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Dynamical Behavior of Two Predators-One Prey Ecological System with Refuge and Beddington –De Angelis Functional Response

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Abstract

The dynamical behavior of an ecological system of two predators-one prey updated with incorporating prey refuge and Beddington –De Angelis functional response had been studied in this work, The essential mathematical features of the present model have been studied thoroughly. The system has local and global stability when certain conditions are met. had been proved respectively. Further, the system has no saddle node bifurcation but transcritical bifurcation and Pitchfork bifurcation are satisfied while the Hopf bifurcation does not occur. Numerical illustrations are performed to validate the model's applicability under consideration. Finally, the results are included in the form of points in agreement with the obtained numerical results.

Keywords: Ecological system, Predator-prey model, Beddington –De Angelis, Refuge, Dynamical behavior.

السلوك الديناميكي لأثنين من المفترسات و فريسة واحدة في نظام بيئي الفريسة تعتمد فيه على ملجاء بوجود دالة افتراس من النوع Beddington –De Angelis

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المستخلص

في هذا البحث تمت دراسة السوك الديناميكي لأثنين من المفترسات و فريسة واحدة وسعي الفريسة للبحث عن ملجاء للإختباء من المفترس متزامنة مع إستجابة وظيفية (دالة افتراس) من النوع Beddington –De Angelis من جانب المفترس اتجاه الفريسة , حيث تمت دراسة السمات الرياضية الأساسية للنموذج الحالي بدقة ، وتعرفنا الى ان النظام الرياضي يتمتع باستقرار محلي وغير محلي عند استيفاء شروط معينة تم إثباتها على التوالي ، وعدم احتواء النظام على تشعب عقدة سرج (saddle–node) ولكن التشعب الحرج (transcritical) موجود مع توافر شروط تسنده وتشعب Pitchfork مستوفى اما بالنسبة لتشعب هوبف (Hopf) فهو غير متحقق، تم تنفيذ الرسوم التوضيحية عدديا من أجل التحقق من قابلية تطبيق النموذج قيد الدراسة ,اخيرا لخصت نتائج العمل مع التركيزعلى ادراج النتائج العددية الحاصلين عليها بشكل نقاط في نهاية البحث .

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1. Introduction

The dynamical study of prey-predator model is one of the most important topics that is studied in both ecology and mathematical ecology. The first well-known classical model was given by Lotka-Volterra in 1927[1], the model was developed by many researchers taking into consideration a number of factors affecting the system like a refuge in [2, 3, 4, 5, 6] and the Beddington–DeAngelis functional response in [6].

Functional response is defined as the rate of consumption of one prey by predators and it plays an important role in population dynamic, there are many types of functional responses that are particularly associated with the work of Holling through his classification of functional responses into three basic types, namely I, II and III, Beddington–DeAngelis functional response is similar to the well-known Holling type II functional response but has an extra term in the denominator which models mutual interference between predators [8], It is well known that refuge and harvesting are two of the most important factors affecting the dynamics of prey-predator systems. By using refuges, the prey population is partially protected against predators. The existence of refuges has a great influence on the coexistence of the prey predator systems [3].

In this research, the system is incorporating of two systems studied in both [4]and [5], where they studied the dynamical behaviour of a two-predator model with prey refuge and the dynamical behaviour of an ecological system with Beddington–DeAngelis functional response, respectively. According to the above, the resulting system has overcrowded with parameters, which are reduced by using the dimensionless technique to simplify the work, while preserving carefully the mathematical properties which are introduced in section 2. Section 3 demonstrates the existence and positive invariance of the resulting system, while section 4 sponsors the persistence of the resulting system. Equilibrium points and their feasibility are discussed in section 5. We represent an analytical study including local and global stability of the resulting dynamical system in section 6. We also explain the bifurcation analysis for certain equilibrium points in sections 7 and 8. Numerical illustrations are performed to validate the model's applicability under consideration shown in section 9. Finally, conclusions are given in section 10.

2. Mathematical model

In this section, a Beddington–De Angelis prey-predator model considers the effect of refuge, the considered model is based on two predators and one prey system that is shown in [4]:

$$\begin{aligned}\frac{dx_1}{dt}&=\alpha x_1\left(1-\frac{x_1}{k}\right)-\frac{\beta_1 x_1 x_2}{1+a_1x_1}-\frac{\beta_2 x_1 x_3}{1+a_2x_1}\\ \frac{dx_2}{dt}&=-d_1x_2+\frac{c_1\beta_1x_1x_2}{1+a_1x_1}-\delta_1x_2x_3\\ \frac{dx_3}{dt}&=-d_1x_3+\frac{c_1\beta_2x_1x_3}{1+a_2x_1}-\delta_2x_2x_3\end{aligned} \tag{2.1}$$

The above system is updated by incorporating prey refuges proportionally to the prey density via mx_1, where $0\le m<1$.

It is considered that the first and the second predator species are compotation for food and other essential resources, respectively, such as shelter. In addition, the predator function response in the model (2.1) is known as Holling type II, which is replaced by Beddington–De Angelis which has extra terms b_1x_2 and b_2x_3 in the denominator that model mutual interference between predators.

Thus our final model is given as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_1\left(1-\frac{x_1}{k}\right)-\frac{\beta_1(1-m)x_1x_2}{a_1+(1-m)x_1+b_1x_2}-\frac{\beta_2(1-m)x_1x_3}{a_2+(1-m)x_1+b_2x_3}\\
\frac{dx_2}{dt} &= -d_1x_2+\frac{c_1\beta_1x_1x_2}{a_1+(1-m)x_1+b_1x_2}-\delta_1x_2x_3\\
\frac{dx_3}{dt} &= -d_1x_3+\frac{c_1\beta_2x_1x_3}{a_2+(1-m)x_1+b_2x_3}-\delta_2x_2x_3
\end{aligned} \tag{2.2}$$

where

i. $x_1(t)$ is the prey population size at time t.

ii. $x_2(t)$ and $x_3(t)$ are the population size of the first and the second predator species at time t, respectively. The prey grows logistically in the absence of the predator, in the same way, that the predator declines directly in the absence of the prey.

iii. The parameters α and k are the growth rate and the environmental carrying capacity of the prey species, respectively.

iv. The parameters d_1 and d_2 are the predators x_1, x_2 death rates, respectively.

v. The parameters $\delta_1$and δ_2 are the rates at which the growth rate of the first predator x_1 is annihilated by the second predator x_2 and vice versa.

vi. The parameters $c_1$and c_2 are the search rate of the first and second predator for each captured prey species, respectively ($0 < c_1$, $c_2 < 1$).

vii. The parameters β_1 and β_2 are the maximum number of prey that can be eaten by the first and second predator per unit time respectively, $\frac{1}{a_1}, \frac{1}{a_2}$ are their respective half saturation rates.

viii. The parameters b_1 andb_2 measure the coefficients of their mutual interference among the first and the second predator, respectively.

ix. m represents the prey refuge where $0 \leq m < 1$, it is considered that the first and the second predator species are competing for food and other essential resources such as shelter.

x. The terms $\frac{\beta_1(1-m)x_1x_2}{a_1+(1-m)x_1+b_1x_2}$ and $\frac{\beta_2(1-m)x_1x_3}{a_2+(1-m)x_1+b_2x_3}$ denote the first and the second predator's response respectively on prey species. This type of predator response function is known Beddington –De Angelis.

Now we will reduce the number of parameters and specify the control set of parameters, so in order to simplify the system, the following dimensionless variables and parameters are used:
$S=\frac{x_1}{k}$, $P_1=\frac{\beta_1x_2}{\alpha k}$, $P_2=\frac{\beta_2x_3}{\alpha k}$, $t=\alpha t$, $\frac{dx_1}{dt}=\alpha k\frac{dS}{dt}$, $\frac{dP_1}{dt}=\frac{\beta_1}{\alpha^2k}\frac{dx_2}{dt}$, $\frac{dP_2}{dt}=\frac{\beta_2}{\alpha^2k}\frac{dx_3}{dt}$, $A_1=\frac{a_1}{k}$, $\epsilon_1=\frac{b_1\alpha}{\beta_1}$, $A_2=\frac{a_2}{k}$, $\epsilon_2=\frac{b_2\alpha}{\beta_2}$, $\theta_1=\frac{d_1}{\alpha}$, $\lambda_1=\frac{\beta_1c_1}{\alpha}$, $\gamma_1=\frac{\delta_1k}{\beta_1}$, $\theta_2=\frac{d_2}{\alpha}$, $\lambda_2=\frac{\beta_2c_2}{\alpha}$, $\gamma_2=\frac{\delta_2k}{\beta_2}$

Then the system (2.2) reduces the following dimensionless system:

$$\begin{aligned}
\frac{dS}{dt} &= S(1-S)-\frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1P_1}-\frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2P_2}\\
\frac{dP_1}{dt} &= -\theta_1P_1+\lambda_1\frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1P_1}-\gamma_1P_1P_2\\
\frac{dP_2}{dt} &= -\theta_2P_2+\lambda_2\frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2P_2}-\gamma_2P_1P_2
\end{aligned} \tag{2.3}$$

Where $S(0) \geq 0, P_1(0) \geq 0$ and $P_2(0) \geq 0$ are evident that the number of parameters is reduced from fifteen in the system (2.2) to eleven in the system (2.3).

