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A New Method for Finding Optimal Solution for Fully Fuzzy Transportation Problems

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Abstract: In the field of transportation problems under fuzzy environments, numerous methods have been proposed where parameters are typically represented by normal fuzzy numbers. However, as S.H. Chen highlighted in "Operations on Fuzzy Numbers with Function Principal" (1985), it is often impractical to confine membership functions to a normal form. Chen introduced the concept of generalized fuzzy numbers to address this limitation. While several studies have employed generalized fuzzy numbers in various real-life applications, their application to transportation problems remains limited.

Recent research has begun to bridge this gap. For instance, a study introduced a fuzzy transportation problem where transportation costs are represented by generalized hexagonal fuzzy numbers. The researchers converted the fuzzy problem into a crisp one using a ranking function, simplifying the computation of initial and optimal solutions without directly solving the original fuzzy transportation problem. therefore; in this paper a novel approach has been introduced for addressing fully fuzzy transportation problems, where the transportation cost, demand, and supply are all represented as trapezoidal fuzzy numbers. The method consists of three primary steps: first, determining an initial fuzzy solution; second, evaluating the optimality of the solution; and third, if the solution is not optimal, refining it to achieve the fuzzy optimal solution. All techniques utilized in this approach are grounded in operations on fuzzy numbers, and the method's effectiveness is demonstrated through a numerical example.

Keywords: Transportation Problem, Trapezoidal Fuzzy Number, Defuzzification, Initial Fuzzy Solution, Optimal Fuzzy Solution.

نـهج جديدة لإيجاد الحل الأمثل لمشاكل النقل الضبابية

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المستخلص: في مجال مشاكل النقل الضبابية، تم اقتراح العديد من الطرق، حيث يتم تمثيل المعلمات عادةً بأرقام ضبابية. ومع ذلك، كما أبرز S.H. Chen في "العمليات على الأرقام الضبابية مع الدوال" في سنة (١٩٨٥)، غالبًا ما يكون من غير العملي حصر الدوال في شكل طبيعي. قدم Chen مفهوم الأرقام الضبابية المعممة لمعالجة هذا القيد. في حين استخدمت العديد من الدراسات أرقامًا ضبابية معممة في تطبيقات مختلفة في الحياة الواقعية، إلا أن تطبيقها على مشاكل النقل لا يزال محدودًا.
بدأت الأبحاث الحديثة في سد هذه الفجوة. على سبيل المثال، قدمت إحدى الدراسات مشكلة نقل ضبابية حيث يتم تمثيل تكاليف النقل بأرقام ضبابية سداسية معممة. قام الباحثون بتحويل المشكلة الضبابية إلى مشكلة غير ضبابية باستخدام دالة الترتيب، مما أدى إلى تبسيط حساب الحلول الأولية والمثلى دون حل مشكلة النقل الضبابية الأصلية بشكل مباشر. لذلك؛ في هذا البحث تم تقديم نهج جديد لمعالجة مشاكل النقل الضبابية بالكامل، حيث يتم تمثيل تكلفة النقل والطلب والعرض على شكل أرقام ضبابية شبه منحرفة. تتكون الطريقة من ثلاث خطوات أساسية: أولاً، تحديد حل ضبابي أولي؛ ثانيًا، تقييم الحل الأمثل؛ وثالثًا، إذا لم يكن الحل مثاليًا، يتم تحسينه لتحقيق الحل الأمثل الضبابي. تعتمد جميع التقنيات المستخدمة في هذا النهج على العمليات على الأرقام الضبابية، ويتم إثبات فعالية الطريقة من خلال أمثلة عددية.

الكلمات المفتاحية: مشكلة النقل، العدد الضبابي شبه المنحرف، إزالة الضبابية، الحل الضبابي الأولي، الحل الضبابي الأمثل.

