

On Rings With Types Of $\gamma(n)$ -Regularity

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Abstract:

Let R be an associative ring with identity . For a fixed integer $n > 1$, an element a in R is said to be $\gamma(n)$ -regular ($\gamma(n)$ -strongly regular) if there exists b in R such that $a = ab^n a$ ($a = a^2 b^n$) . So a ring R is said to be $\gamma(n)$ -regular ($\gamma(n)$ -strongly regular) for a positive integer $n \neq 1$, if every element of R is $\gamma(n)$ -regular ($\gamma(n)$ -strongly regular) .

In this paper we investigate some characterizations and several basic properties of those rings , also the connection between them and rings of some kind of commutivity .

حلقة منتظمة من النمط $\gamma(n)$

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ملخص البحث :

لتكن R حلقة تجميعية بمحايد . لعدد صحيح n اكبر من واحد , العنصر a في R يسمى منتظماً من النمط $\gamma(n)$ (منتظماً بقوة من النمط $\gamma(n)$) اذا وجد عنصر مثل b في R بحيث ان $a = ab^n a$ ($a = a^2 b^n$) . وهكذا فان الحلقة R تسمى حلقة منتظمة من النمط $\gamma(n)$ (حلقة منتظمة بقوة من النمط $\gamma(n)$) حيث ان n عدد صحيح اكبر من واحد, اذا كان كل عنصر من R هو منتظماً من النمط $\gamma(n)$ (منتظماً بقوة من النمط $\gamma(n)$) .
في هذا البحث درسنا بعض الصفات الاساسية لهكذا حلقات و كذلك الروابط بينها وبين حلقات ذات صفات مشابهة للابدالية .

0 : Introduction :

The conception of Von Neumann regular rings occurred in 1936 [13] defined a regular ring as a ring R with property that for each $a \in R$ there exists $b \in R$ such that $a = aba$.

A ring R with property that for each $a \in R$ there exists $b \in R$ such that $a = a^2b$, is called strongly regular ring. This concept has been defined some sixty year ago by R.F.Arens and I.Kaplansky [1] and was studied in recent years by many others (cf. [2], [8], [9]). Note that in such rings $a = ba^2$ if and only if $a = a^2b$ [4].

As a generalization of regularity property, McCoy [6] defined π -regular rings, that is a ring R with every $a \in R$, a^n is regular element for some positive integer n . (cf. [6], [9], [11], [12]). In 1954 Azumaya [2] defined strongly π -regular rings, that is ring R such that for every $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^{n+1}b$. Also Ehrlich [5] defined unit regular ring, that is a ring R with every $a \in R$ there exists a unit $u \in R$ such that $a = aua$.

A ring R is said to be γ -regular if for every $a \in R$ there exists $b \in R$ and an integer $n > 1$ such that $a = ab^n a$. This concept was introduced in 2006 by A.J.Mohammad and S.M.Salih [10]. Similarly definition of γ -strongly regular rings.

A ring R is said to be reduced, if R contains no non-zero nilpotent elements. So every idempotent element in a reduced ring is central [9].

For $a \in R$, $r(a)$ and $l(a)$ denoted the right and left annihilator of a respectively.

Finally, throughout this paper R is associative ring with identity.

§1 : $\gamma(n)$ -regular rings :

In this section we introduce the definition of $\gamma(n)$ -regular rings as special case of regular rings .

Definition 1.1 :

An element a of a ring R is called $\gamma(n)$ -regular (for an integer $n > 1$) if $a = ab^n a$ for some $b \in R$

A ring R is said to be $\gamma(n)$ -regular ring if every element of R is $\gamma(n)$ -regular .

Hence every $\gamma(n)$ -regular ring is γ -regular ring and every γ -regular ring is a regular ring . So every $\gamma(n)$ -regular ring is regular ring .

Examples 1.2 :

(1) The ring Z_3 is $\gamma(3)$ -regular ring .

(2) Boolean rings are $\gamma(n)$ -regular rings for each integer $n > 1$.

Note that every $\gamma(n)$ -regular ring is γ -regular ring , but the converse is not true in general , for example Z_5 is γ -regular ring but not $\gamma(n)$ -regular ring , and the ring Q of rational numbers is regular but not $\gamma(n)$ -regular.

It is clear that a homomorphic image of $\gamma(n)$ -regular ring is $\gamma(n)$ -regular ring .

Proposition 1.3 :

Let R be a $\gamma(n)$ -regular ring then for every $a \in R$, $l(a)$ and $r(a)$ are direct summands .