3. Existence and positive invariance

For $t > 0$, let $Y = (S, P_1, P_2)^T, F = (f_1, f_2, f_3)^T$, then the system (2.3) becomes $\frac{dX}{dt} = F(X)$, here $f_i \in C^\infty$ for i = 1, 2, 3, where:

$$\begin{aligned} f_1 &= S(1-S) - \frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1 P_1} - \frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2 P_2} \\ f_2 &= -\theta_1 P_1 + \lambda_1 \frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1 P_1} - \gamma_1 P_1 P_2 \\ f_3 &= -\theta_2 P_2 + \lambda_2 \frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2 P_2} - \gamma_2 P_1 P_2 \end{aligned} \tag{3.1}$$

Clearly, the interaction functions in the system (2.3) are continuous and have continuous partial derivatives on the positive three dimensional space $\mathbb{R}^3_+ = \{(S, P_1, P_2): S(0) \geq 0, P_1(0) \geq 0, P_2(0) \geq 0\}$. Therefore, these functions are Lipschitzian [9] over $\mathbb{R}^3_+$ and the system (2.3) has a unique solution, see [2],[3],[4]

Theorem1. The solutions of the system (2.3) are uniformly bounded over $X = \{(S, P_1, P_2) \in \mathbb{R}^3_+; w(t) \leq \frac{S^2}{\mu}\}$.

Proof: From the first equation of the system (2.3) ,we observe that :
$\frac{dS}{dt} \leq S(1-S)$,then by solving the above differential inequality, we get that $S(t) \leq e^{\frac{t}{2}} - 1$ thus as $t \to \infty$, we get $S(t) \leq 1$. Now assume that $W(t) = S(t) + \frac{P_1(t)}{\lambda_1} + \frac{P_2(t)}{\lambda_2}$, where W is the total population, we get that $\frac{dW}{dt} = \frac{dS}{dt} + \frac{1}{\lambda_1}\frac{dP_1}{dt} + \frac{1}{\lambda_2}\frac{dP_2}{dt}$ which gives $\frac{dW}{dt} \leq S(1-S) - \frac{\theta_1}{\lambda_1}P_1 - \frac{\theta_2}{\lambda_2}P_2$,by simplifying the last differential inequality and substituting W, we conclude

$$\frac{dW}{dt} \leq S(2-S) - \mu W \tag{3.2}$$

where $\mu = \min\{1, \theta_1, \theta_2\}$ yeilds $\frac{dw}{dt} + \mu w \leq S(2-S)$, finally by solving the differential inequality (3.2) we obtain that $w(t) \leq \max\{w(t_0), \frac{S^2}{\mu}\}$ and $\lim_{t\to\infty} \sup w(t) \leq \frac{S^2}{\mu}$, hence all solutions of the system (2.3) are bounded over $X = \{(S, P_1, P_2) \in \mathbb{R}^3_+; S(0) > 0, P_1(0) > 0, P_2(0) > 0\}$.

4. Persistent

The work of this section is based on the method of Average Lyapunove function ,

Theorem 2. System (2.3) is persistence provided that

$$\theta_1 + \theta_2 < 1 \tag{4.1a}$$

$$\lambda_2 \frac{(1-m)}{A_2+(1-m)} + \lambda_1 \frac{(1-m)}{A_1+(1-m)} > \theta_1 + \theta_2 \tag{4.1b}$$

$$1 + \frac{(\lambda_1 S - P_1)(1-m)}{A_1+(1-m)S+\epsilon_1 P_1} + \lambda_2 \frac{(1-m)S}{A_2+(1-m)S} > S + \theta_1 + \theta_2 \tag{4.1c}$$

$$1 + \frac{(\lambda_2 S - P_2)(1-m)}{A_2+(1-m)S+\epsilon_2 P_2} + \lambda_1 \frac{(1-m)S}{A_1+(1-m)S} > S + \theta_1 + \theta_2 \tag{4.1d}$$

Proof: Considering a function of the form $U(S, P_1, P_2) = S^{\kappa_1} P_1^{\kappa_2} P_2^{\kappa_3}$, where $\kappa_1, \kappa_2, \kappa_3$ are positive constants, obviously $U(S, P_1, P_2) > 0$ for all $(S, P_1, P_2) \in int\mathbb{R}^3$ and $U(S, P_1, P_2) \to 0$ as $S, P_1 or\ P_2 \to 0$, now define the function $Z(S, P_1, P_2)$ such that $Z(S, P_1, P_2) = \frac{U'}{U}$ and

$\frac{U'}{U} = \kappa_1\left((1-S) - \frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} - \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2}\right) + \kappa_2\left(-\theta_1 + \lambda_1\frac{(1-m)S}{A_1+(1-m)S+\epsilon_1 P_1}\right) + \kappa_3\left(-\theta_2 + \lambda_2\frac{(1-m)S}{A_2+(1-m)S+\epsilon_2 P_2}\right)$.

Now we prove that $Z(S,P_1,P_2) > 0$ for all the boundary equilibrium points and for suitable choices of $\kappa_1 > 0$, $\kappa_2 > 0$ and $\kappa_3 > 0$, we get $\frac{U'}{U}(E_0) = \kappa(1-\theta_1-\theta_2) > 0$ if $\theta_1+\theta_2 < 1$ leads to the condition (4.1a) holds for suitable choice of κ where $\kappa = \kappa_i$, $i = 1,2,3$ this gives $\frac{U'}{U}(E_1) = \kappa\left(-\theta_1 + \lambda_1\frac{(1-m)}{A_1+(1-m)} - \theta_2 + \lambda_2\frac{(1-m)}{A_2+(1-m)}\right) > 0$ if $\lambda_2\frac{(1-m)}{A_2+(1-m)} + \lambda_1\frac{(1-m)}{A_1+(1-m)} > \theta_1+\theta_2$ that is the condition (4.1b) holds for a suitable choice of κ, also $\frac{U'}{U}(E_2) = \kappa\left((1-S) + \frac{(\lambda_1-1)(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} - \theta_1 - \theta_2 + \lambda_2\frac{(1-m)S}{A_2+(1-m)S}\right) > 0$ if $1 + \frac{(\lambda_1 S-P_1)(1-m)}{A_1+(1-m)S+\epsilon_1 P_1} + \lambda_2\frac{(1-m)S}{A_2+(1-m)S} > S+\theta_1+\theta_2$, this means the condition (4.1c) holds for suitable choice of κ. Finally, $\frac{U'}{U}(E_3) = \kappa\left((1-S) + \lambda_1\frac{(1-m)S}{A_1+(1-m)S} + \frac{(\lambda_2-1)(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2} - \theta_1 - \theta_2\right) > 0$ as $1 + \frac{(\lambda_2 S-P_2)(1-m)}{A_2+(1-m)S+\epsilon_2 P_2} + \lambda_1\frac{(1-m)S}{A_1+(1-m)S} > S+\theta_1+\theta_2$ then the condition (4.1d) holds for suitable choice of κ. Hence, the proof is completed.

5. Equilibrium Points and their feasibility

The system (2.3) has five equilibrium points they are as the following:
The points $E_0 = (0,0,0)$, $E_1 = (1,0,0)$ are always feasible.
The first planer equilibrium point is $E_2 = (S_2, 0, P_{22})$, where S_2 is a unique positive root, see [3] for the quadratic equation

$$S_2^2 + \left(\frac{(1-m)}{\epsilon_2} - \frac{(1-m)\theta_2}{\epsilon_2\lambda_2} - 1\right)S_2 - \frac{A_2\theta_2}{\epsilon_2\lambda_2} = 0 \quad (5.1)$$

while $$P_{22} = \frac{\lambda_2}{\theta_2}S_2(1-S_2) \quad (5.2)$$

The equilibrium point E_2 exists uniquely in the interior of the positive quadrant of S_2P_{22} _ plan provided that the following sufficient condition holds

$$\left(1 + \frac{(1-m)\theta_2}{\epsilon_2\lambda_2} - \frac{(1-m)}{\epsilon_2}\right) + \sqrt{\left(\frac{(1-m)}{\epsilon_2} - \frac{(1-m)\theta_2}{\epsilon_2\lambda_2} - 1\right)^2 + \frac{4A_2\theta_2}{\epsilon_2\lambda_2}} > 0$$

The second planer equilibrium point is $E_3 = (S_3, P_{13}, 0)$ where S_3 is a unique positive root, see [3] for the quadratic equation

$$S_3^2 + \left(\frac{(1-m)}{\epsilon_1} - \frac{(1-m)\theta_1}{\epsilon_1\lambda_1} - 1\right)S_3 - \frac{A_1\theta_1}{\epsilon_1\lambda_1} = 0 \quad (5.3)$$

while $$P_{13} = \frac{\lambda_1}{\theta_1}S_3(1-S_3) \quad (5.4)$$

The equilibrium point E_3 exists uniquely in the interior of the positive quadrant of S_3P_{13} _ plan provided that the following sufficient condition holds

$$\left(1 + \frac{(1-m)\theta_1}{\epsilon_1\lambda_1} - \frac{(1-m)}{\epsilon_1}\right) + \sqrt{\left(\frac{(1-m)}{\epsilon_1} - \frac{(1-m)\theta_1}{\epsilon_1\lambda_1} - 1\right)^2 + \frac{4A_1\theta_1}{\epsilon_1\lambda_1}} > 0.$$

The last equilibrium point $E_4 = E^* = (S^*, P_1^*, P_2^*)$ exists if the component S^* is a positive root of the equation

$Z_1(S^*)^{12} + Z_2(S^*)^{11} + Z_3(S^*)^{10} + Z_4(S^*)^9 + Z_5(S^*)^8 + Z_6(S^*)^7 + Z_7(S^*)^6 + Z_8(S^*)^5 + Z_9(S^*)^4 + Z_{10}(S^*)^3 + Z_{11}(S^*)^2 + Z_{12}S^* + Z_{13}=0$ (5.5)

While $P_1^* = \frac{-\theta_2}{\gamma_2} + \frac{\lambda_2}{\gamma_2}\frac{(1-m)S^*}{A_2+(1-m)S^*+\epsilon_2 P_2^*}$ (5.6)

$P_2^* = \frac{-\theta_1}{\gamma_1} + \frac{\lambda_1}{\gamma_1}\frac{(1-m)S^*}{A_1+(1-m)S^*+\epsilon_1 P_1^*}$ (5.7)

Where

$\mathbf{Z_1} = H_1^2 H_2^2 > 0$

$\mathbf{Z_2} = 2H_1H_4(-H_1H_5 + H_2H_4) > 0,$

$\mathbf{Z_3} = (-H_1H_5 + H_2H_4)^2 + 2H_1H_4(2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4) > 0$

