag{18}$$

$$u_\circ = 1 - r^2 + 4\gamma_3(1 - r^4) \tag{19}$$

Provided G = 4, and if we set $\gamma_3 = 0$ we will obtain the solution in a case of a straight pipe (Dean, [6]).

$$\Psi_1 = \{\frac{1}{144}\left(r - \frac{9}{4}r^3 + \frac{3}{2}r^5 - \frac{1}{4}r^7\right) + \gamma_3[\frac{1}{69120}(1859\, r - 4707\, r^3 + 3870 r^5 - 1055\, r^7 + 33\, r^9)] + \gamma_3{}^2[\left(\frac{-114717\, r}{512000} + \frac{1}{27648000}(6346285\, r^3 + 338900\, r^5 + 659750 r^7 - 5900750\, r^9 + 1699533\, r^{11})\right)] + \gamma_3{}^3[-\frac{13441590109\, r}{13934592000} + \frac{1}{69672960000}(77681344479\, r^3 + 5827344600\, r^5 + 26411464625\, r^7 - 37061159625\, r^9 - 13580472150\, r^{11} + 7929428616\, r^{13})]\}cos\theta + \frac{\lambda}{\Re e}[\frac{1}{8}\left(1 - r^2\right)^2 +$$

$$\gamma_3\left(\frac{1}{2}-\frac{3}{4}r^2+\frac{1}{4}r^6\right)+{\gamma_3}^2\left(\frac{3589}{490}-\frac{5086}{525}r^2-\frac{576}{1225}r^7+\frac{17}{6}r^8\right)+{\gamma_3}^3\left(\frac{1744}{315}-\frac{332}{49}r^2-\frac{512}{441}r^9+\frac{12}{5}r^{10}\right)+{\gamma_3}^4\left(128-\frac{768}{5}r^2+\frac{128}{5}r^{12}\right)\Big] \qquad (20)$$

Now if we set $\gamma_3 = 0$, in to equation (20) will describe the flow of non Newtonian fluid of second order, [1].

$$u_1 = \Big\{\frac{2413}{3572100}-\frac{r^3}{648}+\frac{r^5}{800}-\frac{r^7}{2352}+\frac{r^9}{23328}+\beta\left[\frac{493}{44100}-\frac{r^3}{36}+\frac{r^5}{50}-\frac{r^7}{294}\right]+\Big[-\frac{1}{140}+\frac{r^3}{48}-\frac{r^5}{60}+\frac{r^7}{336}\Big]\gamma_1+\gamma_3\Big[\frac{596675249}{51866892000}-\frac{1859r^3}{311040}-\frac{679r^5}{32000}+\frac{2281r^7}{94080}-\frac{37691r^9}{3919104}+\frac{4447r^{11}}{4181760}+\beta\left[\frac{864641}{4082400}-\frac{523r^3}{4320}-\frac{181r^5}{400}+\frac{2263r^7}{5040}-\frac{7123r^9}{81648}\right]+\left[-\frac{2171}{18144}+\frac{523r^3}{5760}+\frac{149r^5}{480}-\frac{2861r^7}{8064}+\frac{949r^9}{12960}\right]\gamma_1\Big]+$$
$${\gamma_3}^2\Big[-\frac{245107349704291}{16392632256000000}+\frac{13441590109r^3}{62705664000}-\frac{9250236533r^5}{29030400000}-\frac{r^6}{9}-\frac{35359529r^7}{58060800}+\frac{9r^8}{16}+\frac{12038419643r^9}{22574039040}-\frac{9r^{10}}{20}-\frac{216172771r^{11}}{1021870080}+\frac{7r^{12}}{90}+\frac{314836117r^{13}}{6853017600}-\frac{586000193r^{15}}{108864000000}+$$
$$\beta\left[-\frac{93018484869529}{32056703078400}+\frac{25893781493r^3}{13063680000}+\frac{265327r^5}{345600}-2r^6-\frac{48870443r^7}{5806080}+9r^8+\frac{1489056343r^9}{156764160}-\frac{18r^{10}}{5}-\frac{428326027r^{11}}{78059520}+\frac{563339753r^{13}}{486720000}\right]+\Big[\frac{26410449711223063}{11219846077440000}+\frac{114717r^2}{64000}-\frac{39769949813r^3}{17418240000}-\frac{1269257r^4}{921600}+\frac{30381317r^5}{51840000}+\frac{59413r^6}{62208}+\frac{2017171643r^7}{325140480}-\frac{1677353r^8}{221184}-\frac{2316781123r^9}{313528320}+\frac{144563r^{10}}{38400}+\frac{530731199r^{11}}{133816320}-\frac{2077207r^{12}}{13824000}-\frac{5714525549r^{13}}{6814080000}\Big]\gamma_1\Big]+{\gamma_3}^3\Big[\Big(\frac{314779477711671019}{531978483156120000}+\frac{13441590109r^5}{3628800000}-$$
$$\frac{517390257329r^7}{177811200000}-\frac{14r^8}{3}-\frac{561033997r^9}{41990400}+\frac{558r^{10}}{25}+\frac{12110038331r^{11}}{903260160}-\frac{123r^{12}}{7}-\frac{1085929391r^{13}}{196245504}+\frac{136r^{14}}{45}+\frac{2259476969r^{15}}{2021760000}-\frac{5256329023r^{17}}{43696800000}+\beta\Big[-\frac{107925190514441563}{125221496400000 0}+\frac{25893781493r^5}{648000000}-\frac{543323r^7}{22680}-84r^8-\frac{1142220467r^9}{5598720}+\frac{8928r^{10}}{25}+\frac{5344760417r^{11}}{21954240}-\frac{984r^{12}}{7}-\frac{57395600747r^{13}}{449729280}+\frac{440595703r^{15}}{17062500}\Big]+$$
$$\Big[\frac{1025744282912993}{11925856800000}+\frac{344151r^4}{16000}-\frac{36300907733r^5}{907200000}-\frac{1269257r^6}{57600}+\frac{86046059r^7}{2268000}+\frac{183829r^8}{3456}+\frac{375438079r^9}{2612736}-\frac{689365r^{10}}{2304}-\frac{8258830571r^{11}}{43908480}+\frac{259763r^{12}}{1920}+\frac{2006612321r^{13}}{21415680}-\frac{2077207r^{14}}{672000}-\frac{4643819759r^{15}}{245700000}\Big]\gamma_1\Big]\Big\}\sin\theta$$

