# Freedom as the Physical Notion. Mechanics of a Material Point

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This is a novel conceptual attempt to introduce freedom as a consistent physical notion. The present consideration is limited to mechanics of a material point. Adapted to physical representation, freedom is tested throughout the paper to conform to its general perception. It is first tried on such physical notions (but not restricted to) as potency of a set, Lagrangian, action, degrees of freedom. Defined as angular and essentially nonlinear variable, it appears to be dual as that the domain affords and that the material point possesses within the domain, not necessarily equal to one another. Freedom has to degrade if its bearer experiences an external impact. The two classical problems of Newtonian and relativistic mechanics illustrate the freedom non-linear dynamics whose key factors are the impact power and duration as well as total mechanical energy of the material point. Reversibility of freedom is discussed. Hints to possible relations with quantum mechanics and commonly used numerical techniques as well as to advanced mathematical modeling are proposed.



## Keywords

freedom; material point; mechanics; mechanical energy; multidimensional space

RESEARCH PAPER

# Freedom as the Physical Notion. Mechanics of a Material Point

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**Abstract**

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## 1. Introduction

Although the notion of freedom has been present in the circulation of philosophical thought since ancient Greece [1], the everyday perception of the term does not carry a veil of antiquity, because it is associated with the very essence of a living being in a conscious or unconscious form “here and now” [2]. The discussion of freedom as a philosophical concept is not the subject of this paper, as it requires different research objectives and competences than those I chose and possess. Nevertheless, some definitions of freedom that are widespread outside the circle of philosophical researchers and have become public knowledge are to be given. At the same time, in the context of this work, I have given preference to those definitions that are not related to human consciousness in any form. After all, freedom as a subconscious or instinctive necessity is probably inherent in any living being. Thus, one definition of freedom that allows for some distancing from consciousness, although it has human relationships at its foundation, is *Abwesenheit von Zwang*[1] [3]. The other definition originates from Hegel and equates freedom with autonomy [4]. Particularly interesting in the context of our subject is MacCallum's triadic concept [5]: there is always someone[2] whom[3] freedom is a certain state for, the absence of some possible constraint and unhindered ability to do or not do something.

In the following discussion of the term *freedom*, I shall mean it exclusively in the sense of physical science, unless otherwise specified. One of the guidelines for further reasoning is the expediency of such an understanding of the term that would allow it to be used for solving practical problems. Another guideline is a certain dissatisfaction related, for

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[1] Absence of coercion.
[2] I am also projecting it onto *something*.
[3] Or *what*.

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example, to the fact that the generally accepted notion of “degrees of freedom” relies, in fact, on the term still undefined in physics. Of course, physics is not an axiomatic science (nor is mathematics to a certain extent), but to have a rigorous or at least rational definition of the terms in use (and even more so, of essential terms) is never bad.

## 2. Heuristic

It is of lively interest to introduce some physical quantity that would characterise the freedom in physical systems. To clarify the idea, consider a material point (hereinafter MP) on a straight line (a one-dimensional case in Fig. 1a).

In the absence of external forces, it can have a certain freedom. Received some momentum, it can move infinitely uniformly and rectilinearly either to the left or to the right. Note that if the momentum was directed to the right, MP will never move to the left. Thus, freedom seems to be a certain characteristic of potentialities.

Consider MP on a ray (Fig. 1b). Given momentum to the left, MP will reach the boundary and bounce back (here, we consider a perfectly elastic collision with the boundary as a feature of freedom, whereas a perfectly inelastic adhesion to the boundary would mean a full loss of freedom; still no extra interactions are present as in Fig. 1a). MP cannot enter the region to the left of the boundary when within the framework of classical mechanics. Thus, MP here has less freedom than before, i.e., its potentialities in Fig. 1b are smaller than in Fig. 1a. The objection that, compared to Fig. 1a, MP may actually have a larger range of motion upon receiving the momentum (moving first to the left, then to infinity on the right) seems to be partly motivated. However, if we speak of freedom as a potentiality, such an argument seems to fall short.

### 2.1. Analysis of existing “consonant” concepts

*Potency of a set.* An attempt to use the notion set potency [6] to describe freedom did not appear to be successful, since for potency, only the packing density of elements is important, not the domain itself. In particular, if set A is a segment, its potency is equal to that of the set A $\times$ A. For freedom, this seems to be false as the freedom of MP inside a square (a two-dimensional case) is intuitively expected to be greater than on a segment of length equal to the side of the square (one-dimensional case). Furthermore, for potency, a segment and a straight line are equivalent. However, one cannot completely ignore the ideas related to the notion *potency of a set*, since it may be necessary to consider both continuous and discrete domains. In this paper, however, we consider the continuous cases, which are the most ordinary in physics.

Let us consider the case of an applied force (Fig. 2).

In the one-dimensional case, we can imply Hooke's law (Fig. 2b). Clearly, in the potential sense and in the sense of a common perception of the concept of “freedom”, the presence of a spring limits the MP's potential ability to reach “desired” coordinate magnitudes. This is especially clear when comparing Fig. 2a and b with the spring unstretched and uncompressed when MP receives momentum.

*Lagrangian, action*. It is interesting to consider in this context the Lagrangian and the action [7]. The Lagrangian is hardly suitable as a measure of MP's freedom, since it represents just kinetic energy. Without a dissipation, it hardly fully characterises freedom. Potential energy has the feature of relation to a point of reference, whereas we expect to define freedom as a concept less dependent on a point of reference (but possibly having a reference to some other system). I.e., if the relativity of the notion *freedom* arises here, it is rather in configurational than in calculative sense. Considering such a physical concept as action, we note that there is a principle of least action, which allows us to characterise trajectories of motion. Were it possible to define freedom as a set of all possible trajectories, action could be promising. The difficulty lies, in my opinion, in the fact that calculation of the number of

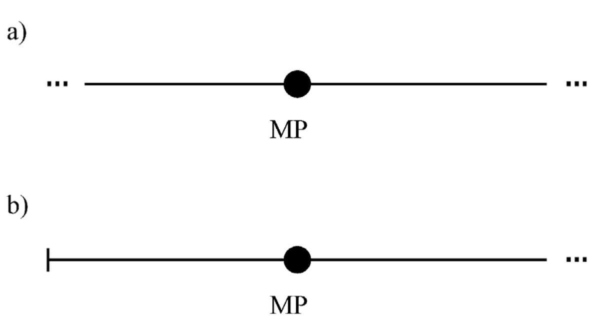


*Fig. 1. Material point (MP) a) on a straight line, and b) on a ray.*

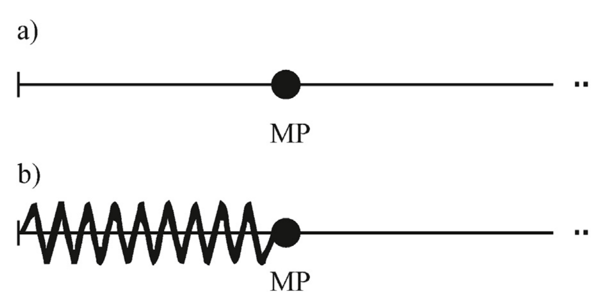


*Fig. 2. MP on a ray with and without a spring.*

trajectories would invoke comparison of infinite quantities, for which it is necessary to define some common measure, including systems of different dimensions. The attempt I had undertook led to results which I took for speculative and rejected. In any case, it is useful to keep in mind the dimensionality of action:

Energy × Time = Moment of Momentum
= Momentum × Distance

Action, when known, fully determines the motion of a system of particles. Freedom characterises not so much the law of motion of the system as the motion domain. Besides, action is not universal. For example, it misses dissipative forces.