$\mathbf{Z_4} = 2H_1H_4(2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5) + 2(-H_1H_5 + H_2H_4)(2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4) > 0$

$\mathbf{Z_5} = (-H_1H_5 + H_2H_4)(2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5) + (2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4)^2 + 2H_1H_4(2H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18}) - \epsilon_2^2 H_9^2 > 0$

$\mathbf{Z_6} = (2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4)(2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5) + 2H_1H_4(2H_{12}V_4 + H_2 + H_3H_7 - H_{13}H_{15} - H_{14}H_{16} - H_{19}) - H_9^2V_2 - 2H_9H_{10}V_1 + 2(-H_1H_5 + H_2H_4)(H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18}) > 0$

$\mathbf{Z_7} = (2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5)^2 - 2H_1H_4(H_{14}H_{15} + H_{10} + V_5) + 2(-H_1H_5 + H_2H_4)(2H_{12}V_4 + H_2 + H_3H_7 - H_{13}H_{15} - H_{14}H_{16} - H_{19}) + (2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4)(2H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18}) - 2H_9H_{10}V_2 - H_9^2 - H_{10}V_1 + 2H_9V_1V_6 > 0$

$\mathbf{Z_8} = (-H_1H_5 + H_2H_4)(H_{14}H_{15} + H_{10} + V_5) + (2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4)(H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18}) + 2(2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5)(H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18}) - H_9^2V_4 - H_9H_{10}V_3 - H_{10}V_1 + 2H_9V_1V_6 - 2H_{10}V_1V_6 < 0$

$\mathbf{Z_9} = -2(2H_{12}V_1 + H_1H_6 - H_2H_5 + H_3H_4)(2H_{12}V_4 + H_2 + H_3H_7 - H_{13}H_{15} - H_{14}H_{16} - H_{19}) + 2(2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5)(2H_{12}V_4 + H_2 + H_3H_7 - H_{13}H_{15} - H_{14}H_{16} - H_{19}) + (2H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18}) - H_9^2V_5 + 2H_9H_{10}V_4 - H_{10}V_3 + 2H_9V_1V_3 - H_{10}V_2V_6 - V_7V_1 < 0$

$\mathbf{Z_{10}} = -2(2H_{12}V_2 + H_1H_7 + H_2H_6 - H_3H_5 + H_{13}H_5)(H_{14}H_{15} + H_{10} + V_5) + 2(2H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18})(2H_{12}V_4 + H_2 + H_3H_7 - H_{13}H_{15} - H_{14}H_{16} - H_{19}) + 2H_9H_{10}V_3 - H_{10}V_4 + 2H_9V_4V_6 - 2H_{10}V_3V_6 - V_2V_7 < 0$

$\mathbf{Z_{11}} = -2(2H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18})(H_{14}H_{15} + H_{10} + V_5) + (2H_{12}V_4 + H_2 + H_3H_7 - H_{13}H_{15} - H_{14}H_{16} - H_{19})^2 - H_{10}V_3 - V_3V_7 + 2H_9V_5V_6 - 2H_{10}V_4V_6 > 0$

$\mathbf{Z_{12}} = -2(2H_{12}V_3 + H_1H_8 + H_2H_7 + H_3H_6 - H_{13}H_{16} + H_{14}H_{15} + H_{18})(H_{14}H_{15} + H_{10} + V_5) - H_{10}V_5V_6 - V_4V_7 < 0$

$\mathbf{Z_{13}} = -V_2V_7 + (H_{14}H_{15} + H_{10} + V_5)^2 > 0$

According to Descarte's rule, see [3] of sign equation (5.3) has three positive real roots in $\mathbb{R}_+^3$ under the following conditions

$Z_1 > 0, Z_2 > 0, Z_3 > 0, Z_4 > 0, Z_5 > 0, Z_6 > 0, Z_7 > 0, Z_8 < 0, Z_9 < 0, Z_{10} < 0, Z_{11} > 0, Z_{12} < 0, Z_{13} > 0$, where

$H_1 = \gamma_2(1-m)^2$, $H_2 = \in_2 \gamma_2(1-m)$, $H_3 = \in_2 \gamma_2 A_1 + \in_1 \in_2 \theta_2$, $H_4 = \frac{1}{4L_2^2}$, $H_5 = \frac{1}{2\in_2 L_2^2}(1-m)(1-L_2)$, $H_6 = \frac{1}{4\in_2^2 L_2^2}(1-m)^2(L_2-1)^2 - 2\in_2^2 - \in_2 L_2 A_2$, $H_7 = \frac{1}{\in_2^2 L_2^2}((L_2-1)(1-m)(\in_2 + L_2 A_2))$, $H_8 = \frac{1}{4\in_2^2 L_2^2}(\in_2 + L_2 A_2)$, $H_9 = \frac{1}{2L_2^2}$, $H_{10} = \frac{1}{2\in_2 L_2^2} + (L_2-1)(1-m)$, $H_{11} = \frac{L_2 A_2}{2\in_2^2 L_2^2}$, $H_{12} = \frac{1}{4\in_2^2 L_2^2}$, $H_{13} = (A_2\gamma_2 + \in_1 \theta_2 + \lambda_2 \in_1 + \theta_2 \in_2)(1-m) - \frac{\in_1 \in_2 \theta_1 \theta_2}{\gamma_2}$, $H_{14} = \gamma_2 A_1 A_2 + \in_1 \theta_2 A_2 + \theta_1 A_1 \in_2$, $H_{15} = \frac{1}{2L_2}$, $H_{16} = \frac{(1-m)(L_2-1)}{2\in_2 L_2}$, $H_{17} = \frac{A_2}{4\in_2^2}$, $H_{18} = (1-m)\theta_1$, $H_{19} = (\theta_1 A_1 + \theta_1 A_2 + \frac{\in_1 \theta_1 \theta_2}{\gamma_2} + \frac{\in_1 \theta_1 \lambda_2}{\gamma_2})(1-m)$, $H_{20} = \theta_1\theta_2 A_2 - \frac{\in_1 \theta_1 \theta_2}{\gamma_2}$, $L_1 = 1 + \frac{\lambda_2\gamma_1}{\lambda_1\gamma_2}$, $L_2 = \frac{2\theta_2}{\lambda_2} + \frac{\gamma_1\theta_2}{\gamma_2\lambda_1}$, $V_1 = \in_2^2$, $V_2 = 4\in_2 L_2(1-m) - 2\in_2(1-m)(L_2-1)$, $V_3 = (1-m)^2(L_2-1)^2 - 2\in_2^2 - 2\in_2^2 L_2 A_2 - 4\in_2 L_2 A_2 + 4L_2 \in_2(1-m) + 4\in_2$, $V_4 = 2(\in_2 + L_2 A_2)(1-m)(L_2-1) + 4\in_2 L_2 A_2 + \in_2 L_1 L_2(1-m)$

$V_5 = (\in_2 + L_2 A_2)^2$, $V_6 = H_{14}H_{17} + H_{11}$, $V_7 = (H_{14}H_{17} + H_{11})^2$

6. Local Stability of Equilibrium points

In this section, we analyze local stability for each equilibrium point of the system (2.3)
The Jacobian matrix of the system (2.3) at any point (S,P_1,P_2) is defined as $J=D_f(X)=[c_{ij}]_{3\times3}$ which is given as follows:

$$\begin{bmatrix} 1-2S-\left[\frac{(A_1+\epsilon_1P_1)(1-m)P_1}{(A_1+(1-m)S+\epsilon_1P_1)^2}\right]-\left[\frac{(A_2+\epsilon_2P_2)(1-m)P_2}{(A_2+(1-m)S+\epsilon_2P_2)^2}\right] & -\left[\frac{A_1(1-m)S+(1-m)^2S^2}{(A_1+(1-m)S+\epsilon_1P_1)^2}\right] & -\left[\frac{A_2(1-m)S+(1-m)^2S^2}{(A_2+(1-m)S+\epsilon_2P_2)^2}\right] \\ \lambda_1\left[\frac{(A_1+\epsilon_1P_1)(1-m)P_1}{(A_1+(1-m)S+\epsilon_1P_1)^2}\right] & -\theta_1-\gamma_1P_2+\lambda_1\left[\frac{A_1(1-m)S+(1-m)^2S^2}{(A_1+(1-m)S+\epsilon_1P_1)^2}\right] & -\gamma_1P_1 \\ \lambda_2\left[\frac{(A_2+\epsilon_2P_2)(1-m)P_2}{(A_2+(1-m)S+\epsilon_2P_2)^2}\right] & -\gamma_2P_2 & -\theta_2-\gamma_2P_1+\lambda_2\left[\frac{A_2(1-m)S+(1-m)^2S^2}{(A_2+(1-m)S+\epsilon_2P_2)^2}\right] \end{bmatrix} \quad (6.1)$$

Local stability of $\mathbf{E_0}$*:* the eigenvalues of the Jacobian matrix J_0 are $1, -\theta_1$ and $-\theta_2$. Therefore, E_0 *is unstable* actually it is a saddle point.

$$J_0 = D_f(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\theta_1 & 0 \\ 0 & 0 & -\theta_2 \end{bmatrix} \quad (6.2)$$

Local stability of $\mathbf{E_1}$*:* the eigenvalues of the Jacobian matrix J_1 are -1, $-\theta_1 + \lambda_1 \frac{A_1(1-m)+(1-m)^2}{(A_1+(1-m)^2)}$ and $-\theta_2 + \lambda_2 \frac{A_2(1-m)+(1-m)^2}{(A_2+(1-m)^2)}$. Therefore, E_1 is locally asymptotically stable for

$$\lambda_1 \frac{A_1(1-m)+(1-m)^2}{(A_1+(1-m)^2)} < \theta_1 \quad (6.3)$$

and

$$\lambda_2 \frac{A_2(1-m)+(1-m)^2}{(A_2+(1-m)^2)} < \theta_2. \quad (6.4)$$

Otherwise, it is a saddle point.