(21)

and

$$\Psi_2 = \{[-\frac{916757r}{311662391040} + \frac{r^2}{62208} - \frac{49661063r^3}{2574483912000} + \frac{2413r^5}{685843200} + \frac{1009r^6}{174182400} - \frac{3103r^8}{762048000} +$$
$$\frac{1849r^{10}}{1738598400} - \frac{19687r^{12}}{129459762432} + \frac{461r^{14}}{83264163840}] + \beta[\frac{401957r}{5708102400} - \frac{10703611r^3}{63567504000} + \frac{493r^5}{8467200} + \frac{r^6}{9450} -$$
$$\frac{43r^8}{496125} + \frac{172r^{10}}{7640325} - \frac{4r^{12}}{2081079}] + \gamma_1[\frac{1640486561477r}{5744561191649280} - \frac{r^2}{864} + \frac{30576305891r^3}{14878428364800} - \frac{509r^4}{311040} - \frac{5309r^5}{23224320} +$$
$$\frac{15101r^6}{14515200} + \frac{8579r^7}{127401984} - \frac{28879r^8}{50803200} - \frac{1457r^9}{159252480} + \frac{133829r^{10}}{754427520} - \frac{293r^{11}}{530841600} - \frac{2815r^{12}}{105380352} + \frac{23r^{13}}{139345920} +$$
$$\frac{361r^{14}}{256988160} + \frac{11r^{15}}{1783627776}] + \gamma_3[[-\frac{30710795212666169r}{4383602838332669952000} + \frac{3455881r^2}{14332723200} - \frac{5799070821818177r^3}{15770418374430720000} +$$
$$\frac{r^4}{34992} + \frac{610015499r^5}{9958443264000} + \frac{1787591r^6}{33443020800} + \frac{299921r^7}{10973491200} - \frac{1263171557r^8}{92177326080000} - \frac{15131r^9}{557383680} -$$
$$\frac{3247622827r^{10}}{260730150912000} + \frac{271057r^{11}}{15676416000} + \frac{516169243r^{12}}{112983065395200} - \frac{43409r^{13}}{7315660800} - \frac{261282706801r^{14}}{47392629831065600} +$$
$$\frac{4157r^{15}}{4096770048} - \frac{39908417r^{16}}{1437267566592000} - \frac{797r^{17}}{12039487488} - \frac{121r^{18}}{285769728000}] + \beta[\frac{6990572977307r}{3926946127104000} -$$