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1. Introduction

One of the early problems used in linear programming is the transportation issue. Transportation models are used in supply chain and logistics, costs, and the aim is to reduce the total costs. For this purpose, effective algorithms have been created to address the transportation problem in situations when the supply and demand quantities, together with the cost factors, are precisely known. But in the actual world, unpredictability and imprecision are certain to occur because of unforeseen circumstances.
In some situations, uncontrolled factors may lead to uncertainty in the cost coefficients and supply and demand quantities of a transportation problem. so, for this purpose, we used trapezoidal fuzzy numbers instead of crisp numbers to approach real situations of transportation problems in real life. Some proposed methods change the fuzzy model to a crisp model and use the existing techniques to solve and provide only crisp solutions for the fuzzy transportation problem. Also, some other methods suggest some ways to find optimal fuzzy solutions, here we are using the same concepts of solving crisp models but instead we are using fuzzy numbers in all the operations.

2. Fuzzy transportation problems

In traditional transportation problems, it is typically assumed that decision-makers have precise information regarding transportation costs, product availability, and demand. However, in real-world scenarios, these parameters often contain inherent uncertainties due to various uncontrollable factors. For instance:

- Transportation Costs: When a product is being transported to a new destination for the first time, there may be a lack of historical data or expert knowledge, leading to uncertainty about the transportation costs.
- Product Demand: The introduction of a new product into the market often results in unpredictable demand patterns, as consumer response is uncertain.
- Product Availability: Suppliers may face situations where the availability of a product is uncertain. For example, a supplier might initially confirm the availability of a product but later retract due to unforeseen stock issues, especially when large quantities are requested.

To effectively address these uncertainties, researchers have applied fuzzy set theory to transportation problems. This approach allows for the modeling of imprecise information, enabling more robust and flexible decision-making processes. By representing transportation costs, availability, and demand as fuzzy numbers, decision-makers can better accommodate the inherent vagueness present in real-life transportation scenarios.
Therefore; the advantages of this approach are:

A. The approach effectively handles uncertainty in real-world transportation problems.
B. It reduces computational complexity by transforming the fuzzy problem into a crisp one.
C. The method is adaptable to different types of fuzzy numbers and practical aspects.

3. Definitions and Theorems

Definition 3.1: Let X be a Universal set, the subset Ã of X is a fuzzy subset with a membership Function $\mu_{\tilde{A}}: X \rightarrow [0,1]$, is defined by: [1][2][3][4][5]

$$\tilde{A} = \{(x, \mu_{\tilde{A}}) \mid x \in X, \ 0 \le \mu_{\tilde{A}}(x) \le 1\} \qquad (1)$$

Definition 3.2: A fuzzy set A in R is called a fuzzy number if it satisfies the following conditions:
(i) A is normal.
(ii) A^{α} is a closed interval for every $\alpha \in (0,1]$,
(iii) The support of A is bounded. [1]

Definition 3.3: A fuzzy number A is a trapezoidal fuzzy number if its membership function is given as the following:[1][6][7][8]

$$\mu_A(x) = \begin{cases} 0 & x \le a_1 \ or \ x \ge a_4 \\ \frac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \frac{a_4-x}{a_4-a_3} & a_3 \le x \le a_4 \\ 1 & a_2 \le x \le a_3 \end{cases} \qquad (2)$$

Where $a_1, a_2, a_3, a_4 \in R$, the trapezoidal fuzzy number denoted by(a_1, a_2, a_3, a_4).

Definition 3.4: Basic arithmetic operations on trapezoidal fuzzy numbers:[1][6][9]
Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers then:
We define addition $\oplus$, subtraction $\ominus$ and Multiplication $\otimes$ as the following:
(1) $(a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$
(2) $(a_1, a_2, a_3, a_4) \ominus (b_1, b_2, b_3, b_4) = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$
(3) $(a_1, a_2, a_3, a_4) \otimes (b_1, b_2, b_3, b_4) = (t_1, t_2, t_3, t_4)$
Where $t_1 = minimum \{a_1 \mathrm{b}_1, a_1\mathrm{b}_4, a_4\mathrm{b}_1, a_4\mathrm{b}_4\}$
$t_2 = minimum \{a_2 \mathrm{b}_2, a_2\mathrm{b}_3, a_3\mathrm{b}_2, a_3\mathrm{b}_3\}$
$t_3 = maximum \{a_1 \mathrm{b}_1, a_1\mathrm{b}_4, a_4\mathrm{b}_1, a_4\mathrm{b}_4\}$
$t_4 = maximum \{a_2 \mathrm{b}_2, a_2\mathrm{b}_3, a_3\mathrm{b}_2, a_3\mathrm{b}_3\}$