$$J_1 = D_f(E_1) = \begin{bmatrix} -1 & -[\frac{A_1(1-m)+(1-m)^2}{(A_1+(1-m))^2}] & -[\frac{A_2(1-m)+(1-m)^2}{(A_2+(1-m))^2}] \\ 0 & -\theta_1 + \lambda_1 \frac{A_1(1-m)+(1-m)^2}{(A_1+(1-m))^2} & 0 \\ 0 & 0 & -\theta_2 + \lambda_2 \frac{A_2(1-m)+(1-m)^2}{(A_2+(1-m))^2} \end{bmatrix} \quad (6.5)$$

Local stability of $\mathbf{E_2}$: the characteristic equation of the Jacobian matrix $J_2 = D_f(E_2)=(a_{ij})_{3\times3}$ is

$\lambda^3+\Omega_1\lambda^2+\Omega_2\lambda+\Omega_3=0$, where $\Omega_1=-[a_{11}+a_{22}+a_{33}]$, $\Omega_2=a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}+a_{22}a_{33}$ and $\Omega_3=-a_{33}(a_{11}a_{22}-a_{21}a_{12})$, hence by Routh-Hurwitz criterion [10] E_2 is locally asymptotically stable if $\Omega_1>0, \Omega_3>0$ *and* $\Delta>0$ where $\Delta=\Omega_1\Omega_2-\Omega_3=-a_{11}^2(a_{22}+a_{33})-a_{22}^2(a_{11}+a_{33})-a_{33}^2(a_{11}+a_{22})-2a_{11}a_{22}a_{33}+a_{21}a_{12}(a_{11}+a_{22})$, so that E_2 is locally asymptotically stable point if
$a_{11}<0,\quad a_{22}<0,\quad a_{33}<0$ that is :

$$1-2s_2-\left(\frac{(A_1+\epsilon_1P_{22})(1-m)P_{22}}{(A_1+(1-m)S_2+\epsilon_1P_{22})^2}\right)<0 \tag{6.6}$$

$$-\theta_2+\lambda_2\left(\frac{A_2(1-m)S_2+(1-m)^2S_2^2}{(A_2+(1-m)S_2+\epsilon_1P_{22})^2}\right)<0 \tag{6.7}$$

$$-\theta_1-\gamma_1P_{22}+\lambda_1\left(\frac{A_1(1-m)S_2+(1-m)^2S_2^2}{(A_1+(1-m)S_2)^2}\right)<0 \tag{6.8}$$

$$\boldsymbol{J_2=D_f(E_2)=}$$
$$\begin{bmatrix} \mathbf{1-2S_2-(\frac{(A_1+\epsilon_1P_{22})(1-m)P_{22}}{(A_1+(1-m)S_3+\epsilon_2P_{22})^2})} & \mathbf{-(\frac{A_1(1-m)S_2+(1-m)^2S_2^2}{(A_1+(1-m)S_2)^2})} & \mathbf{\frac{A_2(1-m)S_2+(1-m)^2S_3^2}{(A_2+(1-m)S_2+\epsilon_2P_{22})^2}} \\ \mathbf{0} & \mathbf{-\theta_1-\gamma_1P_{22}+\lambda_1(\frac{A_1(1-m)S_2+(1-m)^2S_2^2}{(A_1+(1-m)S_2)^2})} & \mathbf{0} \\ \mathbf{\lambda_2(\frac{(A_2+\epsilon_2P_{22})(1-m)P_{22}}{(A_2+(1-m)S_2+\epsilon_2P_{22})^2})} & \mathbf{-\gamma_1P_{22}} & \mathbf{-\theta_2+\lambda_2(\frac{A_2(1-m)S_2+(1-m)^2S_2^2}{(A_2+(1-m)S_2+\epsilon_2P_{22})^2})} \end{bmatrix}$$
(6.8)

Local stability of $\mathbf{E_3}$: The characteristic equation of the Jacobian matrix $J_3=D_f(E_3)=(b_{ij})_{3\times3}$ is $\lambda^3+\Psi_1\lambda^2+\Psi_2\lambda+\Psi_3=0$, where $\Psi_1=-[b_{11}+b_{22}+b_{33}]$, $\Psi_2=b_{11}b_{22}-b_{31}b_{13}+b_{11}b_{33}+b_{22}b_{33}$ and $\Psi_3=-b_{22}(b_{11}b_{33}-b_{31}b_{13})$,so by Routh-Hurwitz criterion E_3 is locally asymptotically stable point if $\Psi_1>0, \Psi_3>0$ *and* $\Delta>0$ where $\Delta=\Psi_1\Psi_2-\Psi_3=-b_{11}^2(b_{22}+b_{33})-b_{22}^2(b_{11}+b_{33})-b_{33}^2(b_{11}+b_{22})-2b_{11}b_{22}b_{33}+b_{31}b_{13}(b_{11}+b_{33})$, thus E_2 is locally asymptotically stable if $b_{11}<0$,
$b_{22}<0, b_{33}<0$, that is :

$$1-2S_3-\left(\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right)<0 \tag{6.10}$$

$$-\theta_2-\gamma_2P_{13}+\lambda_2\left(\frac{A_2(1-m)S_3+(1-m)^2S_3^2}{(A_2+(1-m)S_3)^2}\right)<0 \tag{6.11}$$

$$-\theta_1+\lambda_1\left(\frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right)<0 \tag{6.12}$$

$$J_3=D_f(E_3)=$$
$$\begin{bmatrix} 1-2S_3-(\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}) & -[\frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}] & \frac{A_2(1-m)S_3+(1-m)^2S_3^2}{(A_2+(1-m)S_3)^2} \\ \lambda_1\left(\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right) & -\theta_1+\lambda_1(\frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}) & -\gamma_1P_{13} \\ 0 & 0 & -\theta_2-\gamma_2P_{13}+\lambda_2(\frac{A_2(1-m)S_3+(1-m)^2S_3^2}{(A_2+(1-m)S_3)^2}) \end{bmatrix}$$

(6.13)

Local stability of $\boldsymbol{E^*}$**.** Let $J^*=D_f(E^*)=J$ as shown in (6.1) (After substituting S *with* S^*, P_1 *with* P_1^* *and* P_2 *with* P_2^*)

Theorem 3: The system (2.3) is locally asymptotically stable around the equilibrium point $E^{*}=(S^{*}, P_1^{*}, P_2^{*}) = (S, P_1, P_2)$ if the following conditions are satisfied :

$$2S + \left[\frac{(A_1+\epsilon_1 P_1)(1-m)P_1}{(A_1+(1-m)S+\epsilon_1 P_1)^2}\right] + \left[\frac{(A_2+\epsilon_2 P_2)(1-m)P_2}{(A_2+(1-m)S+\epsilon_2 P_2)^2}\right] > 1 \quad (6.14)$$

$$\lambda_1\left[\frac{A_1(1-m)S+(1-m)^2S^2}{(A_1+(1-m)S+\epsilon_1 P_1)^2}\right] < \theta_1 + \gamma_1 P_2 \quad (6.15)$$

$$\lambda_2\left[\frac{A_2(1-m)S+(1-m)^2S^2}{(A_2+(1-m)S+\epsilon_2 P_2)^2}\right] < \theta_2 + \gamma_2 P_1 \quad (6.16)$$

Proof: Let us define the characteristic equation of the Jacobian matrix $J^{*} = D_f(E^{*}) = (c_{ij})_{3\times 3} = Df\ (X)$ as $\Lambda^3 + \Theta_1\Lambda^2 + \Theta_2\Lambda + \Theta_3 = 0$, where $\Theta_1 = -[c_{11} + c_{22} + c_{33}]$, $\Theta_2 = c_{11}c_{22} - c_{21}c_{12} - c_{31}c_{13} + c_{11}c_{33} + c_{22}c_{33} - c_{32}c_{23}$ and $\Theta_3 = -c_{33}(c_{11}c_{22} - c_{21}c_{12}) - c_{12}c_{23}c_{31} - c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31} + c_{11}c_{23}c_{32}$, so by Routh-Hurwitz criterion E^{*} is locally asymptotically stable if $\Theta_1 > 0, \Theta_3 > 0\ and\ \Delta > 0$,where $\Delta = \Theta_1\Theta_2 - \Theta_3 = -(c_{11} + c_{22} + c_{33})[c_{11}c_{22} - c_{21}c_{12} + c_{11}c_{33} + c_{22}c_{33} - c_{31}c_{13} - c_{32}c_{23}] + c_{33}(c_{11}c_{22} - c_{21}c_{12}) + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32}$.
So E^{*} is locally asymptotically stable if $c_{11} < 0, c_{22} < 0, c_{33} < 0$, that is: (6.14) ,(6.15) and (6.16) holds . Therefore, the proof is complete.