*Degrees of freedom*. Freedom has to be associated with degrees of freedom, and not only in the sense of space (beyond the scope of our consideration) as, for example, rotational (applied to a solid body) and oscillatory (applied to a system of MPs, starting from two) degrees of freedom, spin (beyond the scope of classical mechanics; it implies, by the way, the ability to work with freedom on discrete domains). In our context, the concept of degrees of freedom is remarkable in the sense that it is used as a physical concept in solving physical problems as, in particular, for determination of the number of holonomy bonds in a system with contacting bodies [8].

### 2.2. Some inherent properties of the notion "freedom"

Since both finite and infinite motion are to be considered, we need a measure whose calibration would suit both cases. We would rather be able to answer the question of how many times less freedom MP has on a segment than on a straight line. In doing so, we should expect that "here and now" should have a prevailing weight over that of "there and then", because in the practical world, no one uses infinite possibilities. MP can go to infinity rather in a mathematical than in a physical or computational sense. Hence, there has to envisage some reduction of "infinite freedom" into some measure, infinite (certainly calibrated) or finite.[4]

Another important aspect is superposition of freedoms in different dimensions (or in different degrees of freedom). Without introducing a definition of freedom, some reasoning on this topic has sense. Suppose that MP has two degrees of freedom (it moves in two dimensions). Let MP's freedom in one of these dimensions be characterised by a relatively big value, while in the second it is small. If the measure of freedom allows expression in numbers both smaller than and greater than one, then in the case of multiplicativity, the superposition of freedoms may turn out to result in a smaller value of the measure than the freedom in one of the dimensions is, which seems absurd. Therefore, in the case of multiplicativity, the scale of the measure technically should not contain numbers smaller than 1. In fact, this means that a measure of freedom equal to 1 would correspond to a loss of freedom, while higher values would correspond to a nontrivial freedom. This would diverge, in some part, from the usual perception that the loss of some attribute would rather correspond to zero while a positive value would mean the presence of that. Besides multiplicativity, one can consider additivity as well as more complex superposition rules like, e.g., the definition of the cumulative experimental error as a function of a number of factors A, B, C … as

$$\| \; \| = \sqrt{\|A\|^2 + \|B\|^2 + \|C\|^2 + \dots}$$

Anyway, we believe that MP in Fig. 3a and c possesses more freedom than in Fig. 3b.

Another useful fact is that almost any problem in mechanics can be solved in different coordinate systems. Probably, the freedom value should be invariant in this sense. As an example, consider a fractional linear function: it can translate a bounded set into an unbounded set, and vice versa. The question is, is it the mapping of the physical space onto itself? Probably, space compression is not an acceptable kind of mapping if freedom is desired as an invariant, since it changes the density of points on a line or in space i.e., the metrics. Acceptable transformations are rotation (in multidimensional cases) and translation.

## 3. Measure of freedom

The parameter characterising freedom has to have a connection with the space metrics. While within the classical physics framework, we consider connected sets only. The one-dimensional set of real numbers along each axis of Cartesian coordinates is to be upgraded now with the set of imaginary numbers to a complex plane.[5] In the one-

[4] Note that in the case of potency of sets, this turned out to be possible (see, for example, sets of measure zero).
[5] Like an ascension to the space of higher dimensionality.

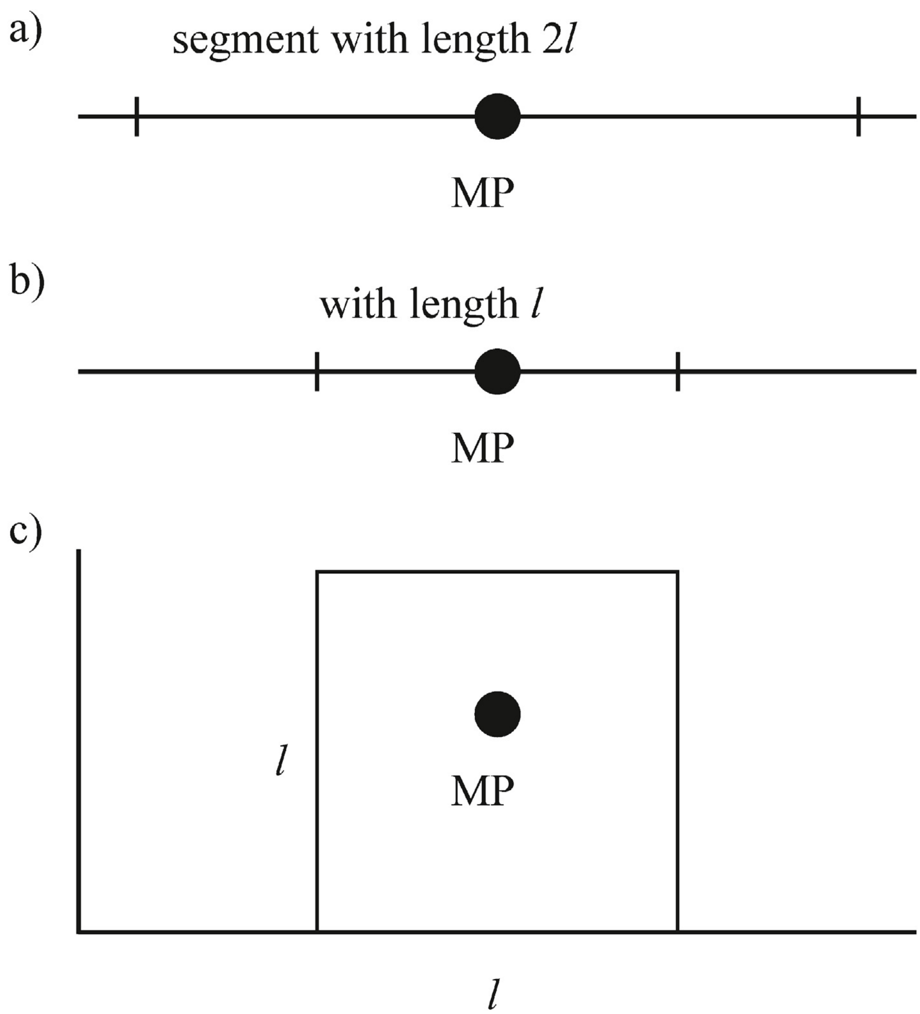


*Fig. 3. Comparative examples of MP domains in the context of freedom. l is a characteristic unit of length.*

dimensional case, we arrive thereby at the configuration in Fig. 4a. If the free movement of MP is restricted within the segment [$a$; $b$], we define the freedom the segment affords (hereinafter freedom of the domain) as follows

$$\mathcal{F}_d = æ = \tan^{-1}\|[a,b]\| \tag{1}$$

If $b \to +\infty$, then $\mathcal{F}_d \to \pi/2$. If $a \to -\infty$, we can make the coordinate transformation $x' = -x$ out of those admissible, and then have $a' = -a \to +\infty$, and $\mathcal{F}_d \to \pi/2$ takes place again. It follows that under natural additivity of freedom in the framework of one-dimensional free motion, we obtain $\mathcal{F}_d \to \pi$ at $a \to -\infty$, $b \to +\infty$. Thus, an infinite straight line endows MP with freedom of measure $\pi$. Obviously, the measure zero has to correspond to the complete loss of freedom, since it is the constriction of segment $[a,b]$ to a point.