Definition 3.5: Consider we have a trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, the defuzzied number or crisp value of $\tilde{A}$ called *magnitude* of (A) denoted by:[1]

$$Mag(\tilde{A}) = \frac{a_i + 5a_2 + 5a_3 + a_4}{12} \qquad (3)$$

Definition 3.6: Let $\tilde{u}$ and $\tilde{v}$ be two trapezoidal fuzzy numbers. Then by using the ranking of $\tilde{u}$ and $\tilde{v}$, the set of trapezoidal fuzzy numbers is defined as follows:[1][6][7]

(1) $Mag(\tilde{u}) > Mag(\tilde{v})$ if and only if $\tilde{u} \succ \tilde{v}$;
(2) Mag $(\tilde{u}) <$ Mag $(\tilde{v})$ if and only if $\tilde{u} \prec \tilde{v}$ and
(3) Mag $(\tilde{u}) =$ Mag$(\tilde{v})$ if and only if $\tilde{u} \approx \tilde{v}$

Definition 3.7: The ordering $\succeq$ and $\preceq$ between any two trapezoidal fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are defined as follows: [1][7][8][10]
(1) $\tilde{u} \succeq \tilde{v}$ if and only if $\tilde{u} \succ \tilde{v}$ or $\tilde{u} \approx \tilde{v}$ and
(2) $\mathrm{u} \preceq \mathrm{v}$ if and only if $\tilde{u} \prec \tilde{v}$ or $\tilde{u} \approx \tilde{v}$.

(3) $\tilde{u} = (a, b, c, d) \approx \tilde{0}$ if and only if Mag $(\tilde{u}) = 0$;
(4) $\tilde{u} = (a, b, c, d) \succeq \tilde{0}$ if and only if Mag $(\tilde{u}) \geq 0$ and
(5) $\tilde{u} = (a, b, c, d) \preceq \tilde{0}$ if and only if Mag $(\tilde{u}) \leq 0$.

Definition 3.8: Let's have the following Fuzzy Transportation problem with fuzzy costs, fuzzy sources and fuzzy demands:[1][2][3][4][7][8]

$$\begin{aligned} & \textit{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \otimes \tilde{x}_{ij} \\ & \textit{Subject to:} \\ & \sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i \qquad \textit{for } i = 1,2, \dots, m \\ & \sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j \qquad \textit{for } j = 1,2, \dots, n \\ & \quad \tilde{x}_{ij} \geq 0 \qquad \textit{for } i = 1,2, \dots, m \textit{ and } j = 1,2, \dots, n, \end{aligned} \tag{4}$$

where
m: the number of supply points.
n: the number of demand points.
$\tilde{x}_{ij} \approx \left(x_{ij}{}^1, x_{ij}{}^2, x_{ij}{}^3, x_{ij}{}^4\right)$ is the uncertain number of units shipped from supply point i to demand point j.
$\tilde{c}_{ij} \approx \left(c_{ij}{}^1, c_{ij}{}^2, c_{ij}{}^3, c_{ij}{}^4\right)$ is the uncertain cost of shipping one unit from supply point i to the demand point j.
$\tilde{a}_i \approx (a_i{}^1, a_i{}^2, a_i{}^3, a_i{}^4)$ is the uncertain supply at supply point i and
$\tilde{b}_j \approx \left(b_j{}^1, b_j{}^2, b_j{}^3, b_j{}^4\right)$ is the uncertain demand at demand point j.
we can represent the transportation problem in a tabular form:

Table (1): Tabular form of transportation problem

	1	. . .	n	Supply
1	$\tilde{c}_{11}$	. . .	$\tilde{c}_{1n}$	$\tilde{a}_1$
⋮	⋮	⋮	⋮	⋮
m	$\tilde{c}_{m1}$	. . .	$\tilde{c}_{mn}$	$\tilde{a}_n$
Demand	$\tilde{b}_1$		$\tilde{b}_n$	

Definition 3.9: A set of non-negative allocations $\tilde{x}_{ij} \approx \left(x_{ij}{}^1, x_{ij}{}^2, x_{ij}{}^3, x_{ij}{}^4\right)$which satisfies (in the sense equivalent) the row and the column restrictions is known as fuzzy feasible solution. [1]
Definition 3.10: A fuzzy feasible solution to a fuzzy transportation problem with m sources and n destinations is said to be a fuzzy basic feasible solution if the number of positive allocations are $(m + n - 1)$. If the number of allocations in a fuzzy basic solution is less than $(m + n - 1)$, it is called fuzzy degenerate basic feasible solution.

Definition 3.11: A fuzzy feasible solution is said to be fuzzy optimal solution if it minimizes the total fuzzy transportation cost.

Theorem 3.1:

The necessary and sufficient condition for a Fuzzy transportation problem to have a solution is that the total of fuzzy demand equals the total of fuzzy supply.

Proof: (Necessary condition)

Consider a fuzzy transportation problem as a fuzzy linear programming problem:

$$Minimize\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \otimes \tilde{x}_{ij}$$

Subject to:

$$\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i \qquad for\ i = 1,2, \dots, m$$

$$\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j \qquad for\ j = 1,2, \dots, n$$

$$\tilde{x}_{ij} \geq 0 \qquad for\ i = 1,2, \dots, m\ and\ j = 1,2, \dots, n,$$

Clearly each of the fuzzy supplies $(\tilde{a}_i)$ meets one of the following constraints:

$$\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i$$

If we take sum all the supplies we get:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{x}_{ij} \approx \sum_{i=1}^{m} \tilde{a}_i \quad - - - (5)$$

By the same way for the fuzzy demands $\tilde{b}_j$ satisfy the following constraints:

$$\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j$$

If we Take sum all the demands we get:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \tilde{x}_{ij} \approx \sum_{j=1}^{n} \tilde{b}_j \quad - - - -(6)$$

From (5) and (6) we get that $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$ Necessary condition proof completed.

Sufficient condition as given above:

let $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j = \tilde{\beta}$

let's choose our fuzzy feasible solution $\tilde{x}_{ij} = \frac{\tilde{a}_i \tilde{b}_j}{\tilde{\beta}}\ \ \forall i, j$ then for supply constraints:

$$\sum_{j=1}^{n} \tilde{x}_{ij} = \sum_{j=1}^{n} \frac{\tilde{a}_i \tilde{b}_j}{\tilde{\beta}} = \frac{\tilde{a}_i}{\tilde{\beta}} \sum_{j=1}^{n} \tilde{b}_j = \frac{\tilde{a}_i}{\tilde{\beta}} (\tilde{\beta}) = \tilde{a}_i$$

For demand constraints:

$$\sum_{i=1}^{m} \tilde{x}_{ij} = \sum_{i=1}^{m} \frac{\tilde{a}_i \tilde{b}_j}{\tilde{\beta}} = \frac{\tilde{b}_i}{\tilde{\beta}} \sum_{j=1}^{n} \tilde{a}_i = \frac{\tilde{b}_i}{\tilde{\beta}} (\tilde{\beta}) = \tilde{b}_j$$

The proof is complete.

Theorem 3.2: Existence of optimal solution

Optimal solution always exist for any fuzzy transportation problem.

Proof:

By Theorem (3.1) we have $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$

And clearly $\tilde{O} \leq \tilde{x}_{ij} \leq \min(\tilde{a}_i, \tilde{b}_j)$ so my Fuzzy decision variables are bounded

Hence that is mean my feasible Region is Bounded and closed.