Global Stability

In this subsection, the global stability is studied for each locally stable equilibrium point using a suitable Lyapunov function that is given in the following theorems:

Theorem 4 Assume that the equilibrium point E_1 =(1,0,0) is locally asymptotically stable in $\mathbb{R}^3$. Then it is globally asymptotically stable if the following conditions are satisfied:

$$(1+\lambda_1)\frac{(1-m)(1-s)P_1}{A_1+(1-m)S+\epsilon_1 P_1} \le 0 \quad (6.17)$$

$$(1+\lambda_2)\frac{(1-m)(1-S)P_2}{A_2+(1-m)S+\epsilon_2 P_2} \le 0 \quad (6.18)$$

Proof. Using an appropriate Lyapunov consider $W_1 = (S - 1 - lnS) + P_1 + P_2$ (6.19).
Clearly, $w_1(S, P_1, P_2) > 0$ is a continuously differentiable real-valued function for all $(S, P_1, P_2) \in \mathbb{R}^3$ with $(S, P_1, P_2) \neq (1,0,0)$ and $W_1(1,0,0) = 0$, we get

$$\frac{dW_1}{dt} = \left(\frac{S-1}{S}\right)\frac{dS}{dt} + \frac{dP_1}{dt} + \frac{dP_2}{dt}.$$

$$\frac{dW_1}{dt} = -(1-s)^2 + (1+\lambda_1)\frac{(1-m)(1-s)P_1}{A_1+(1-m)S+\epsilon_1 P_1} + (1+\lambda_2)\frac{(1-m)(1-S)P_2}{A_2+(1-m)S+\epsilon_2 P_2} - \theta_1 P_1 - \theta_2 P_2 - (\gamma_1+\gamma_2)P_1P_2 .$$

In order to get $\frac{dW_1}{dt} < 0$ the following inequalities must be satisfied

$(1+\lambda_1)\frac{(1-m)(1-s)P_1}{A_1+(1-m)S+\epsilon_1 P_1} \le 0$ and $(1+\lambda_2)\frac{(1-m)(1-S)P_2}{A_2+(1-m)S+\epsilon_2 P_2} \le 0$, for this (6.17) and (6.18) holds.
Hence E_1 is globally asymptotically stable.

Theorem 5: Assume that the equilibrium point $E_2=(S_2, 0, P_{22})$ is locally asymptotically stable in $\mathbb{R}^3$. Then it is globally asymptotically stable if the following conditions are satisfied:

$$\theta_1 P_1 + \gamma_1 P_1 P_2 > 0 \quad (6.20)$$

$$(S-1) > \frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} + \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2} \quad (6.21)$$

$$\theta_2 + \gamma_2 P_1 > \lambda_2 \frac{(1-m)S}{A_2+(1-m)S+\epsilon_2 P_2} \tag{6.22}$$

Proof. Applying suitable Lyapunov function at $E_2 = (S_2, 0, P_{22})$ we get:

$$W_2 = \frac{(S-S_2)^2}{2} + P_1 + \frac{(P_2-P_{22})^2}{2} \tag{6.23}$$

Clearly, $w_2(S, P_1, P_2) > 0$ is a continuously differentiable real-valued function for all $(S, P_1, P_2) \in \mathbb{R}^3$with $(S, P_1, P_2) \neq (S_2, 0, P_{22})$ and $W(S_2, 0, P_{22})_2 = 0$, moreover we have that $\frac{dW_2}{dt} = (S - S_2)\frac{dS}{dt} + \frac{dP_1}{dt} + (P_2 - P_{22})\frac{dP_2}{dt}$ we get by Substituting $\frac{dS}{dt}$, $\frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ we get $\frac{dW_2}{dt} = (S - S_2)\left[S(1-S) - \frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1 P_1} - \frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2 P_2}\right] - \theta_1 P_1 + \lambda_1 \frac{(1-m)SP_1}{A_1+(1-m)S+\epsilon_1 P_1} - \gamma_1 P_1 P_2 + (P_2 - P_{22})\left[-\theta_2 P_2 + \lambda_2 \frac{(1-m)SP_2}{A_2+(1-m)S+\epsilon_2 P_2} - \gamma_2 P_1 P_2\right]$

Now straightforward computations give

$$\frac{dW_2}{dt} \leq -\tau_1 (S - S_2)^2 - \tau_2 (P_2 - P_{22}) - (\theta_1 P_1 + \gamma_1 P_1 P_2)$$

Where $\tau_1 = (S - 1) - \frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} - \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2}$

$$\tau_2 = \theta_2 + \gamma_2 P_1 - \lambda_2 \frac{(1-m)S}{A_2+(1-m)S+\epsilon_2 P_2}$$

So according to conditions (6.20), (6.21) and (6.22) we guarantee $\frac{dW_2}{dt} < 0$

Hence E_2, is globally asymptotically stable**.**

Theorem 6*:* Assume that the equilibrium point $E_3 = (S_3, P_{13}, 0)$ is locally asymptotically stable in $\mathbb{R}^3$. Then it is globally asymptotically stable if the following conditions are satisfied

$$(1-m)S\left(\frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} + \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2}\right) > 0 \tag{6.24}$$

$$S_3 + \theta_1 P_1 + \theta_2 P_2 > 0 \tag{6.25}$$

$$P_1 P_2 (\gamma_1 + \gamma_2) > 0 \tag{6.26}$$

Proof. Applying a suitable Lyapunov function on E_3 we get :

$$W_3 = \left(S - S_3 - S_3 ln\frac{S}{S_3}\right) + \left(P_1 - P_{13} - P_{13} ln\frac{P_1}{P_{13}}\right) + P_2 \tag{6.27}$$

Clearly, $W_3(S, P_1, P_2) > 0$ is a continuously differentiable real-valued function for all $(S, P_1, P_2) \in \mathbb{R}^3$with $(S, P_1, P_2) \neq (S_3, P_{13}, 0)$ and $W_3(S_3, P_{13}, 0)_2 = 0$, moreover we have that

$\frac{dW_3}{dt} = \frac{(S-S_3)}{S}\frac{dS}{dt} + \frac{(P_1-P_{13})}{P_1}\frac{dP_1}{dt} + \frac{dP_2}{dt}$.

$\frac{dW_3}{dt} = S(1-S) - (1-m)K_1 SP_1 - (1-m)K_2 SP_2 - S_3(1-S) + (1-m)K_1 P_1 S_3 + (1-m)K_2 P_2 S_3 - \theta_1 P_1 + (1-m)\lambda_1 SP_1 - \gamma_1 P_1 P_2 + \theta_1 P_{13} - (1-m)\lambda_1 K_1 SP_{13} + \gamma_1 P_{13} P_2 - \theta_2 P_2 + (1-m)\gamma_2 K_2 SP_2 - \gamma_1 P_1 P_2$.

where $K_1 = \frac{1}{A_1+(1-m)S+\epsilon_1 P_1}$, $K_2 = \frac{1}{A_2+(1-m)S+\epsilon_2 P_2}$

$\frac{dW_3}{dt} \leq -(S - S_3)^2 - (S_3 - P_1)^2 - (S_3 - P_2)^2 - (S - P_1)^2 - (S - P_2)^2 - (P_{13} - P_2)^2 - (1-m)K_1 SP_1 - (1-m)K_2 SP_2 - S_3 - \theta_1 P_1 - \theta_2 P_2 - \gamma_1 P_1 P_2 - \gamma_1 P_1 P_2$.

So according to conditions (6.26),(6.25) and (6.24) the condition $\frac{dW_3}{dt} < 0$ is guaranteed.

Hence, E_3 is globally asymptotically stable.

Theorem 7. Assume that the equilibrium $E^* = (S^*, P_1^*, P_2^*)$ point is locally asymptotically stable in $\mathbb{R}^3$.

Then it is globally asymptotically stable if the following conditions are satisfied:

$$S + \frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} + \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2} > 0 , \tag{6.28}$$

$$\theta_1 + \gamma_1 P_2 - \lambda_1 \frac{(1-m)S}{A_1+(1-m)S+\epsilon_1 P_1} > 0 \quad , \tag{6.29}$$

$$\theta_2 + \gamma_2 P_1 - \lambda_2 \frac{(1-m)S}{A_2+(1-m)S+\epsilon_2 P_2} > 0 \, , \tag{6.30}$$

Proof. Consider the following Lyapunov function $W^* = \frac{(S-S^*)}{2} + \frac{(P_1-P_1^*)}{2} + \frac{(P_2-P_2^*)^2}{2}$ *(6.31)*
where $W^*$is a function of (S^*,P_1^*,P_2^*) and $W^* > 0$, Now by differentiating W^* with respect to time t , this gives that:

$$\frac{dW^*}{dt} = (S - S^*)\frac{dS}{dt} + (P_1 - P_1^*)\frac{dP_1}{dt} + (P_2 - P_2^*)\frac{dP_2}{dt}$$

$$\frac{dW^*}{dt} = (S^2 - SS^*)\left[-S - \frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} - \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2}\right] - (P_1^2 - P_1P_1^*)\left[\theta_1 + \gamma_1 P_2 \lambda_1 - \frac{(1-m)S}{A_1+(1-m)S+\epsilon_1 P_1}\right] - (P_2^2 - P_2P_2^*)\left[\theta_2 + \gamma_2 P_1 - \lambda_2 \frac{(1-m)S}{A_2+(1-m)S+\epsilon_2 P_2}\right].$$

After using the method of completing square and taking common factors of resulting algebraic terms and simplifying them, we get

$$\frac{dW^*}{dt} \leq -(S - S^*)^2\left[S + \frac{(1-m)P_1}{A_1+(1-m)S+\epsilon_1 P_1} + \frac{(1-m)P_2}{A_2+(1-m)S+\epsilon_2 P_2}\right] - (P_1 - P_1^*)^2\left[\theta_1 + \gamma_1 P_2 \lambda_1 - \frac{(1-m)S}{A_1+(1-m)S+\epsilon_1 P_1}\right] - (P_2 - P_2^*)^2\left[\theta_2 + \gamma_2 P_1 - \lambda_2 \frac{(1-m)S}{A_2+(1-m)S+\epsilon_2 P_2}\right]$$

So according to conditions (6.28), (6.29) and (6.30), the condition $\frac{dW^*}{dt} < 0$ is guaranteed. Therefore E^* is globally asymptotically stable.

7. Bifurcation Analyses

The occurrence of local bifurcation is well known that non-hyperbolic equilibrium point property is a necessary but not sufficient condition for the occurrence of bifurcation around that point. In the following theorems, the candidate bifurcation parameter is selected so that the equilibrium point under study will be a non-hyperbolic point, we study in this section the local bifurcation for the equilibrium points E_1 ,E_2 and E_3 by applying the Sotomayor's theorem [11] ,while E^* is selected to analyze the Hopf -bifurcation occurrence around certain parameter λ_1.