$$\frac{662567266769r^3}{164766970368000} + \frac{r^4}{1944} + \frac{883541r^5}{783820800} + \frac{379r^6}{453600} + \frac{196043r^7}{406425600} - \frac{54091r^8}{166698000} - \frac{1343r^9}{2903040} - \frac{1495481r^{10}}{6601240800} +$$
$$\frac{73453r^{11}}{290304000} + \frac{778727r^{12}}{7224317100} - \frac{6217r^{13}}{101606400} - \frac{158491r^{14}}{15937281360} + \frac{629r^{15}}{113799168}] + \gamma_1[-\frac{2123558539838419r}{457769719959552000} +$$
$$\frac{8341r^2}{737280} - \frac{14338861175753r^3}{1300799737036800} + \frac{267779r^4}{37324800} - \frac{8408119r^5}{4459069440} - \frac{15433417r^6}{3483648000} + \frac{7743539r^7}{1981808640} + \frac{18106811r^8}{7315660800} -$$
$$\frac{64119517r^9}{19110297600} - \frac{3620083r^{10}}{5912248320} + \frac{19031893r^{11}}{15925248000} + \frac{476527r^{12}}{13699445760} - \frac{2114813r^{13}}{13377208320} + \frac{2999443r^{14}}{1295220326400} +$$
$$\frac{212185r^{15}}{149824733184} + \frac{583r^{16}}{10998374400} + \frac{121r^{17}}{53508833280}]\}\sin\theta\cos\theta + \frac{\lambda}{\Re e}\{[\frac{44442527r}{1393459200} - \frac{7r^2}{864} - \frac{713129r^3}{11612160} +$$
$$\frac{43r^4}{17280} + \frac{4757r^5}{110592} + \frac{223r^6}{201600} - \frac{3163r^7}{442368} - \frac{157r^8}{51840} + \frac{3103r^9}{4423680} + \frac{115r^{10}}{217728} - \frac{433r^{11}}{11059200} + \frac{r^{13}}{7741440}]\gamma_1 +$$
$$\gamma_3[-\frac{5279r}{6220800} + \frac{2981r^3}{2419200} - \frac{r^5}{3456} + \frac{r^7}{2560} - \frac{109r^9}{179200} + \frac{481r^{11}}{3628800} - \frac{r^{13}}{108864} + \beta[\frac{79r}{1120} - \frac{1109r^3}{6720} + \frac{29r^5}{240} -$$
$$\frac{193r^7}{6720} + \frac{r^9}{420}] + [-\frac{89253672928451r}{920602755072000} + \frac{2915r^2}{82944} + \frac{3971662847r^3}{24908083200} - \frac{3689r^4}{1036800} - \frac{593701r^5}{6635520} - \frac{115601r^6}{8064000} -$$
$$\frac{298819r^7}{26542080} + \frac{643709r^8}{60963840} + \frac{561731r^9}{53084160} + \frac{15271r^{10}}{6531840} - \frac{1015021r^{11}}{663552000} - \frac{232279r^{12}}{217451520} + \frac{45769r^{13}}{464486400}]\gamma_1\}\sin\theta$$

(22)

7. Results & Discussion

In this section we study the effects of the parameters L, β, γ_1, γ_3, and $\frac{\lambda}{\Re e}$ upon the components of the secondary flow and the axial velocity. Since the pseudo stream- function, for duct with non zero torsion dose not represent the secondary flow as described by v and w, therefore vector plots are employed to present the secondary flow.

7.1. The Secondary Flow Motion

A helical pipe characterized by non zero torsion. We have more than 40 cases to a certain how the parameters L, $\frac{\lambda}{\Re e}\beta, \gamma_1, \gamma_3$ effects on the secondary flow in helical and straight pipes.In equation (21) , if $\beta = 0, \gamma_1 = 0, \gamma_3 = 0$, we recover the first order results in L of Gremano, [9]for Newtonian flow in a helical pipe with an elliptical cross- section, and if $\gamma_1 = 0, \gamma_3 = 0$ in that equation we recover the flow of non Newtonian fluid of second order.

Figure (2) shows the effects of a material moduli γ_3 on the secondary flow. For $\lambda/\Re e = 0.01$, $\beta = 0.5$, $\gamma_1 = 1$, and $\gamma_3$increases from 0.01 to 1 we observed:-

- There is new secondary flow which increases when $\gamma_3$increased.
- There is a shifting toward the left side of cross- section. That is means the intensity of flow in the right side is increased and consequently begins to push the main flow to the left, figure (3).
- When γ_3=0.07, there is a secondary flow which near the center of the cross- section, figure (4).
- When γ_3=1, the effect of this disappears, figure (5).
    - When$\gamma_1 = 0, \gamma_3 = 0$,$\beta = 0$,and torsion equal zero ,the flow is in a straight pipe.

Figure(6)illustrates the effects of $\gamma_1$on the secondary flow. Here $\Re e = 2$, $\beta = 0$, $\gamma_3 = 0$ and $\gamma_1$varies from 0.1 to 4.