In the two-dimensional case (Fig. 4b), the measures of freedom each of the segments $[a_{x_1}, b]$ and $[a_{x_2}, c]$ affords along axes $x_1$ and $x_2$ respectively, are equal to (I am writing the norm in the most used way in Cartesian coordinates) ${}_1\mathcal{F}_d = æ_1 = \tan^{-1}(b - a_{x_1})$ and ${}_2\mathcal{F}_d = æ_2 = \tan^{-1}(c - a_{x_2})$ respectively. The question arises, in which way one is to define a superposition operation in the equation for measure of freedom $\mathcal{F}_d = {}_1\mathcal{F}_d \times {}_2\mathcal{F}_d$ for the domain in Fig. 4b, keeping in mind that as in just considered one-dimensional case, ${}_1\mathcal{F}_d \to \pi/2$ takes place at $b \to +\infty$ and ${}_2\mathcal{F}_d \to \pi/2$ at $c \to +\infty$. Additivity in the context of dimensions does not seem to be an adequate solution, since some square of finite size would turn out to afford less freedom than a segment sufficiently longer than the side of the square, what is in contradiction with the general sense. Really, it is easy to imagine a segment

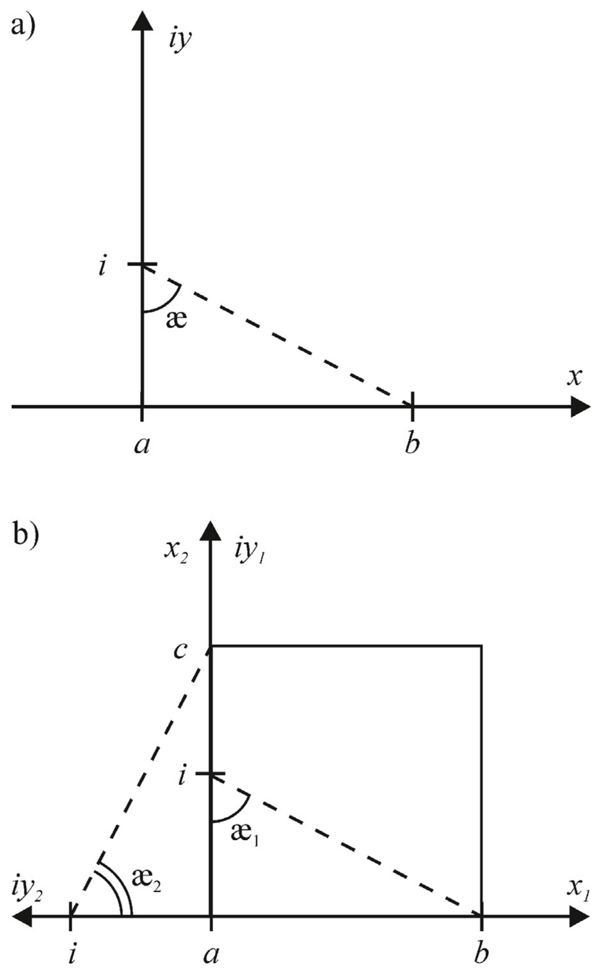


*Fig. 4. Upgrade of real dimensions to corresponding complex planes. a) one-dimensional case, b) two-dimensional case (point a has coordinates ($a_{x_1}$, $a_{x_2}$)).*

characterised by freedom $\pi/4$, with its length $(b - a_{x_1}) = \tan\frac{\pi}{4} = 1$, and it suffices to present a square with the side shorter than $\sqrt{2} - 1$ as the inequality ${}_2\mathcal{F}_d < 2\tan^{-1}(\sqrt{2} - 1) = \pi/4$ for the freedom it affords takes place. The expectation is that a two-dimensional domain of any finite size together with its boundary is to provide more freedom than any one-dimensional segment of finite length. An algebraic superposition $\mathcal{F}_d = \sqrt[\gamma]{({}_1\mathcal{F}_d)^\gamma + ({}_2\mathcal{F}_d)^\gamma}$ with $\gamma > 0$ does not do much for that either. Looking for a better definition, consider an extension of the method proposed in Fig. 4a.

For the two-dimensional case, we define the quasi-complex space by some analogy with the complex plane, namely by introducing the $iz$ axis orthogonal to the plane $xy$ of pairs of real numbers (Fig. 5).

The most compact convex figure bounding a part of the plane is a triangle. Let us raise a perpendicular parallel to axis $iz$ from vertex A to point $P_A = (a_x, a_y, i)$ in the quasi-complex space. The solid angle $æ_A$ (or $P_A$ABC) then arises. Similarly, there arise solid angles $æ_B$ and $æ_C$ (not shown in Fig. 5 to avoid cluttering) due to the perpendiculars to $(b_x, b_y, i)$ and $(c_x, c_y, i)$, and in general, the set $K_2$ of all possible angles $\{æ\}$ with vertices on raised perpendiculars from all points of the figure boundary does. Thus, the following equation seems to be a natural extension of the freedom definition from one- to two-dimensional case:

$$\mathcal{F}_d = \pi + \frac{1}{2}\inf K_2 \tag{2}$$

where $\inf K_2$ is the greatest lower bound of set $K_2$, equal to the smallest[6] of all solid angles subtended by the figure in the plane, with a vertex at the corresponding point of the quasi-complex space with imaginary coordinate $iz = i$. Evidently, $\inf K_2 \in K_2$. The freedom definition (2) meets inequality $\mathcal{F}_d > \pi$. Really, for any figure in plane $xy$, the solid angle $\inf K_2 \to 0$, if the figure square $S \to 0$. If, on the contrary, the figure fills the whole plane with $S \to \infty$, we obtain $\inf K_2 \to 2\pi$. Thus, freedom of a finite domain in the two-dimensional case necessarily meets the condition $\pi < \mathcal{F}_d < 2\pi$.