By fundamental theorem of linear Programming, implies that optimal solution exists.

Theorem 3.3:

A balanced Fuzzy transportation problem always has a basic feasible solution. Such a solution consists of m + n − 1 positive Fuzzy variables at most.

Proof: clearly, we have $\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i$ and $\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{x}_{ij} \approx \sum_{i=1}^{m} \tilde{a}_i \ \text{ and } \sum_{j=1}^{n-1} \sum_{i=1}^{m} \tilde{x}_{ij} \approx \sum_{j=1}^{n-1} \tilde{b}_j$$

So

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{x}_{ij} - \sum_{j=1}^{n-1} \sum_{i=1}^{m} \tilde{x}_{ij} \approx \sum_{i=1}^{m} \tilde{a}_i - \sum_{j=1}^{n-1} \tilde{b}_j$$

This implies that $\sum_{i=1}^{m} \tilde{x}_{in} \approx \tilde{b}_n$ which is the last constraint of the Fuzzy Transportation model.

The Total of Constraints is (m+n) and last constrain can be obtained from the other constraints

So its linearly dependent, but the basic feasible solutions must be independent hence we conclude that the basic feasible solutions must be m+n-1.

4. Steps of New method for solving Fuzzy Transportations problems:

This technique is consisting of three main steps:
Step 1: Check if the transportation problem is balanced, if it's not balanced add a dummy row or column with zero fuzzy cost to make it balanced.
Step2: Find the initial solution for the fuzzy transportation problem by using the Vogel approximation method but applying the operations in the fuzzy environment or i.e. using fuzzy arithmetic operations
As the following:
Calculate the row fuzzy penalty by subtracting a second minimum fuzzy number from the first minimum fuzzy number for each row.
Calculate the column fuzzy penalty by subtracting a second minimum fuzzy number from the first minimum fuzzy number for each column.
In row and column determine the maximum penalty select this row or this column and then go to the cell that has minimum fuzzy cost allocates fuzzy shipping units as much as possible.
Repeat the above steps until we can allocate for all possible cells in such a way that demand and supply conditions are satisfied. Then, we will get an initial solution for the problem.
Step 3: Checking the optimality of the initial solution as the following:
First: Check the condition to see if the initial basic feasible solutions equals m+n-1.
Second: calculating the penalties of non-basic cells as the following, take each non-basic cell and make a close Loop with basic solution cells, start from the non-basic fuzzy cost cell with a positive sign and then take the next corner cell which is the basic cell and negative sign, and then go to another corner of the loop till we take all the corner cells of the loop with alter sign and we must create the loop using the vertical and horizontal route along our transportation problem table.
Doing the above operation for all non-basic cells, if all fuzzy results have positive or zero magnitude value then the initial solution is optimal, if there exists a fuzzy result with a negative magnitude value then the solution is not optimal going to step 4.
Step4: To refine the initial solution of a fuzzy transportation problem towards optimality, the following steps can be implemented:
Find the non-basic cell that has the most negative magnitude value starting with this cell to make a close loop with the basic solution list and calculate fuzzy value θ:
$\theta = min\ \{negative\ sing\ corner\ fuzzy\ value\}$
Adding fuzzy value θ to the positive corner fuzzy value and subtracting it from the negative corner fuzzy value of the loop and then we will get a new set of fuzzy solutions and the go to step 3.

5. Numerical Examples:

Example 5.1 Consider the following fuzzy transportation problem:

	B1	B2	B3	B4	(supply)
A1	(17,18,20,21)	(27,28,32,33)	(48,49,51,52)	(7,9,11,13)	(5,6,8,9)
A2	(68,69,71,72)	(28,29,31,32)	(37,39,41,43)	(58,59,61,62)	(7,8,10,11)
A3	(38,39,41,42)	(6,7,9,10)	(68,69,71,72)	(18,19,21,22)	(16,17,19,20)
(Demand)	(2,4,6,8)	(6,7,9,10)	(5,6,8,9)	(12,13,15,16)	

Step1: Now, the total fuzzy supply, $\sum \widetilde{a_\iota} = (28, 31, 37, 40)$ and the total fuzzy demand, $\sum \widetilde{b_\iota} = (25, 30, 38, 43)$. Since $\text{Mag}(\widetilde{a_\iota}) = \text{Mag}(\widetilde{b_\iota}) = 34$, the given problem is a balanced.