Theorem 8*:* The system (2.3) has no transcretical bifurcations and neither pitchfork bifurcation nor saddle node bifurcation can occur near the equilibrium point E_1 passes through the parameter $\theta_2^* = \lambda_2 \frac{A_2(1-m)+(1-m)^2}{(A_2+(1-m))^2}$.

Proof. It is easy to verify that the Jacobain matrix of system (2.3) at (E_1, θ_2^*) can be written as

$$J_1^{\theta_2^*} = \begin{bmatrix} -1 & -R_1 & -R_2 \\ 0 & -\theta_1 + \lambda_1 R_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; where \; R_1 = \frac{A_1(1-m)+(1-m)^2}{\left(A_1+(1-m)\right)^2} \, ,$$

$$R_2 = \frac{A_2(1-m)+(1-m)^2}{(A_2+(1-m))^2}$$

Clearly, the third eigenvalue ς_{3P_2} in the P_2 direction is zero while the first eigenvalue $\varsigma_1 = -1 < 0,$ the second eigenvalue $\varsigma_2 = -\theta_1 + \lambda_1 R_1 < 0$ if conditions (6.3) is satisfied, further the eigenvector $v = (v_1, v_2, v_3)^T$ corresponding to ς_{3P_2} satisfies the following $J_1^{\theta_2^*} v = \varsigma\, v$ then $J_1^{\theta_2^*} v = 0$ we get

$$-v_1 - R_1 v_2 - R_2 v_3 = 0 \tag{7.1}$$

$$(-\theta_1 + \lambda_1 R_1)v_2 = 0 \tag{7.2}$$

so by solving the above system of equations, we get $v_2 = 0$, $v_1 = -R_2 v_3$ where v_3 is a nonzero value number thus :

$v=\begin{bmatrix}-R_2v_3\\0\\v_3\end{bmatrix}$, similarly we take the eigenvector $\omega=(\omega_1,\omega_2,\omega_3)^T$ corresponding to the eigenvalue ς_{3P_2} of $[J_1^{\theta_2^*}]^T$ can be written as

$$\begin{bmatrix}-1&0&0\\-R_1&-\theta_1+\lambda_1R_1&0\\-R_2&0&0\end{bmatrix}\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\end{bmatrix}=0,\text{ this gives }\quad \omega=(0,0,\omega_3)^T \qquad (7.3)$$

Here ω_3 is any nonzero real number.

Now rewrite the system in vector form as $\frac{dx}{dt}=f(X)$ where $X=(S,P_1,P_2)^T$, $f=(f_1,f_2,f_3)^T$

And $\frac{\partial f}{\partial\theta_2^*}=f_{\theta_2^*}$, we get that $f_{\theta_2^*}=[0,0,-P_2]^T$ obviously $f_{\theta_2^*}(E_1,\theta_2^*)=[0,0,0]^T$. Therefore,

$$\omega^T f_{\theta_2^*}(E_1,\theta_2^*)=0 . \qquad (7.4)$$

Consequently, according to the Sotomayor theorem, the system has no saddle-node bifurcation near *E₁* through θ_2^*, now in order to investigate the occurrence of the other types of bifurcation, the derivative of $f_{\theta_2^*}$ with respect to vector *X* say $Df_{\theta_2^*}(E_1,\theta_2^*)$ is computed

$$Df_{\theta_2^*}(E_1,\theta_2^*)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}\text{ And }\quad \omega^T Df_{\theta_2^*}(E_1,\theta_2^*)v'=0$$

Again, according to Sotomayor theorem if in addition to the above, the following holds

$$\omega^T[D^2f_{\theta_2^*}(E_1,\theta_2^*)(v',v')]=0 \qquad (7.5)$$

And
$$\omega^T[D^3f_{\theta_2^*}(E_1,\theta_2^*)(v',v',v')]=0 \qquad (7.6)$$

Then the system (2.3) has neither transcritical bifurcation nor pitchfork bifurcation at E_1.

Theorem 9: The system (2.3) has transcretical bifurcations and pitchfork bifurcation can occur near the equilibrium point E_3 passes through the parameter $\theta_2^*=\lambda_2\frac{A_2(1-m)S_3+(1-m)^2S_3^2}{A_2+(1-m)S_3+\epsilon_2P_{13}}$. However, the saddle node bifurcation cannot occur.

Proof. The Jacobian matrix of the system (2.3) at (E_3,θ_2^*) can be written as:

$$J_3\theta_2^*=\begin{bmatrix}1-2S_3-\left(\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right) & \frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_2+\epsilon_2P_{13})^2} & \left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)\\ \lambda_1\left[\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right] & -\theta_1+\lambda_1\left[\frac{A_1(1-m)S_3+(1-m)S_3^2}{(A_1+(1-m)S_2+\epsilon_2P_{13})^2}\right] & -\gamma_1P_{13}\\ 0&0&0\end{bmatrix}$$

In $J_3\theta_2^*$ clearly the third eigenvalue in $\tau_{3\,P_2}$ in the P_2 direction is zero $\tau_{3\,P_2}=0$, while both of the first and the second eigenvalues $\tau_1=1-2S_3-\left(\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right)<0$, $\tau_2=-\theta_1+\lambda_1\left[\frac{A_1(1-m)S_3+(1-m)S_3^2}{(A_1+(1-m)S_2+\epsilon_2P_{13})^2}\right]<0$ when conditions (6.11) and (6.10) satisfied, respectively. Further, the eigenvector $H=(\eta_1,\eta_2,\eta_3)^T$ corresponding to $\tau_{3\,P_2}$ satisfies the following: $J_3\theta_2^*H=\tau H$, we get $J_3\theta_2^*H=0$ thus :

$$\left(1-2S_3-\left(\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right)\right)\eta_1-\frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_2+\epsilon_2P_{13})^2}\eta_2-\left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)\eta_3=0 \qquad (7.7)$$

$$\left(\lambda_1\left[\frac{(A_1+\epsilon_1P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1P_{13})^2}\right]\right)\eta_1+\left(-\theta_1+\lambda_1\left[\frac{A_1(1-m)S_3+(1-m)S_3^2}{(A_1+(1-m)S_2+\epsilon_2P_{13})^2}\right]\right)\eta_2-(\gamma_1P_{13})\eta_3=0 \qquad (7.8)$$

We have that: $H=(Z_1\eta_2,\eta_2,Z_2\eta_2)^T$ where

$$Z_1 = \frac{\gamma_1 P_{22}\left(\frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_2+\epsilon_2 P_{13})^2}\right)+\left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)\left(-\theta_1+\lambda_1\left[\frac{A_1(1-m)S_3+(1-m)S_3^2}{(A_1+(1-m)S_2+\epsilon_2 P_{13})^2}\right]\right)}{\lambda_1\left[\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right]\left(\frac{(A_1+\epsilon_1 P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1 P_{13})^2}\right)-\gamma_1 P_{22}\left[1-2S_3-\left(\frac{(A_1+\epsilon_1 P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1 P_{13})^2}\right)\right]}$$

(7.9)

$$Z_2 = \left\{\frac{\left[1-2S_3-\left(\frac{(A_1+\epsilon_1 P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1 P_{13})^2}\right)\right]}{\left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)}\right\} Z_1 \quad (7.10)$$

Similarly $\mathcal{E} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$ the eigenvector corresponding to $\tau_{3\,P_2}$ (the third eigenvalue) of $(J_3\theta_2^*)^T$ can be written as $(J_3\theta_2^*)^T\,\Upsilon = 0$, so we have

$$\left(1-2S_3-\left(\frac{(A_1+\epsilon_1 P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1 P_{13})^2}\right)\right)\varepsilon_1 + \left(\lambda_1\left[\frac{(A_1+\epsilon_1 P_{13})(1-m)P_{13}}{(A_1+(1-m)S_3+\epsilon_1 P_{13})^2}\right]\right)\varepsilon_2 = 0\ , \quad (7.11)$$

$$-\left(\frac{A_1(1-m)S_3+(1-m)^2S_3^2}{(A_1+(1-m)S_2+\epsilon_2 P_{13})^2}\right)\varepsilon_1 + \left(-\theta_1+\lambda_1\left[\frac{A_1(1-m)S_3+(1-m)S_3^2}{(A_1+(1-m)S_2+\epsilon_2 P_{13})^2}\right]\right)\varepsilon_2 = 0\ ,$$

(7.12)

$$-\left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)\varepsilon_1 - \gamma_1 P_{13}\varepsilon_2 = 0\ .$$

(7.13)

Solving the above equations we get $\mathcal{E} = (0,0,\varepsilon_3)^T$, here ε_3 is any nonzero real number. Now rewrite the system (2.3) in term of vector form as $\frac{dX}{dt} = g(x)$ where $X=(S, P1, P_2)^T$ and $G = (g_1, g_2, g_3)^T$ and $\frac{\partial g}{\partial \theta_2^*} = g_{\theta_2^*}$ subsequently $g_{\theta_2^*} = [0,0,-P_2]^T$ then $g_{\theta_2^*}(E_3, \theta_2^*) = (0,0,0)^T$ and

$$\mathcal{E}^T g_{\theta_2^*}(E_3, \theta_2^*) = 0 \quad (7.14)$$

From the Sotomayor theorem, E_3 has no saddle node bifurcation through θ_2^*.

Now in order to investigate the occurrence of the other types of bifurcation the derivative of $g_{\theta_2^*}$ with respect to the vector X say $Dg_{\theta_2^*}(E_3, \theta_2^*)$ is computed $Dg_{\theta_2^*}(E_3, \theta_2^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and

$$\mathcal{E}^T Dg_{\theta_2^*}(E_3, \theta_2^*) = -\theta_2 + \lambda_2 \frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\eta_3\varepsilon_3 \neq 0\ , \quad (7.15)$$

$$\mathcal{E}^T D^2 g_{\theta_2^*}(E_3, \theta_2^*)(H, H) = \varepsilon_3\lambda_2 \frac{\partial\left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)}{\partial S_1}\eta_1\eta_3 + \lambda_2 \frac{\partial\left(\frac{A_2(1-m)S_3+(1-m)S_3^2}{(A_2+(1-m)S_3)^2}\right)}{\partial P_{13}}\eta_3\eta_2 \neq 0\ ,$$

(7.16)

Subsequently, E_3 has transcritical bifurcation through θ_2^*.