We briefly discuss the notion of freedom and define it in the three-dimensional case. Similar to those above, we introduce a quasi-complex 4-dimensional space whose axis $i\pitchfork$ is orthogonal to each of the three spatial axes $x$, $y$ and $z$. The full solid angle in the 4-dimensional space is equal to $2\pi^2$, the three-dimensional subspace appears to fully subtend its half. The following equation generalises the freedom definition (2) to the three-dimensional case:

$$\mathcal{F}_d = 2\pi + \left(1 - \frac{2}{\pi}\right)\inf K_3 \tag{3}$$

where $K_3$ is the set of all possible solid angles with vertices with the imaginary coordinate $i\pitchfork = i$, and thus positioned over the spatial boundary of the three-dimensional figure considered. Like above, $\inf K_3 \in K_3$. As in definition (2), the tendency of the three-dimensional figure volume to zero means $\mathcal{F}_d \to 2\pi$, and the expansion of this figure with its

[6] Should not be believed unique which is especially obvious in one-dimensional case.

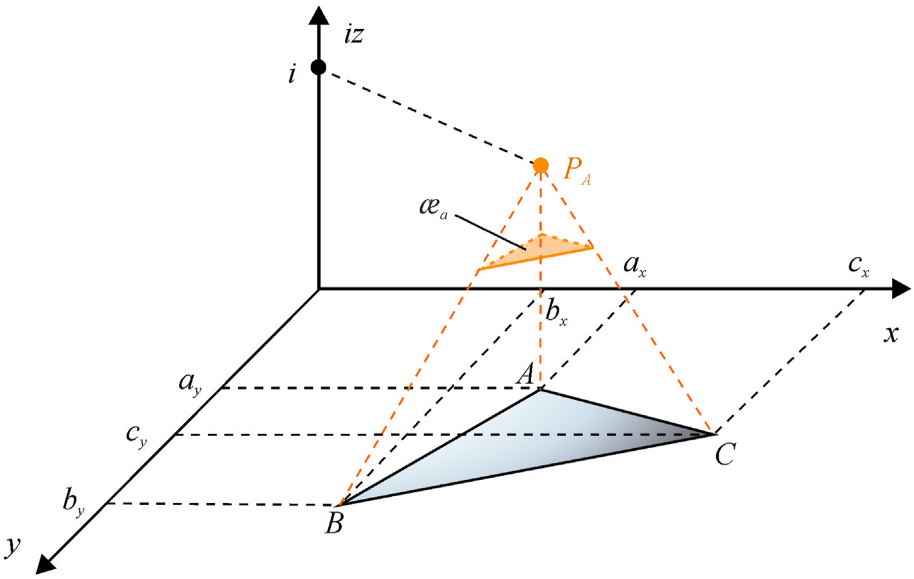


*Fig. 5. Upgrade of the plane of pairs of real numbers to a quasi-complex space.*

volume $V \to \infty$ up to filling the whole three-dimensional sub-space corresponds to $\mathcal{F}_d \to \pi^2$.

Let's make five points:

(i) By chance or not, π in definition (2) corresponds to the sum of interior angles of a triangle as the most compact convex figure bounding a part of the plane, and 2π in definition (3) corresponds to the smallest possible value of the sum[7] of all dihedral angles of a tetrahedron as the most compact convex figure bounding a part of 3D-space;

(ii) Equations (2) and (3) are written based on linearity of freedom with respect to the solid angle, in analogy with Equation (1);

(iii) Equations (1)−(3) convey the principle of favouring "here and now" over "there and then". Really, because of arctangent, points adjacent to the reference position contribute more to the angle value than those remote;

(iv) the cases with segment length $l = 0$, 2D-figure square $S = 0$ and 3D-figure volume $V = 0$ may require especial discussion. Formally, we get from Equations (1)−(3) the respective results $\mathcal{F}_d = 0, \pi$ and $2\pi$ out of which only the first one is correct. The situation is similar to that someone is to meet in the quasi-classical limit in quantum mechanics while trying to undergo a passage to the limit $\hbar \to 0$ by letting $\hbar = 0$ and thereby destroying the non-stationary Schrödinger equation [10]. The physical MP is a point only in that model when the corresponding object size is small compared to the characteristic length of the problem under consideration. For example, the earth is MP in the context of the Universe problems. While the earth size is then considered as tending to zero, to take it equal to zero would be a mistake. Therefore, for existing domains with concern with Equations (2) and (3), the limit $S, V \to 0$ simultaneously means $S, V \neq 0$ respectively;

(v) I have admitted addition of angle values to which by convention are assigned different angle dimensionalities, namely radian in two- and steradian in three-dimensional space. Entering the four-dimensional space (one of whose dimensions is imaginary), we would face the angle values unspecified in SI. Indeed, on the one part, the angle values appear to be dimensionless by their nature (beyond any conventions), contrariwise, they can be characterised as incoherent [11]. However, the coherence of plane or spatial angles in spaces with different dimensionality matters, according to Ref. [11], if it deals with the dimensional analysis what is not the subject in the context of freedom as the physical notion.

To summarise: angle æ is defined with reference to either of the two finite boundary points of a

[7] Depending on a tetrahedron, it is between $2\pi$ and $3\pi$ [9].

segment in the one-dimensional case. If there are no finite boundaries, then æ $= \pi, 2\pi$ or $\pi^2$ depending on the space dimensionality. Were angle æ measured in some other way than based on one of the boundary points, it would lead to a bigger (if the perpendicular-originating point is inside the domain under consideration) or smaller (if outside) value of æ. Since the domain freedom value should not depend on the observer's position, I chose an unambiguous anchoring of the angle vertex.

## 4. Freedom of a material point

The above convention for the reference point of a solid angle concerns the freedom $\mathcal{F}_d$ the region inside a figure with its boundary (altogether domain) affords. Thus, the datum is bound to the domain, not to MP. At the same time, it is expedient to define the notion of MP freedom $\mathcal{F}_{MP}$, since this is the object of the present study. As it follows from what was pointed out to at the end of the previous section, freedom MP is expected to meet inequality $\mathcal{F}_{MP} \geq \mathcal{F}_d$. Further consideration will be carried out in the one-dimensional case, whose results extrapolate organically to the plane and space, but their exact obtaining in the two- and three-dimensional cases would be associated with somehow cumbersome calculations.

### 4.1. MP on a segment

Let MP with coordinate $p$ be positioned within segment $[a, b]$ (Fig. 6).