- The effects of $\gamma_1$appear when $\gamma_1$is greater than 1, figure (7).
- When $\gamma_1$increases there is new secondary flow, figure (8).
- The intensity of fluid which is found in the lower part and near the center of cross- section is stronger that is the secondary flow of fluid in the upper part is weaker ,figure (9).

Figure (10) shows the effects of β upon the secondary flow, we noted

- That the parameter βinfluouns the secondary velocity of fluid when it is very large since, it is product by small values.
- There is a displacement to the towardupper part of the cross- section .That is due to the increasing in the intensity of fluid the lower part of cross- section, figure(11).
- There is new secondary flow in the lower part of cross section, figure (12).

Figure (13) explains the effects of $\lambda/\Re e$ on the secondary flow.

These effects are:-

- There is a displacement to the left toward of cross- section, There is a small secondary flow near the center the cross-section.

The intensity of fluid in the lower part and near of the center cross- section increases.

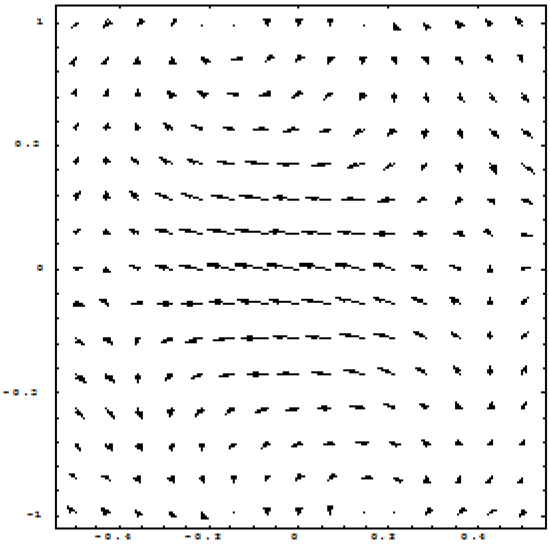


Figure (2), β=0.5, γ1=0,γ 3=0,λ=0

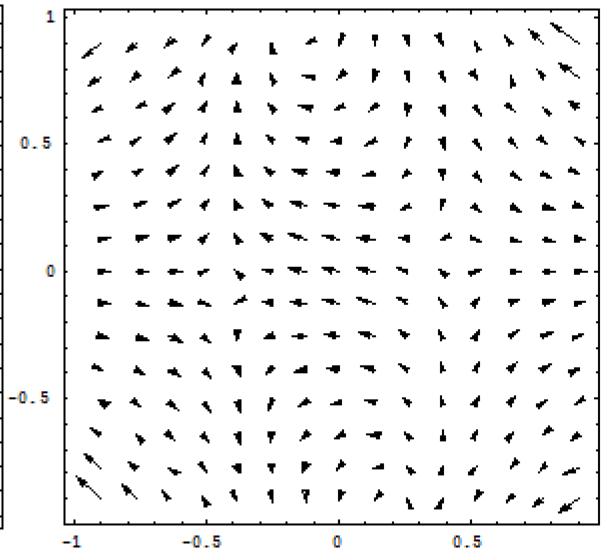


Figure (3), β=0.5, γ1=1,γ3=0.01,λ=2

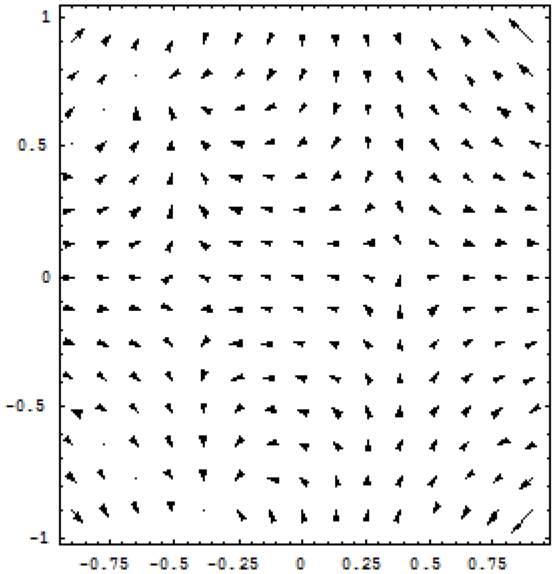


Figure (4), β =0.5, γ1=1,γ3=0.07,λ=0.2

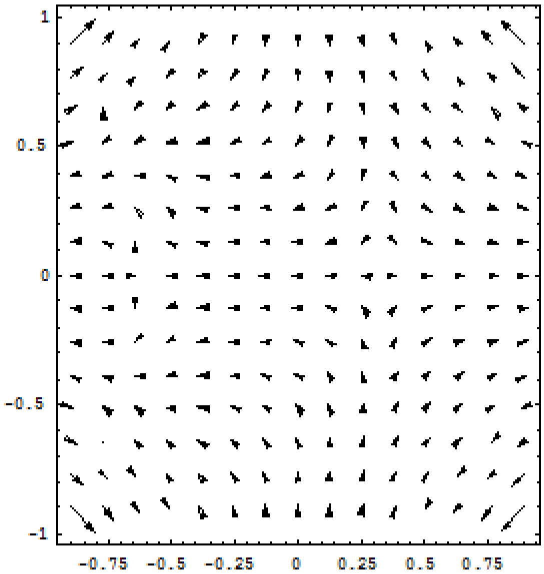


Figure (5), β=0.5, γ1=1, γ3=1,λ=0.2

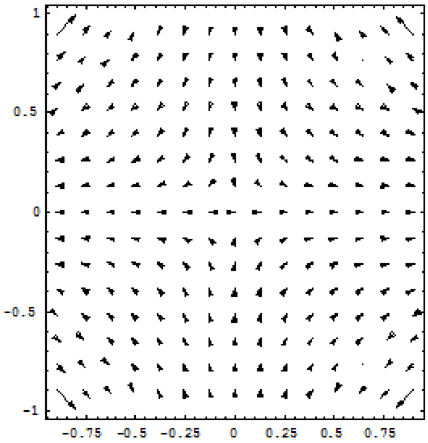


Figure (6), $\beta=0$, $\gamma 1=1$, $\gamma 3=0$,$\lambda=0.2$

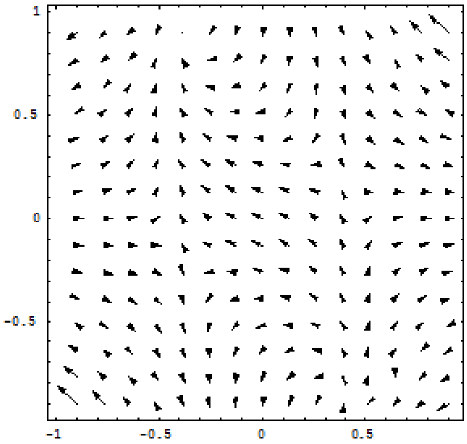


Figure (7) $\beta=0$, $\gamma 1=2$ $\gamma 3=0$,$\lambda=0$.