Finally, as the same way we compute $\mathcal{E}^T D^3 g_{\theta_2^*}(E_3, \theta_2^*)(H, H, H) \neq 0$.

(7.17)

Thus from (7.15),(7.16) and (7.17), we get that E_3 has pitchfork bifurcation through θ_2^*.

By the same way, we could prove that $E_2=(S_2,0,P_{22})$ has transcritical bifurcation through θ_1^* and pitchfork bifurcation through θ_1^* such that $\theta_1^* = \lambda_1 \frac{A_1(1-m)S_2+(1-m)^2S_2^2}{A_1+(1-m)S_2+\epsilon_1 P_{22}}$.

8. Hopf-bifurcation.

Theorem 10. The equilibrium point E^* of the system(2.3) has no Hopf-bifurcation around the parameter λ_1.

Proof. According to the local stability analysis of system (2.3) at E^*, we have that the coefficients of the characteristic equation $\Theta_i; i = 1,2,3$ are positive provided that

$$\Lambda^3 + \Theta_1\Lambda^2 + \Theta_2\Lambda + \Theta_3 = 0 \quad (8.1)$$

However, $\Delta = \Theta_1\Theta_2 - \Theta_3$ is positive provided that $c_{22} < 0$ in J^*

That is $-\theta_1 - \gamma_1 P_2 + \lambda_1 \left[\frac{A_1(1-m)S+(1-m)^2S^2}{(A_1+(1-m)S+\epsilon_1 P_1)^2}\right] < 0$ and hence there is no Hopf- bifurcation in this case.

Now suppose that $\Delta = \Theta_1\Theta_2 - \Theta_3 = 0$ then according to [12] there is possibility to occurrence of Hopf-bifurcation if and only if the Jacobian matrix of system (2.3) near E^* has two complex conjugate eigenvalues, say $\kappa_i = \rho_1 \pm i\rho_2$ with the third eigenvalue is real and negative, in addition, the following two conditions are held in specific parameter say $l = l^*$ and

$$\rho_1(l^*) = 0\,, \tag{8.2}$$

$$\frac{d\rho_1}{dl}\Big|_{l=l^*} \neq 0\,, \tag{8.3}$$

Now from $\Delta = \Theta_1\Theta_2 - \Theta_3 = 0$ we obtain that

$$Mc_{22}^2 + Bc_{22} + C = 0 \tag{8.4}$$

Where

$M = -(c_{11} + c_{33}) > 0$,

$B = (-(c_{11} + c_{33})^2 + c_{21}c_{12} + c_{32}c_{23})$,

$C = (c_{11} + c_{33})(c_{13}c_{31} + c_{11}c_{33}(c_{11} + c_{33}) + c_{11}c_{12}c_{21} + c_{33}c_{32}c_{23} + c_{13}c_{21}c_{32} + c_{12}c_{23}c_{31})$

Clearly, for $C < 0$ we have two real roots of equation (8.4) say

$c_{22} = \frac{-B}{2M} \pm \frac{\sqrt{B^2-4MC}}{2M}$, since $c_{22} < 0$ then we get $c_{22} = \frac{-B}{2M} - \frac{\sqrt{B^2-4MC}}{2M}$ and hence

$$-\theta_1 - \gamma_1 P_2 + \lambda_1 \left[\frac{A_1(1-m)S+(1-m)^2S^2}{(A_1+(1-m)S+\epsilon_1 P_1)^2}\right] + \frac{B}{2M} + \frac{\sqrt{B^2-4MC}}{2M} = 0\,, \tag{8.5}$$

Which gives $f(\lambda_1^*) = 0$ *and* $\lambda_1 = \lambda_1^*$ represent a root of equation (8.5) consequently for $\lambda_1 = \lambda_1^*$ we get $\Theta_1\Theta_2 = \Theta_3$ from which the characteristic equation can be written as

$$\rho(\Lambda) = (\Lambda + \Theta_1)(\Lambda^2 + \Theta_2) = 0\,, \tag{8.6}$$

Hence, in such case $\lambda_1 = \lambda_1^*$ the eigenvalues $\Lambda_1 = -\Theta_1 < 0$ *and* $\Lambda_{2,3} = \pm i\sqrt{\Theta_2}$ so the first condition of Hopf-bifurcation is satisfied at $\lambda_1 = \lambda_1^*$ that is $\rho_1(\lambda_1^*) = 0$ while $\rho_2 = \sqrt{\Theta_2}$, that is $\Lambda_{2,3} = \rho_1(\lambda_1) \pm i\rho_2(\lambda_1)$, substituting $\Lambda = \rho_1 + i\rho_2$ in equation (8.6) we get after some algebraic computations

$$\text{Ŋ}\rho_1' - \phi\rho_2' = -\Theta\,, \tag{8.7}$$

where $\frac{d\rho_3(\Lambda)}{d\lambda_1} = \rho_3'(\Lambda)$

$$\phi\rho_1' - \text{Ŋ}\rho_2' = -\Gamma \tag{8.8}$$

Such that

$$\begin{aligned} \text{Ŋ} &= 3\rho_1^2 + 2\Theta_1\rho_1 + \Theta_2 - 3\rho_2^2 \\ \phi &= 6\rho_1\rho_2 + 2\Theta_1\rho_2 \\ \Theta &= \rho_1^2\Theta_1' + \Theta_2'\rho_1 + \Theta_3' - \Theta_1'\rho_2^2 \\ \Gamma &= 2\rho_1\rho_2\Theta_1' + \Theta_2'\rho_2 \end{aligned} \tag{8.9}$$

Solving the linear system (8.7) and (8.8) for the unknowns ρ_1', ρ_2' it is obtained that $\rho_1' = \frac{\text{Ŋ}\Theta+\Gamma\phi}{\text{Ŋ}^2+\phi^2}$, $\rho_2' = \frac{-\Gamma\text{Ŋ}+\Theta\phi}{\text{Ŋ}^2+\phi^2}$. Hence, the second condition of Hopf-bifurcation will be reduced to verify that

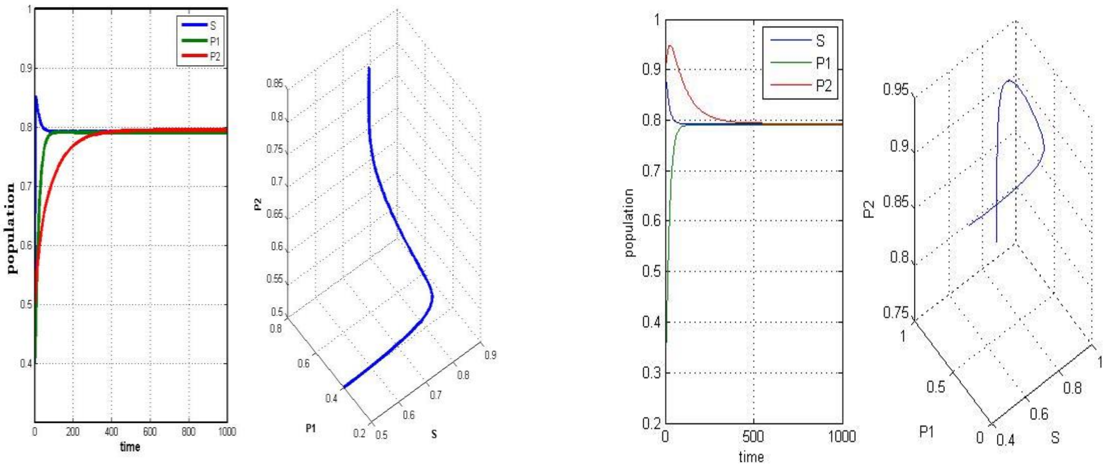
$$\text{Ŋ}\Theta + \Gamma\,\phi \neq 0\,, \tag{8.10}$$

But $\Theta_1' = -1$, $\Theta_2' = c_{11} + c_{33}$ *and* $\Theta_3' = -\Theta_2 + \Theta_1(c_{11} + c_{33})$ thus $\text{Ŋ} = -2\Theta_2$, $\phi = 2\Theta_1\sqrt{\Theta_2}$, $\Theta = \Theta_1(c_{11} + c_{33})$, $\Gamma = (c_{11} + c_{33})\sqrt{\Theta_2}$ substituting in (8.8), we get

$\text{Ŋ}\Theta + \Gamma\,\phi = 0$. Hence the system (2.3) does not undergo a Hopf-bifurcation through E^*.