Angle æ characterises freedom of the domain. Geometrically, it is clear for angles in Fig. 6 that æ $\leq \phi_1 + \phi_2 = \phi$. The equality takes place only when MP is a boundary point of the segment, with $\phi_1 = 0$ or $\phi_2 = 0$. I define freedom $\mathcal{F}_{MP}$ of MP with coordinate $p$ as the angular size of the segment as seen from point $(p + i)$ on the complex plane:

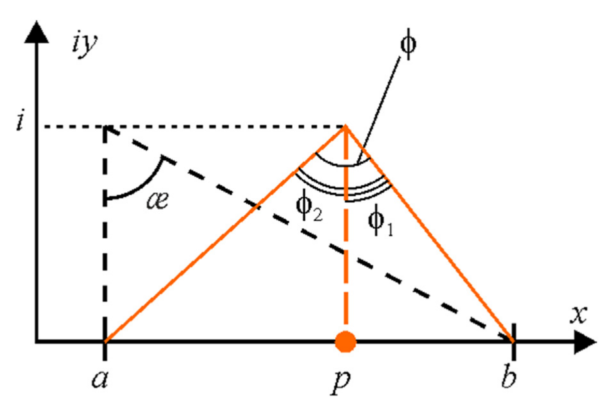


*Fig. 6. To the definition of MP freedom on a segment.*

$$\mathcal{F}_{\mathrm{MP}} = \phi \tag{4}$$

If MP is an internal point of the segment, then $\phi = \tan^{-1}(b - p) + \tan^{-1}(p - a)$, and if it is a boundary one then $\phi =$ æ. The interpretation of MP freedom not smaller than the domain freedom may have a philosophical connotation: at the beginning of free motion, the MP as the segment's interior point has the freedom to choose the direction, absent for MP as a boundary point. MP has maximum freedom when it is the midpoint of the segment, then

$$\mathcal{F}_{\mathrm{MP}}^{\max} = 2\tan^{-1}(\tfrac{1}{2}(b - a))$$

The relationship between the maximum MP freedom in the domain and the freedom of the domain is expressed by the equation:

$$\mathcal{F}_{\mathrm{MP}}^{\max} = 2\tan^{-1}(\tfrac{1}{2}\tan \mathcal{F}_d)$$

It is easy to obtain an expression showing how many times the maximum MP freedom exceeds the freedom of the domain:

$$\frac{\mathcal{F}_{\mathrm{MP}}^{\max}}{\mathcal{F}_d} = \frac{2\tan^{-1}(\tfrac{1}{2}(b - a))}{\tan^{-1}(b - a)}$$

In the case of MP in spaces, the freedom of choice of direction is even richer than on a segment, and some interior point has to exist in the spatial domain D bounded by the figure under study, in which MP has the maximum freedom:

$$\mathcal{F}_{\mathrm{MP}}^{\max} = \sup_{\mathrm{D}}\{\mathcal{F}_{\mathrm{MP}}\}$$

where the right-hand side of the equality indicates the lowest upper bound of the set of MP freedom values in D.

### 4.2. MP on a closed path of motion

Let us consider a potentially possible motion of MP along some circular orbit (Fig. 7a).

This motion is one-dimensional, like the motion considered earlier, but not rectilinear. At such motion, MP would be constantly under the action of centrifugal force and would experience centrifugal acceleration. This does not have to reduce the MP freedom, since such a state is natural for this particular geometry of one-dimensional motion. The length of a segment coiled into a circle of radius $r$ is $2\pi r$. To unfold the circle into a segment would require transformations like “translation” and “rotation” applied to each infinitesimal fraction of the trajectory. However, the action of some force has to be preserved, which in the new coordinates is

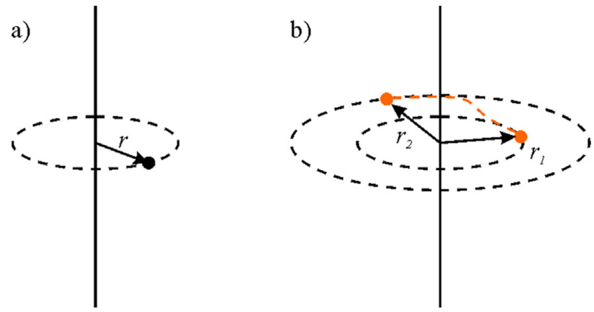


*Fig. 7. MP motion along an orbit. Trajectory: a) circle, b) change of circular orbit at continuous motion.*

difficult to identify as centrifugal. A certain equivalence between the geometry of space and the forces acting in this case is not a news in physics [12,13]. Let us discuss other differences. MP motion along a circle is one-way, since there are neither boundaries nor bounce from them. Therefore, one cannot distinguish between the freedom the circle affords and the MP freedom. The philosophical implication accompanying this circumstance is perhaps some cost for that MP cannot bounce and change the direction of motion, and also perhaps the constant experience of the centrifugal force.

One-dimensional are also the motion along a closed trajectory with variable curvature (oval) and a continuous transition between closed trajectories of different curvature (Fig. 7b). In this case, $r = r(t)$ is not only piecewise (motion along circles), but also in the sense of continuous change of the radius of curvature along the transition path. Consideration of such motion is also possible with the trajectory decomposition into sub-segments of lengths $2\pi r_1\lambda_1$ and $2\pi r_2\lambda_2$, where $\lambda_1$ и $\lambda_2$ are the number of revolutions (possibly, non-integer) on each of the circles, and also $\int_{\varphi_1}^{\varphi_2} r(\varphi)d\varphi(t)$, where $\varphi_1$ and $\varphi_2$ are the angles of the beginning and end of the transition to a circular orbit with a different radius. At the same time, changing the tangential velocity by changing the radius does not necessarily change the freedom, which can be unambiguously affected only by nonzero tangential acceleration.

### *4.3. One-dimensional piecewise rectilinear motion in the presence of external forces*

Consider an MP experiencing gravity when moving within the segment [*a*, *b*] (Fig. 8).

Under new circumstances, let us continue the discussion of the issue raised in the previous section. Does MP have more freedom on the sub-segment [*c*, *b*] than on [*a*, *c*]? The motion within [*a*, *b*] occurs in the field of gravity.

Let us first consider the idealised case assuming no frictional force (Fig. 8a). On the sub-segment [*a*, *c*], gravity accelerates MP while on the sub-segment [*c*, *b*] MP moves rectilinearly before and after bounce from the boundary, and so on. Considering the gravity field as a natural and inherent factor for such a case, I assume MP motion in Fig. 8a to be free. In doing so, I also appeal to the term “free fall” as if MP were moving vertically downwards only under the influence of gravity. The situation in Fig. 8a has a direct analogy to this as a projection of such motion onto the vertical axis exists. In Fig. 8b, the gravity force both favours and hinders the motion of the body along the inclined plane in the sub-segment [*a*, *c*] by inducing a bearing reaction force with thereby emerging frictional force $\mathbf{F}_f$. In the sub-segment [*c*, *b*], the gravity force only hinders the motion by the same mechanism until MP stops (possibly upon bouncing off the boundary as depending on the initial velocity of MP and the magnitude of the friction coefficient). At the point with coordinate *c*, MP will experience δ-overload, but it is as natural as if there were a smooth roundabout at this point (see “MP on a closed path of motion” above).

Let us assess in this context two alternative descriptive definitions of freedom: (*i*) a state without any influences brought into the ordinary environment; (*ii*) the full absence of any influences. In the second case, travelling along a circle would be associated with some limitation of freedom compared to travelling along a straight line. I believe that the first definition is closer to the usual everyday meaning. In this case, MP on earth under the action of gravity and in space without any forces is equally free (with a reservation concerning the region of motion). Thus, freedom is a relative notion whose sense is attached to some usual state.[8] Another conclusion to be drawn is that freedom is limited not by individual forces, but their superposition in a particular situation that differs from the usual state of motion.