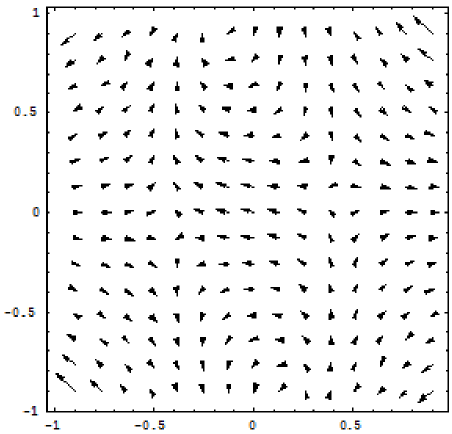


Figure (8) $\beta=1.4$, $\gamma 1=0$, $\gamma 3=0$,$\lambda=0.2$

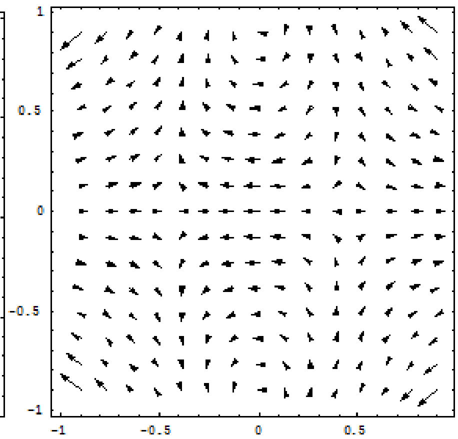


Figure (9), $\beta=0.5$, $\gamma 1=1$, $\gamma 3=1$,$\lambda=0.2$

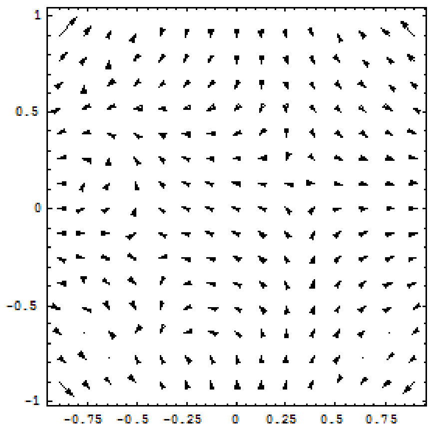


Figure (10) $\beta=0.5$, $\gamma 1=1$, $\gamma 3=0.01$,$\Re e=4$

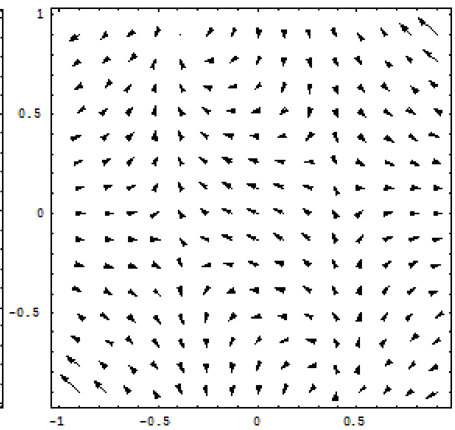
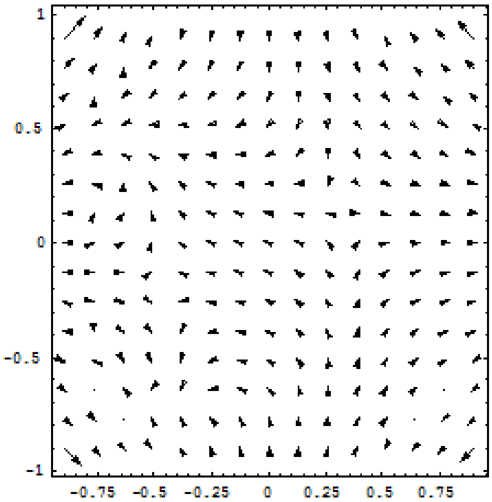


Figure (11) $\beta=1.4$, $\gamma 1=2$, $\gamma 3=0$,$\lambda=0.2$



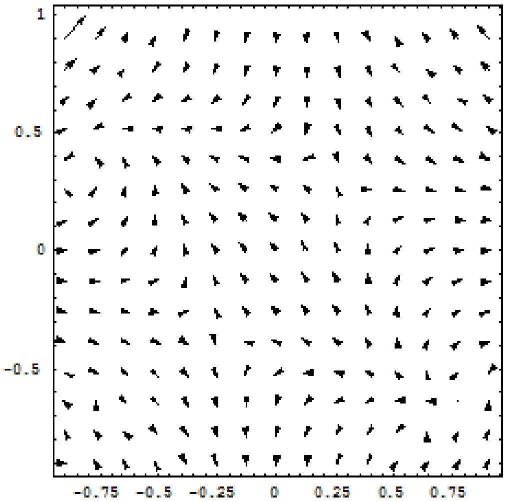


Figure (12) β=0.5, γ1=1, γ3=0.01,ℜe=4 **Figure (13) β=0, γ1=0, γ3=0,ℜe=10**

7.2. The Axial Flow

In this we analyze the axial flow by studies the effects of $\gamma_1, \gamma_3$and β.If $\beta = 0, \gamma_1 = 0, \gamma_3 = 0,$ in equation(19)we go to the flow in a straight pipe, figure (14)

Figure (15) shows the effects of γ_3.For L=80, $\beta = 0.01$, $\gamma_1 = 1$, as $\gamma_3$increases from 0.01 to 2.

- Thereis a displacement toward upper part of the pipe. That is the velocity of fluid in the lower part of the pipe is stronger, figure (16).
- When γ_3= 1, in which there a continuous a displacement and a stagnation region short to appear in the middle of the pipe, figure (17).
- In addition there are two vortexes in the upper and lower part of the pipe.

Figure (18) gives the axial flow under the effects of γ_1, as increases from 0.1 to 3.

- Thereis a displacement toward the upper wall of the pipe, figure (19).
- There is a stagnation region in the middle of pipe, figure (20).
- When γ_1=1.5, the stream lines becomes thicker near the stagnation region, figure (21).

Figure (22) expresses the effects of β, as it varies from 0.01 to 1.5.

- Thereis a displacement toward the upper wall of the pipe . That is the axial velocity in the lower part of pipe is stronger then it pushes the fluid to the upper part of pipe, figure (23).
- There is a stagnation region in a center plane of pipe, figure (24).

For $\beta = 1.7$, we note that there is a displacement toward the lower wall of the pipe, and the intensity of the fluid in the upper part of cross-section becomes stronger.