9. Numerical Analysis.

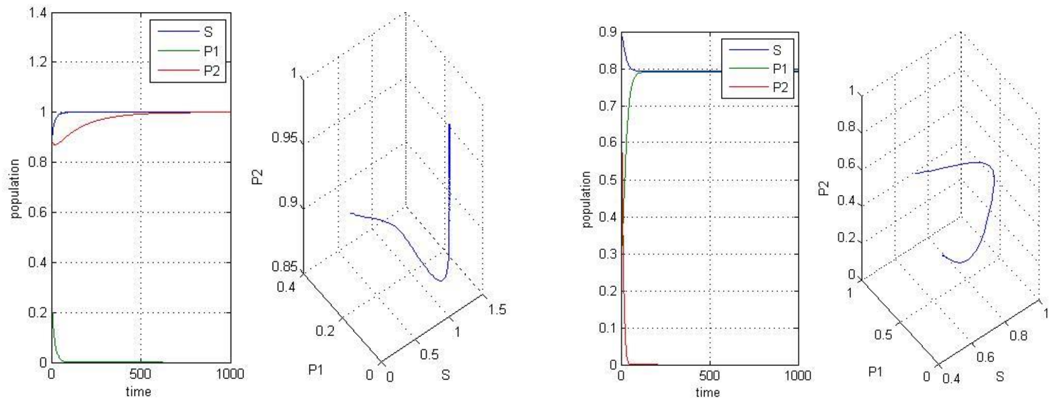
In this section, we studied the global dynamics of the system (2.3) numerically to verify the obtained analytical results and specify the control set of parameters. For the following hypothetical set of parameters system (2.3) is solved numerically and the obtained trajectories are drawn in the form of phase portrait and time series. First, we examine varying the value of each parameter on the dynamical behavior of the system (2.3). Second, we assure our obtained analytical results. It is spotted that, the following set of parameters that satisfies stability conditions of the positive equilibrium point E^* of system 3. System 3 has a globally asymptotically stable positive equilibrium point as shown in Figure 1.a and Figure 1.b , for: A_1 = 0.5, A_2 = 0.1 , $\epsilon_1 = 0.9$, $\epsilon_2 = 0.9$, $\gamma_1 = 0.001$, $\gamma_2 = 0.01$, m = 0.6 , $\theta_1 = 0.1 \theta_2 = 0.01, \lambda_1 = 0.486, \lambda_2 = 0.064$ (9.1)



(a) **(b)**

Figure 1: Time series of the solution of system (2.3) that start from two different initial points (0.5,0.4,0.5) , (0.4 ,0.3, 0.9) shown in (a) and (b), respectively, for the data that are given by (9.1), population size= (S ,P_1 ,P_2) the vertical line at a specific continuous time is the horizontal line.

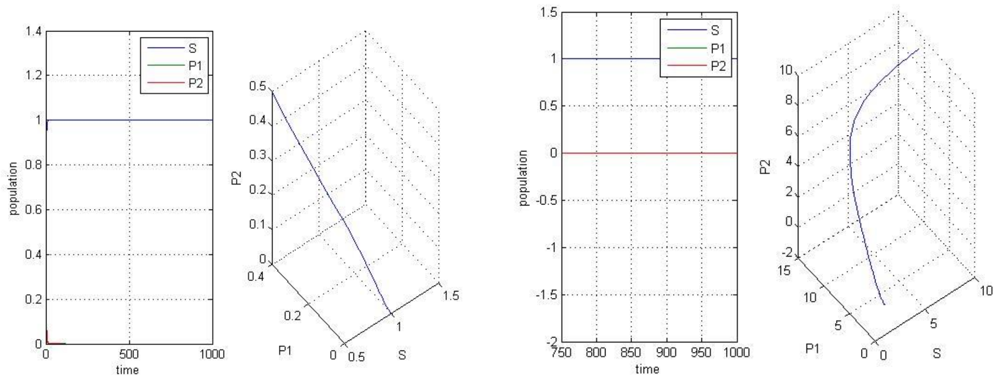
Clearly, Figure 1(a) and Figure 1(b) show that system (2.3) has a globally asymptotically stable solution approaches asymptotically positive equilibrium points E^* =(0.792,0.791,0.79), respectively according to data that are given in (9.1) and for two different initial points (0.5,0.4,0.5) and (0.4,0.3,0.9) shown in Figure.1b and Figure.1a, respectively, for more accurately when $\Theta_1 \in (0.095, 0.107)$,$\Theta_2 \in (0.0097, 0.0103)$, $\gamma_1 \in (0, 0.004)$,$\gamma_2 \in (0.0097, 0.0103)$, $\lambda_1 \in (0.48, 0.49)$ *and* $\lambda_2 \in (0.063, 0.065)$. In order to explain the effect of the above parameters values of the system (2.3) on the dynamical behavior of the system, the system is solved numerically for the data given in (9.1) with varying the two parameters $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ by decreasing to $\lambda_1 = 0.1$, $\lambda_2 = 0.035$ the equilibrium point E^* approaches E_2 = *(S,0,P₂)* = (1.002, 0 ,0.997) , see Figure.2a**.** taking into the fixation of the rest of the values for the parameters in (9.1) and initial point (0.4,0.3,0.9) **,** while when we vary $\boldsymbol{\gamma_1}$ *and* $\boldsymbol{\gamma_2}$ by increasing to $\gamma_1 = 0.03$ and $\gamma_2 = 0.24$, $\lambda_1 = 0.482$ **,** $\lambda_2 = 0.01$. We find that the equilibrium point E^*approaches E_3 =*(S,P₁,0)*= (0.793, 0.791, 0) see Figure 2.b taking into the fixation of the rest of the values for the parameters in (9.1) and initial point (0.4,0.3,0.9).



(a) (b)

Figure 2: Time series of the solution of system (2.3) with initial point (0.4,0.3,0.9) of varying parameters $\lambda_1 = 0.1$, $\lambda_2 = 0.035$ shown in (a), and varying parameters $\gamma_1 = 0.03$ and $\gamma_2 = 0.24$ shown in (b)

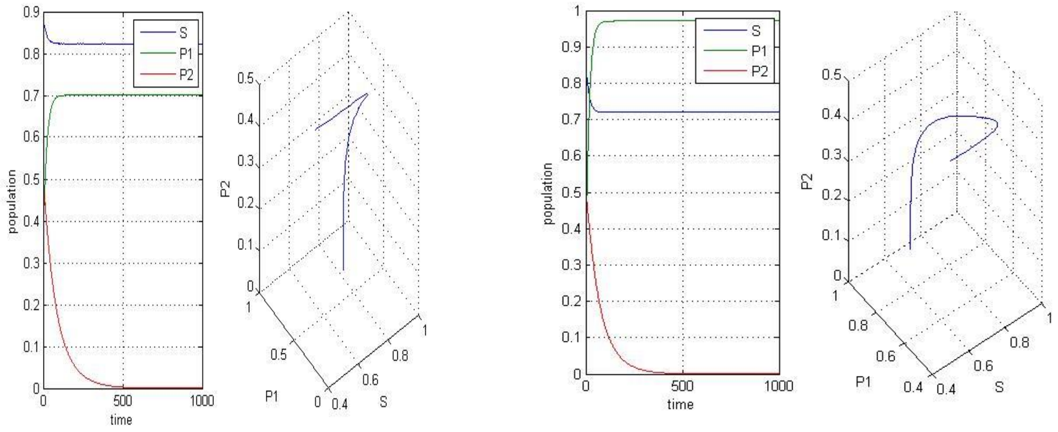
When θ_1 and θ_2 increase, that $\theta_1 = 0.5$, $\theta_2 = 0.4$ the equilibrium point E^* approaches E_1=(1,0,0) taking into the fixation of the rest of the values for the parameters in (9.1) and initial point (0.5,0.4,0.5) see (Figure.3a) and initial point (10,11,9) see (Figure.3b)



(a) (b)

Figure 3: Time series solution of system (2.3),E^* approaching E_1 with varying parameters $\theta_1 = 0.5$,$\theta_2 = 0.4$ for initial point (0.4,0.5,0.4) shown in (Figure 3.a), and for initial point (10,11,9) shown in (Figure 3.b)

Finally, regarding the refuge parameter **m**, the system (2.3) is globally stable around E^* at $\mathbf{m} \in$ (0.56, 0.63) which is explained in Figure.4, (Figure.4a) for m= 0.5 and when m=0.64 shown in (Figure.4b) with initial point (0.5,0.4,0.5)



(a) **(b)**

Figure 4 : Time series of the solution for system (2.3), global stability of E^* differ when m=0.64 in Figure.4a , and m= 0.5 in Figure.4b, for initial point (0.5,0.4,0.5).

10. Results and Conclusion

In this paper, an ecological system consisting of one prey- two predators with Beddington –De Angelis functional response and refuge is proposed and studied. The model is assumed to be held the effect of prey refuge on the absence of Beddington –De Angelis. The existence, uniqueness and bounded condition of the solution of the proposed model are discussed. All possible equilibrium points with their local stability conditions are obtained using the Routh-Hurwitz criterion. Suitable Lyapunov functions are used to investigate the global dynamics of the equilibrium points. The persistence of the system is investigated with the help of the average Lyapunov method. The Local bifurcation analysis around the equilibrium points E_2 and E_3 are carried out depending on Sotomayor's theorem. Finally, the appearance of the Hopf bifurcation around the positive equilibrium point E^* is also investigated. For the suitable set of biologically feasible hypothetical data, the proposed system is solved numerically to verify the obtained analytical results and specify the control set of parameters. Also, the obtained numerical results depending on the data given by (9.1) can be summarized as follows:

1. We found out that the system (2.3) has a globally asymptotically stable in the endemic equilibrium point E^* =(0.792,0.791,0.79) at certain intervals to the parameters $\Theta_1 \in (0.095, 0.107)$, $\Theta_2 \in (0.0097, 0.0103)$, $\gamma_1 \in (0, 0.004)$, $\gamma_2 \in (0.0097, 0.0103)$, $\lambda_1 \in (0.48, 0.49)$, *and* $\lambda_2 \in (0.063, 0.065)$.
2. The influence of varying the two parameters $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ is clear by decreasing them from $\lambda_1 = 0.486$ and $\lambda_2 = 0.064$ to $\lambda_1 = 0.1$ and $\lambda_2 = 0.035$, we found out that the equilibrium point E^* approaches $E_2 = (S,0,P_2)$= (1.002, 0 ,0.997) for the initial point (0.4,0.3,0.9).
3. As we vary $\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}$ by increasing them to $\gamma_1 = 0.03$ and $\gamma_2 = 0.24$ the equilibrium point E^* approaches $E_3 = (S,P_1,0)$= (0.793,0.791, 0) according to the initial point (0.4,0.3,0.9).
4. Finally, the influence of θ_1 and $\theta_2$by increasing them from $\theta_1 = 0.1$, $\theta_2 = 0.01$ to $\theta_1 = 0.5$, $\theta_2 = 0.4$ leads to the equilibrium point E^* approaches E_1=(1,0,0).
5. The previous results are not affected mainly by changing initial values.
6. The refuge parameter *m* satisfies the above results for $0.56 \geq m \geq 0.63$.

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