Thus, one can speak of freedom only if there is motion or potentiality to move (for example, in a state of unstable equilibrium).[9] Possession of some energy becomes an integral property of freedom, while the

[8] Perhaps, a certain parallel with such quantum-mechanical notion as the ground state is appropriate here.
[9] See *Introduction* for MacCallum's concept with ability to do or not do something [5].

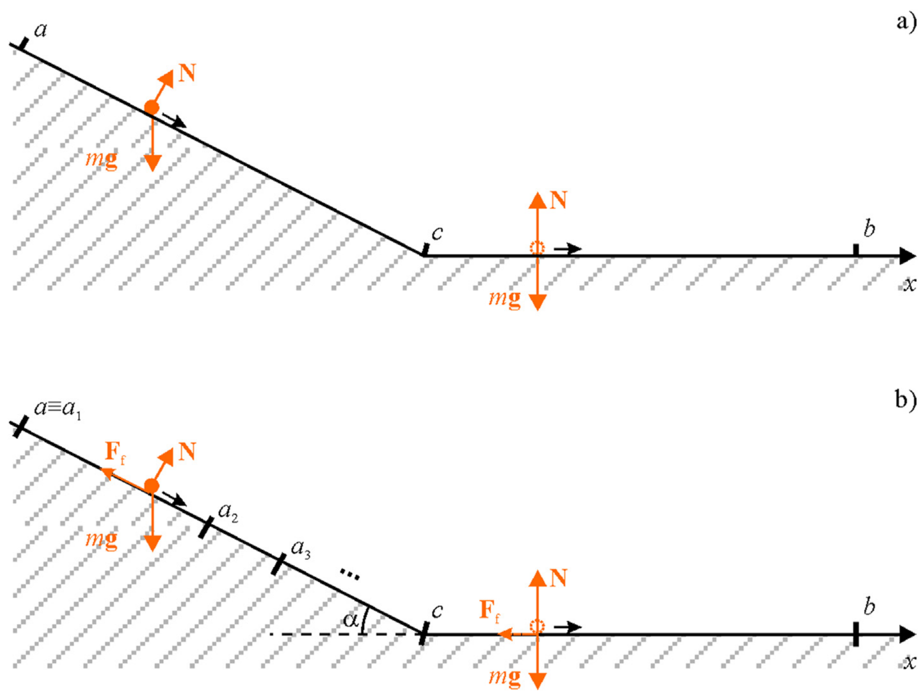


*Fig. 8. Piecewise rectilinear motion in the field of gravity. Frictional force $\mathbf{F}_f$ is: a) absent (idealisation), b) present.*

amount of this energy is less important from the point of view of freedom than the fact of its possession itself. For example, in the absence of dissipation, it is irrelevant how fast a freely moving MP would travel an interval or a certain distance in infinite motion.

While considering and formulating these circumstances, I felt that it was very important to comply with the commonly established categories.

### *4.4. Harm to freedom caused by third forces*

Following the above, I deliberately chose the present subtitle in a form not exclusively related to physical phenomena. Let some external force start to act on a moving or possessing some potential energy MP. It seems very reasonable to associate the current harm caused to the freedom of its bearer with the power $N$ of the resulting impact. According to the generally accepted definition of power, the equation

$$N(t) = \pm \frac{d(T+H)}{dt}$$

takes place, where $T$ and $H$ are current kinetic and potential energy of MP so that with sign «+» the impact contributes to MP's total mechanical energy, while with sign «−» reduces it. Let us divide the left and right parts by $(T_0 + H_0)$, where $T_0$ and $H_0$ are the corresponding values of the energy components at some previous moment, perhaps at the moment of impact inception:

$$\frac{d}{dt}\frac{T+H}{T_0+H_0} = \pm \frac{N(t)}{T_0+H_0} \tag{5}$$

With the intention of considering the quantity associated with $\frac{T+H}{T_0+H_0}$ as expressing the extent which freedom is restricted to, we keep in mind that under any influence whether it increases or decreases MP's total mechanical energy, freedom decreases. Therefore, let us integrate Equation (5), choosing in any case the sign "+" and further treating the extent of impact by its absolute value:

$$\frac{T+H}{T_0+H_0} = \frac{1}{T_0+H_0}\int_{t_0}^{t} N(\xi)d\xi + 1 \tag{6}$$

where $t - t_0 = \Delta t$ is the time interval within which the impact is considered, the last term is the integration constant. The integral in Equation (6) is the work done on MP during such an impact:

$$W = \int_{t_0}^{t} N(\xi)d\xi \geq 0$$

Equation (6) at $T+H>0$ can be easily transformed to the form

$$\frac{T_0+H_0}{T+H}=\frac{T_0+H_0}{T_0+H_0+W}$$

The following is proposed as a definition of freedom evolution due to work $W$ by an external force on MP:

$$\mathcal{F}_{\mathrm{MP}}=\mathcal{F}_{MP}^{(0)}\frac{T_0+H_0}{T_0+H_0+W}\times[T+H>0] \quad (7)$$

where $\mathcal{F}_{MP}^{(0)}$ is MP freedom at the initial moment when $T=T_0$ and $H=H_0$, Iverson's bracket represents the considerations made earlier about the necessity of possessing energy, with the reference point of potential energy taken to be zero. Defined so, $\mathcal{F}_{\mathrm{MP}}=\mathcal{F}_{MP}^{(0)}$ in the case $W=0$ and $T_0+H_0>0$. If work reduces MP's mechanical energy, then at some position, MP loses its freedom ($T+H=T_0+H_0-W=0$). Further force action means fully dependent behaviour of MP (the prehistory is erased by reaching zero of its own energy). If the work increases MP's mechanical energy, then MP behaviour is treated as controlled by an external force in an ever-increasing fraction, and its freedom tends to zero asymptotically.

## 4.5. Examples

1. The problem in Fig. 8b is a simple example where the sliding frictional force acts as an external force, which reduces MP's total mechanical energy down to zero. Let MP be initially positioned at $a$. Then, unless MP's energy is zero, the current freedom $\mathcal{F}_{\mathrm{MP}}$ is equal to

$$\mathcal{F}_{\mathrm{MP}}=\mathcal{F}_{MP}^{(0)}\frac{H_0}{W+H_0}$$

where $H_0=mg(c-a)\sin\alpha$, $\mathcal{F}_{MP}^{(0)}=\tan^{-1}(b-a)$. If the friction coefficient[10] $k>\tan\alpha$, MP will not move and $W=0$. $k=\tan\alpha$ is the boundary value, below which the MP motion on the inclined plane begins. At the boundary value, fluctuations can give an impetus to the motion[9] uniform due to the resultant force equal to zero. At $k<\tan\alpha$, MP will move along the inclined plane with acceleration and, depending on the exact relationship between the friction coefficient and inclination angle $\alpha$, will stop on the horizontal section (possibly after elastic bounce from the right boundary) or will rise to some height smaller than $(c-a)\sin\alpha$ when moving back along the inclined plane. MP velocity at the end of descent down the inclined plane can be easily found from the energy conservation condition $H_0=T+W$, travel to a complete stop from the equations of motion, then calculate the work of frictional force:

$$W(x)=mg\left[k\cos\alpha(c-a)+\begin{cases}(c-a)(\sin\alpha-k\cos\alpha), \text{if } \sin\alpha-k\cos\alpha\le 2k\dfrac{b-c}{c-a}\\ 2k(b-c)+\tan\alpha((c-a)(\sin\alpha-k\cos\alpha)-2k(b-c))+\ldots, \text{otherwise}\end{cases}\right]$$