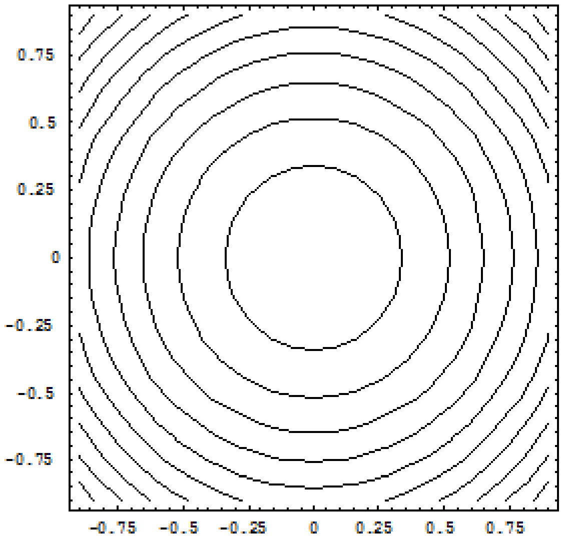


Figure (14)β=0, γ1=0,γ 3=0

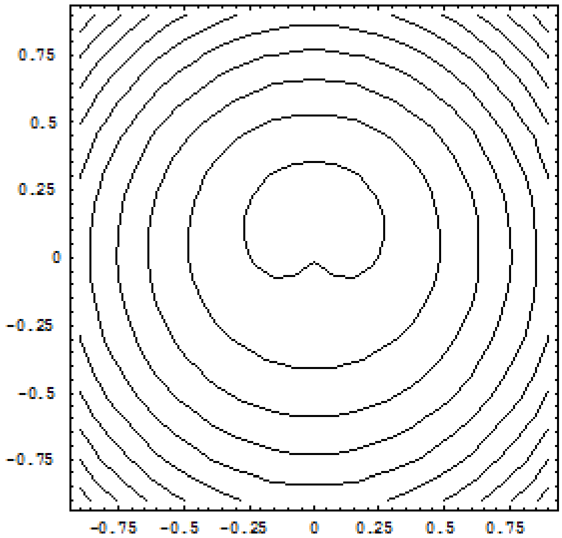


Figure (15) β=0.01, γ1=0, γ3=0.3,L=80

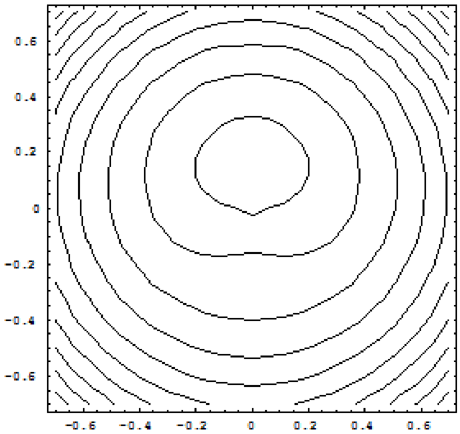


Figure (16) β=0.01, γ1=0, γ3=0.3,L=80

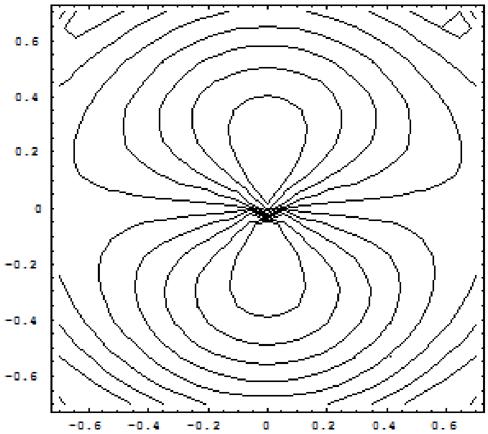


Figure (17) β=0.01, γ1=0, γ3=1,L=80

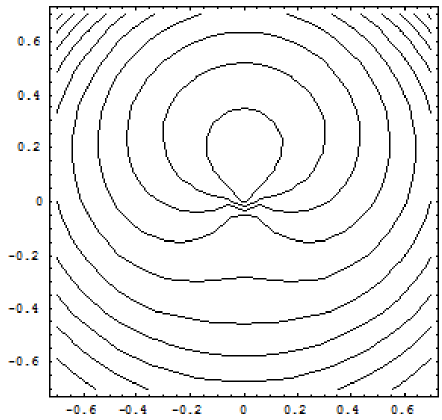


Figure (18) β=0.01, γ1=0.3, γ3=0.01,L=80

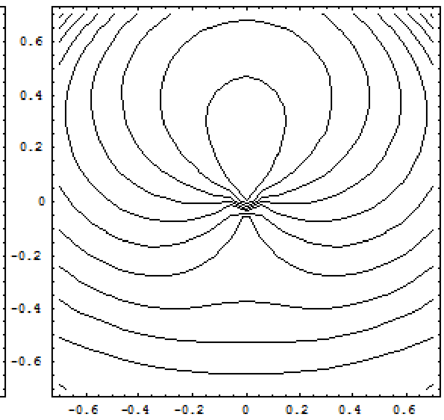


Figure (19) β=0.01, γ1=0.5, γ3=0.1,L=80