Step 2: Applying Vogel app

	B1		B2		B3		B4	
A1	(17,18,20,21)	(2,4,6,8)	(27,28,32,33)		(48,49,51,52)		(7,9,11,13)	(−3,0,4,7)
A2	(68,69,71,72)		(28,29,31,32)		(37,39,41,43)	(5,6,8,9)	(58,59,61,62)	(−9,−3,7,13)
A3	(38,39,41,42)		(6,7,9,10)	(6,7,9,10)	(68,69,71,72)		(18,19,21,22)	(6,8,12,14)

proximation method in fuzzy environment to find initial fuzzy solution, we get:

Calculating the objective function:
$\tilde{z}$ = (-234, 324, 1252, 1860), and
Mag = (-234, 324, 1252, 1860) =792.1666666666666
Step 3: clearly, we have M+N-1=4+3-1=6 basic feasible fuzzy solution cells.
Checking the optimality of the initial solution, we should check the penalty for all non-basic if there is negative value.
For cell(1,2) starting form cell (1,2) creating a close loop with cell (1,4), (3,4) and (3,2)
$P(1,2) = (27,28,32,33)\Theta(7,9,11,13) \oplus (18,19,21,22)\ \Theta(6,7,9,10) = (22,\quad 27,\quad 37,\quad 42)$, and $\text{Mag}(22, 27, 37, 42) = 32 > 0$
By the same way check all other non-basic cells.
The only cell penalty $\text{P}(2,2) = (-26, -22, -14, -10)$
$\text{Mag}(-26, -22, -14, -10) = -18 < 0$ which is negative.
so, we can say this solution is not optimal so go to step 4.
Step 4: Modifying the initial solution and going toward the optimal solution.
From cell (1,3) we make close loop with cells (3,3), (3,1), and (1,1):
$\theta = min(\,cell(2,4), cell(3,2)) = (-9, -3,7,13)$
and then the solution will be change to:

	B1		B2		B3		B4	
A1	(17,18,20,21)	(2,4,6,8)	(27,28,32,33)		(48,49,51,52)		(7,9,11,13)	(−3,0,4,7)
A2	(68,69,71,72)		(28,29,31,32)	(−9,−3,7,13)	(37,39,41,43)	(5,6,8,9)	(58,59,61,62)	
A3	(38,39,41,42)		(6,7,9,10)	(-7, 0, 12, 19)	(68,69,71,72)		(18,19,21,22)	(-3, 5, 19, 27)

Again, we check for optimality by the same as step 3, we can see that:
The penalty of All non- basic cell values are positive, so we can say that the solution is optimal.
Calculating the objective function:
$\tilde{z}$ = (-244, 308, 1216, 1846), and Mag (-244, 308, 1216, 1846) =768.5

Example 5.2 Consider the following fuzzy transportation problem:

	B1	B2	B3	B4	(supply)
A1	(42,44,46,48)	(18,19,21,22)	(19,20,22,23)	(28,29,31,32)	(12,14,16,18)
A2	(12,13,15,16)	(16,17,19,20)	(28,29,31,32)	(29,30,32,33)	(11,12,14,15)
Demand	(7,8,10,11)	(4,5,7,8)	(5,6,8,9)	(7,8,10,11)	

Step1: Now, the total fuzzy supply, $\sum \widetilde{a_\iota} = (23, 26, 30, 33)$ and $\text{Mag}(\sum \widetilde{a_\iota}) = 28$ the total fuzzy demand, $\sum \widetilde{b_\iota} = (23, 27, 35, 39)$, $\text{Mag}(\sum \widetilde{b_\iota}) = 31$, since $\text{Mag}(\sum \widetilde{a_\iota}) \neq \text{Mag}(\sum \widetilde{b_\iota})$, the given problem is a unbalanced. We should make it balanced as the following:

	B1	B2	B3	B4	(supply)
A1	(42,44,46,48)	(18,19,21,22)	(19,20,22,23)	(28,29,31,32)	(12,14,16,18)
A2	(12,13,15,16)	(16,17,19,20)	(28,29,31,32)	(29,30,32,33)	(11,12,14,15)
A3	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(1,2,4,5)
Demand	(7,8,10,11)	(4,5,7,8)	(5,6,8,9)	(7,8,10,11)	

Step 2: Applying Vogel approximation method in fuzzy environment to find initial fuzzy solution, we get:

	B1	B2	B3	B4
A1	(42,44,46,48)	(18,19,21,22) $\boxed{(-4, -1, 5, 8)}$	(19,20,22,23) $\boxed{(5, 6, 8, 9)}$	(28,29,31,32) $\boxed{(2, 4, 8, 10)}$
A2	(12,13,15,16) $\boxed{(7,8,10,11)}$	(16,17,19,20) $\boxed{(0, 2, 6, 8)}$	(28,29,31,32)	(29,30,32,33)
A3	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0) $\boxed{(1,2,4,5)}$

Step 3: Checking the optimality of the initial solution, clearly, we have M+N-1=4+3-1=6 basic feasible fuzzy solution cells

Now we should check the penalties of all non-basic if there is negative value.

All non- basic cell values are positive, so we can say that the solution is optimal.

Calculating the objective function:

$\tilde{z} = (147, 353, 793, 1039)$, and

Mag (147, 353, 793, 1039) =576.3333333333334.

Example 5.3: Consider the following fuzzy transportation problem:

A1	(1,2,4,5)	(4,6,8,10)	(3,4,6,7)	(3,4,6,7)	(36,37,39,40)
A2	(2,4,6,8)	(3,4,6,7)	(0,2,4,6)	(2,3,5,6)	(8,9,11,12)
A3	(0,1,3,4)	(2,3,5,6)	(2,3,5,6)	(2,4,6,8)	(14,15,17,18)
A4	(1,2,6,7)	(7,8,10,11)	(1,2,6,7)	(4,5,7,8)	(21,22,24,25)
Demand	(22,24,26,28)	(27,28,30,31)	(19,20,22,23)	(19,20,22,23)	

Step1: Now, the total fuzzy supply, $\sum \widetilde{a_\iota} = (79, 83, 91, 95)$ and $\mathrm{Mag}(\sum \widetilde{a_\iota}) = 87$ the total fuzzy demand, $\sum \widetilde{b_\iota} = (87, 92, 100, 105)$, $\mathrm{Mag}\left(\sum \widetilde{b_\iota}\right) = 96$ Since $\mathrm{Mag}(\widetilde{a_\iota}) \neq \mathrm{Mag}(\widetilde{b_\iota})$, the given problem is a un balanced. We should make it balanced as the following:

	B1	B2	B3	B4	supply
A1	(1,2,4,5)	(4,6,8,10)	(3,4,6,7)	(3,4,6,7)	(36,37,39,40)
A2	(2,4,6,8)	(3,4,6,7)	(0,2,4,6)	(2,3,5,6)	(8,9,11,12)
A3	(0,1,3,4)	(2,3,5,6)	(2,3,5,6)	(2,4,6,8)	(14,15,17,18)
A4	(1,2,6,7)	(7,8,10,11)	(1,2,6,7)	(4,5,7,8)	(21,22,24,25)
A5	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(7, 8, 10, 11)
Demand	(22,24,26,28)	(27,28,30,31)	(19,20,22,23)	(19,20,22,23)	

Step 2: Applying Vogel approximation method in fuzzy environment to find initial fuzzy solution, we got:

	B1	B2	B3	B4
A1	(1,2,4,5) $\boxed{(4, 7, 11, 14)}$	(4,6,8,10) $\boxed{(-3, 4, 16, 23)}$	(3,4,6,7)	(3,4,6,7) $\boxed{(13, 16, 22, 25)}$
A2	(2,4,6,8)	(3,4,6,7) $\boxed{(8,9,11,12)}$	(0,2,4,6)	(2,3,5,6)
A3	(0,1,3,4) $\boxed{(14,15,17,18)}$	(2,3,5,6)	(2,3,5,6)	(2,4,6,8)
A4	(1,2,6,7)	(7,8,10,11)	(1,2,6,7) $\boxed{(19,20,22,23)}$	(4,5,7,8) $\boxed{(-2, 0, 4, 6)}$
A5	(0,0,0,0)	(0,0,0,0) $\boxed{(7,8,10,11)}$	(0,0,0,0)	(0,0,0,0)

Calculating the objective function:
$\tilde{z}$ = (56, 193, 553, 792), and
Mag = (56, 193, 553, 792) =381.5
Step 3: Checking the optimality of the initial solution, clearly, we have M+N-1=5+4-1=8 basic feasible fuzzy solution cells
Now we should check the penalties of all non-basic if there is negative value.
We saw that the only cell penalty $P(3,2) = (-11, -6, 2, 7)$
$Mag(-11, -6, 2, 7) = -2 < 0$ which is negative.
so, we say that the solution is not optimal, so go to step 4.
Step 4: Modifying the initial solution and going toward the optimal solution.
From cell (3,2) we make close loop with cells (1,2),(1,1) and (3,1)

$$\theta = min(cell(1,2), cell(3,1)) = \min\{(-3,4,16,23),(14,15,17,18)\} = (-3,4,16,23)\ mag = 10$$

and then the solution will be change to:

	B1	B2	B3	B4
A1	(1,2,4,5) $\boxed{(1,11,27,37)}$	(4,6,8,10)	(3,4,6,7)	(3,4,6,7) $\boxed{(13,16,22,25)}$
A2	(2,4,6,8)	(3,4,6,7) $\boxed{(8,9,11,12)}$	(0,2,4,6)	(2,3,5,6)
A3	(0,1,3,4) $\boxed{(-9,-1,13,21)}$	(2,3,5,6) $\boxed{(-3,4,16,23)}$	(2,3,5,6)	(2,4,6,8)
A4	(1,2,6,7)	(7,8,10,11)	(1,2,6,7) $\boxed{(19,20,22,23)}$	(4,5,7,8) $\boxed{(-2,0,4,6)}$
A5	(0,0,0,0)	(0,0,0,0) $\boxed{(7,8,10,11)}$	(0,0,0,0)	(0,0,0,0)

Again, we check for optimality by the same as step 3, we can see that:
The penalty of All non- basic cell values are positive, so we reach to optimal solution
Calculating the objective function:
$\tilde{z}$ = (29, 171, 557, 827), and Mag = (29, 171, 557, 827) =374.6666666666667

6. Results and discussion

In many studies, generalized fuzzy numbers are transformed into normal fuzzy numbers through a normalization process before being applied to real-life problems. although this procedure is mathematically correct; it can lead to a loss of information inherent in the original data. Specifically, when a measurement of an objective value is converted to a valuation of a subjective value, certain nuances and details may be diminished or lost. Therefore, to preserve the richness of the original information, it is advisable to avoid such transformations when possible.

7. Conclusions

In this paper, we introduce novel method for determining the Initial Fuzzy Basic Feasible Solution (IFBFS) and the fuzzy optimal solution of transportation problems where transportation costs, availability, and product demand are represented as generalized trapezoidal fuzzy numbers. Unlike traditional approaches that convert generalized fuzzy numbers into normal fuzzy numbers, the novel method operates directly on generalized fuzzy numbers, and it’s very easy to understand and to apply for solving the fuzzy transportation problems occurring in real life situations as well as and cannot lead to a loss of information inherent in the original data. Therefore; The proposed method is theoretically sound, practical, and innovative, offering a robust solution to fully fuzzy transportation problems. However, further research is needed to address its computational complexity and generalizability to larger-scale problems.

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