The multipoint means that MP, having stopped at the turning point $a_2$ located below $a_1\equiv a$, will start moving downwards again, and if now the relation $\sin\alpha-k\cos\alpha\le 2k\frac{b-c}{c-a_2}$ is fulfilled, the movement will end within the horizontal section (possibly upon bouncing from the boundary $b$), otherwise MP will rise again along the inclined plane by some height to a new turning point $a_3>a_2$, and so on. We explore the behaviour of work $W$ at infinitesimal sliding friction coefficient $k$, letting thereby the number of cycles consisting of MP round trips between the variable turning point on the left and the stationary on the right, become infinitely big. In the limit $k\to 0$, it is easy to calculate the coordinate of the turning point on the left at the beginning of the $n$th cycle:

$$a_n=c-\left(\frac{\tan\alpha}{k}\right)^{n-1}(c-a_1)+O\left(\frac{1}{k^{n-2}}\right),$$

$$n=2,3,\ldots$$

When $n\to\infty$, MP will make an asymmetric periodic motion about point $b$ with the amplitude asymptotically tending to zero. Work $W$ can be represented in this case as the sum of works $W_n$: $W=\sum W_n$. The expression for the work of the frictional force in an $n$th cycle is derived easily:

$$W_n=mg(c-a_1)\tan\alpha\sin\alpha\left(\frac{\tan\alpha}{k}\right)^{n-1}+O\left(\frac{1}{k^{n-2}}\right)$$

Now let us sum the dominant terms of the expansions of $W_n$:

[10] Considered independent of possible motion velocity.

$$W \sim mg(c-a_1)\tan\alpha\sin\alpha\sum_{n=1}^{\infty}\left(\frac{\tan\alpha}{k}\right)^{n-1}$$

The infinite series on the right is a progression with the geometric ratio $\frac{\tan\alpha}{k}$. It converges only when $\tan\alpha < k$, i.e., when $\alpha\to 0$ faster than $k\to 0$, which means that there is no inclination from $a$ to $c$. In other words, at any practically significant inclination angle, the series diverges, $W\to\infty$ and, consequently, $\mathcal{F}_{\mathrm{MP}}\to 0$. Thus, in our hypothetical example with an infinitesimal friction coefficient, an infinitely large number of cycles appears to have a higher rate of divergence than the rate with which the friction coefficient tends to zero. If the friction coefficient is a nonvanishing number, the number of cycles is finite, and $\mathcal{F}_{\mathrm{MP}}$ becomes zero within finite time.

2. Let MP of mass $m$, performing one-dimensional unbounded motion with velocity $\mathbf{v}_0$, begin to experience the action of a co-directional constant force $\mathbf{F}$. MP's freedom is expressed by the equation

$$\mathcal{F}_{\mathrm{MP}}(t) = \pi\frac{T_0}{W+T_0} \tag{8}$$

Since MP accelerates infinitely, the relativistic case is a must. To keep generality, we will assume that the velocity modulus $v_0$ can also be commensurable with light velocity $c$. Then

$$T_0 = mc^2\left(\frac{1}{\sqrt{1-v_0^2/c^2}}-1\right) \tag{9}$$

Work in the relativistic case is expressed through the integral

$$W = \int vdp = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}}-\frac{1}{\sqrt{1-v_0^2/c^2}}\right) \tag{10}$$

where $p$ is momentum. The change of MP's momentum originates from the constant acceleration preserving its form in the relativistic case: $w = F/m$. The velocity of a particle at motion with acceleration in the special relativity theory is equal to

$$v(t) = \frac{u_0+wt}{\sqrt{1+(u_0+wt)^2\big/c^2}}$$

where $u_0 = v_0\big/\sqrt{1-v_0^2/c^2}$. Substituting the velocity into Equation (10), we obtain an expression for the work done on MP:

$$W = mc^2\left(\frac{1}{\sqrt{1-\frac{1}{c^2}\frac{(u_0+Ft/m)^2}{1+(u_0+Ft/m)^2\big/c^2}}}-\frac{1}{\sqrt{1-v_0^2/c^2}}\right) \tag{11}$$

Substituting Equations (9) and (11) into Equation (8), we obtain the expression for freedom:

$$\mathcal{F}_{\mathrm{MP}}(t) = \pi\frac{-1+1\Big/\sqrt{1-v_0^2/c^2}}{-1+1\Bigg/\sqrt{1-\frac{1}{c^2}\frac{(u_0+Ft/m)^2}{1+(u_0+Ft/m)^2\big/c^2}}} \tag{12}$$

As it follows from Equation (12), $\mathcal{F}_{\mathrm{MP}}(t)$ is an asymptotically decreasing function of time. We find the order of decreasing at $t\gg mu_0/F$ by expanding the expressions in Equations (11) and (12) by powers of $1/t$. For work, it's easy to obtain the representation

$$W = mc^2\left[\frac{Ft}{mc}+\frac{u_0}{c}\left(1-2\frac{u_0^2}{c^2}\right)-\frac{1}{\sqrt{1-v_0^2/c^2}}\right]+O\left(\frac{1}{t}\right)$$

Substituting the expansion into Equation (12), we obtain the ultimate expansion for freedom:

$$\mathcal{F}_{\mathrm{MP}}(t) = \pi\left(\frac{1}{\sqrt{1-v_0^2/c^2}}-1\right)\frac{mc}{Ft}\left[1-\frac{mc}{Ft}\left(\frac{u_0}{c}-2\frac{u_0^3}{c^3}-1\right)\right]+O\left(\frac{1}{t^3}\right)$$

Thus, the dominant term is decreasing $\sim t^{-1}$.