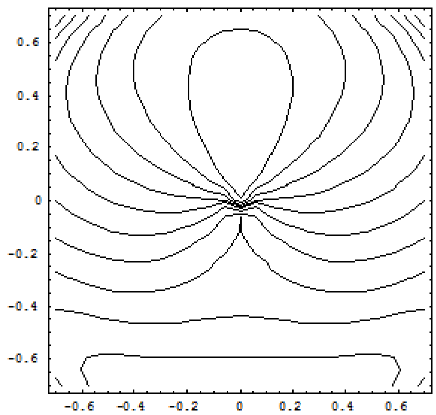


Figure (20) β=0.01, γ1=0.7, γ3=0.1,L=80

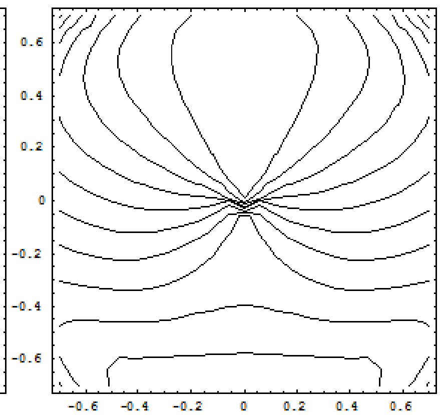


Figure (21) β=0.01, γ1=1.5, γ3=0.1,L=80

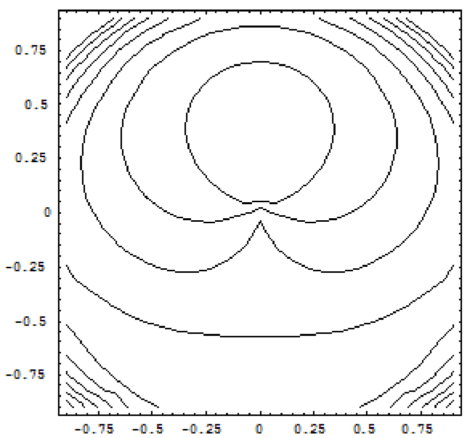


Figure (22) β=0.01, γ1=0.3, γ3=0.07,L=80

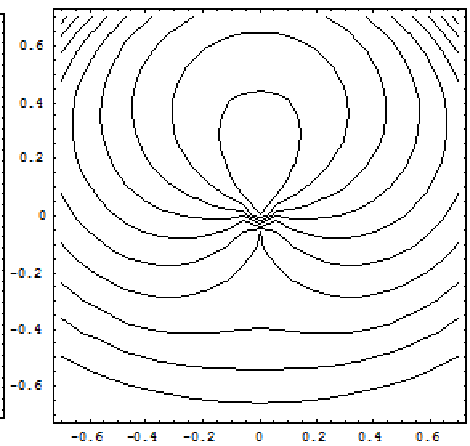


Figure (23) β=0.05, γ1=0.3, γ3=0.07, L=80

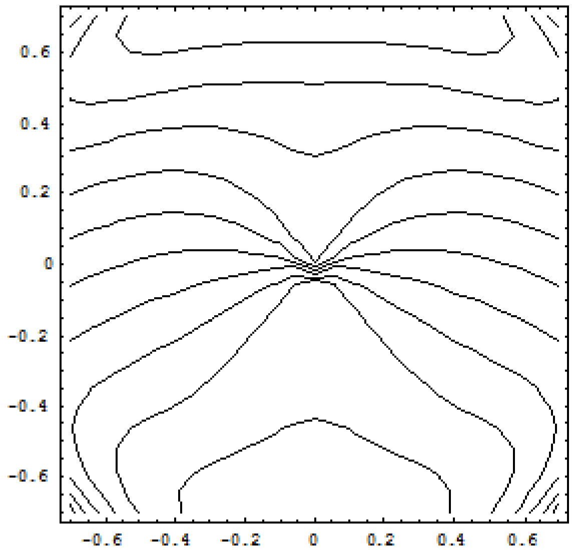


Figure (24) β=0.05, γ1=0.3, γ3=0.07, L=80

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