### 4.6. The freedom derivative

Let us write equation in differentials while MP mechanical energy is positive (in case the work being done on MP were aimed at reducing that), keeping in mind in general that the force may not be constant[11]:

$$\delta\mathcal{F}_{\mathrm{MP}} = -\mathcal{F}_{MP}^{(0)}\frac{T_0+H_0}{(W+T_0+H_0)^2}|\mathbf{F}d\mathbf{l}|$$

Thus, the freedom is a curvilinear integral of the second kind along the MP trajectory with the cosine between $\mathbf{F}$ and $d\mathbf{l}$ (or $\mathbf{v}$) always taken positive. Let us write the equation for the derivative:

$$\frac{\delta\mathcal{F}_{\mathrm{MP}}(t)}{dt} = -\mathcal{F}_{\mathrm{MP}}^{(0)}\frac{T_0+H_0}{(W(t)+T_0+H_0)^2}|\mathbf{F}(\mathrm{t})\mathbf{v}(\mathrm{t})| \tag{13}$$

[11] I write $\delta\mathcal{F}$ instead of $d\mathcal{F}$ because freedom at production of work on MP by an external force is not a total differential, exactly as mechanical work is not either.

that is the derivative of MP freedom is proportional to modulus of the scalar product of the force acting on MP by its velocity, which has the dimension of power, but is not power, since the velocity in this case is not an exclusive result of the force action. The proportionality coefficient contains MP's initial freedom and mechanical energy, as well as the total work done on MP by the time $t$. It follows from Equation (13), in particular, that freedom cannot increase smoothly. Any impact leads to a more or less rapid decrease of freedom. The only possibility to increase the freedom of MP is to completely eliminate the impact of external forces with preservation of its positive total mechanical energy.

### 4.7. On freedom reversibility

A reasonable question arises as to what is the cost of returning MP to its former freedom after the impact has been exhausted. Is this cost, in particular, equal to the work previously done? Let us turn to the events above in Example 2. Suppose that at some point the force **F** has ceased its action. MP continues to perform uniform and rectilinear motion, only with a different velocity. If we determine its freedom again at this moment, it will be equal to $\pi$, as it had been before force **F** emerged. Indeed, the "coercion" is over, freedom is restored. However, to return MP to its initial position in phase space, much more work would have to be done, namely stopping MP, sending it in the opposite direction, then stopping it again and accelerating it back to the initial velocity. Freedom is not related to MP's position on an infinite interval. However, in a finite space, the cancellation of the impact does not mean that freedom is restored, since it is the position relative to the boundary(s) that is significant. In addition, it follows from the above reasoning that it may require more work to restore freedom in [semi-]bounded space than was done with the initial impact. An exception, in particular, may be the case when the impact had been made on MP orthogonally approaching the boundary and, after its velocity had been changed as a result of the impact, it was bounced in the opposite direction of motion. The geometric factor is so significant that it should not be surprising to see the change of freedom of a uniformly and rectilinearly moving MP in [semi-] bounded space. In this case, $\inf\{\mathcal{F}_{MP}\} = \mathcal{F}_d$ (see the beginning of the present section). Practically, because of dissipative processes, recovery of MP's pre-existing freedom may require even more work than in their absence.

## 5. Conclusions

This paper introduces the concept of freedom as applicable to mechanics of a material point. When formulating definitions and investigating the properties of freedom as a physical concept, I traced its connection with ordinary perception of freedom as an integral part of existence of various objects of both animate and inanimate matter, as well as the implications of these ideas in physics. It seems reasonable to distinguish between such notions as freedom $\mathcal{F}_\mathrm{d}$ the domain (possibly unbounded) containing MP affords, as introduced in Equations (1)−(3), and freedom $\mathcal{F}_\mathrm{MP}$ MP itself possesses in this domain as in Equation (4). MP freedom in some spatial domain of dimension $n$ as well as the freedom this domain affords, are associated with solid angles with a vertex in some space of dimension $n+1$, constructed according to certain rules. A rigorous definition allows us to introduce the common measure of freedom independently of the dimensionality of space, and for a freely moving MP, the regions of freedom values are transparently distinguished: the highest freedom on an infinite line is equal to $\pi$, freedom in the plane is contained within the interval from $\pi$ to $2\pi$, and in three-dimensional space from $2\pi$ to $\pi^2$. An obvious extension of the formulated definitions of freedom to the case of $n$-dimensional spaces with $n>3$ is allowed. The relation $\mathcal{F}_\mathrm{d} \le \mathcal{F}_\mathrm{MP}$ is valid. The equality takes place in the two cases, when either MP is positioned in those domain boundary points where the greatest lower bound of set $\mathrm{K}_n$ of solid angles subtended by the domain boundary is reached, or the domain is unbounded.

Freedom is defined as a concept relative with respect to the MP's usual lifeline in each particular situation i.e., in the context of some problem. The degradation of MP's freedom due to external forces and the evolution of freedom generally depending on the circumstances related to the mechanics of MP are considered. The properties of freedom as a physical concept are illustrated in various situations typical for mechanics of MP. The notion is defined irrespectively of the coordinate system and withstands any coordinate transformation which does not deal with scale change. Therefore, freedom appears to be invariant, in particular, with respect to Galilean transformation.

Conformity to a number of common notions is ensured, in particular "free motion", "degrees of freedom", "free fall", "equilibrium state". On the basis of defined notion "freedom", it is easy to formally define the notion "degrees of freedom" in the context of translational motion. It is the number of dimensions corresponding to MP's freedom. Indeed, if, for example, a domain affords MP freedom $^3/_2\,\pi$, then the number of degrees of freedom is 2.

At present, I consider it premature to outline the scope of how freedom as a physical concept can be put into circulation when analysing physical phenomena or mathematical properties of objects. However, one application is obvious: freedom as physical concept can be used to compare MP states in spaces of different dimensionality. The study of objects' behaviour (e.g., solutions of differential equations in the form of trajectories [14]) under a substantial restriction of their freedom may be a promising direction for analysis. Particular considerations related to unreasonable behaviour of numerical solutions of differential equations near finite boundaries due to incorrectly set boundary conditions [15] make me think about it. Thus, perhaps operating on the freedom properties of objects would help in finding true solutions of differential equations. Upgraded to elastic bodies,[12] the concept of freedom is to be applied e.g., to deformation of beams [16−18] where it is especially interesting to trace freedom of infinitesimal elements subjected to mechanical displacements, from one bending theory to another and finally judged by measurements. It seems promising to characterise with freedom the mechanical displacements due to Rayleigh, Love and SH-waves in piezo- and flexoelectric structures [19−21]. Another potential field of application can be related to the family of Monte Carlo methods [22] in the sense that their convergence rate is determined by, among other things, the dimensionality of the space, and thus may explicitly depend on the freedom the domain affords.

The concept of freedom can be further developed in the direction of its extension from MP to a solid body [23] as a set of MPs by integration over volume, introduction of rotational (with respect to the axes passing through the solid body) and vibrational degrees of freedom, and also to discrete spaces and quantum objects. For the latter, we note that in quantum mechanics [24] the square of the wave function modulus represents the probability density of finding a particle in a point of space that gives the particle certain possibilities i.e., the corresponding freedom. In this case, the probability is normalised in space by one, and the connectivity of the region in which the particle is considered is irrelevant. The non-zero probability of finding a particle at any point in space represents a qualitatively different freedom, substantially greater than in the classical case. Thus, in a situation similar to Example 1 discussed above in connection with Fig. 8b, the turning points [25] would not completely restrict the MP motion. I believe that clarification of the notion *freedom* in these cases is possible, as well as answering possible questions related to circumstances beyond the classical MP mechanics.

The motivation for the present work was my reflections on a concept so important both in human society and in relation to other subjects of animate and inanimate nature [26], *freedom*, which seems to be still underrepresented in physics.

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