Proof :

Since R is $\gamma(n)$ -regular ring, then for each $a \in R$ there exists $b \in R$ such that $a = ab^n a$. So $(ab^n - 1)a = 0$. That is $ab^n - 1 \in l(a)$. So $1 - ab^n \in l(a)$. Since $1 = ab^n + (1 - ab^n)$, then $R = (ab^n)R + l(a)$.

Now let $x \in (Rab^n) \cap l(a)$, then $x \in (Rab^n)$ and $xa = cab^n a = 0$. That is $x = cab^n$ for some $c \in R$. Hence $xa = cab^n a$, then $ca = 0$. So $cab^n = 0$. Hence $x = 0$ there for $l(a)$ is direct summand.

Similarly, $r(a)$ is direct summand. ■

Proposition 1.4 :

If R is a ring with every nonzero element $a \in R$ there is a unique element $b \in R$ such that $a^n = a^n b a^n$ where n is an integer greater than one, then b is $\gamma(n)$ -regular element.

Proof :

Since $a^n = a^n b a^n$ for each $a \in R$, then R has no divisors of zero. then by cancellation law, $1 = a^n b$. So, $b = b a^n b$.

Therefore b is $\gamma(n)$ -regular element. ■

Theorem 1.5 :

Let I be an ideal of a $\gamma(n)$ -regular ring R , then the ring R/I is also $\gamma(n)$ -regular ring.

Proof :

Let $a + I \in R/I$, then $a \in R$. Since R is $\gamma(n)$ -regular ring then there exists $b \in R$ such that $a = ab^n a$. So

$$a + I = ab^n a + I = (a + I)(b + I)^n(a + I).$$

Therefore R/I is $\gamma(n)$ -regular ring. ■

Definition 1.6 :

An ideal I of a ring R is said to be $\gamma(n)$ -regular ideal if for every element $a \in I$ there exists $b \in I$ such that $a = ab^n a$.

Lemma 1.7 :

Let x and y be two elements of a ring R such that $xy^t x - x$ is $\gamma(n)$ -regular element, with $1 \neq t \in \mathbb{Z}^+$, then x is regular element.

Proof :

Since $xy^t x - x$ is $\gamma(n)$ -regular element, then there exists an element $b \in R$ such that

$$xy^t x - x = (xy^t x - x)b^n(xy^t x - x). \text{ Hence}$$

$$xy^t x - x = xy^t x b^n xy^t x - xy^t x b^n x - x b^n xy^t x + x b^n x.$$

$$\begin{aligned} \text{So } x &= xy^t x - xy^t x b^n xy^t x + xy^t x b^n x + x b^n xy^t x - x b^n x \\ &= x(y^t - y^t x b^n xy^t + y^t x b^n + b^n xy^t - b^n)x. \end{aligned}$$

Therefore x is a regular element. ■

Theorem 1.8 :

Let I be a $\gamma(n)$ -regular ideal of a ring R and R/I is also $\gamma(n)$ -regular ring. Then R is regular ring.

Proof :

Let $a \in R$. So $a + I \in R/I$. Since R/I is $\gamma(n)$ -regular ring, then there exists $b + I \in R/I$ such that $a + I = (a + I)(b + I)^n(a + I)$. Hence $a + I = (ab^n a) + I$. So $a - ab^n a \in I$. That is, $a - ab^n a$ is $\gamma(n)$ -regular element. Then a is regular element [Lemma 1.7].

Therefore R is a regular ring. ■

Corollary 1.9 :

Let R_1 and R_2 be two $\gamma(n)$ -regular rings . Then $R_1 \oplus R_2$ is regular ring .

Proof :

Since $(R_1 \oplus R_2)/R_1 \cong R_2$ and R_2 is $\gamma(n)$ -regular ring , then $R_1 \oplus R_2$ is regular ring [Theorem 1.8] . ■

Theorem 1.10 :

A finite direct sum of $\gamma(n)$ -regular rings is a regular ring .

Proof :

The proof is by mathematical induction ■

Theorem 1.11 :

Every ideal in a reduced $\gamma(n)$ -regular ring is $\gamma(n)$ -regular ideal .

Proof :

Let I be an ideal of a reduced $\gamma(n)$ -regular ring R , and $a \in I$. Then there exists $b \in R$ such that $a = ab^n a$. Since $b^n a$ is idempotent element , it is central .

$$\begin{aligned} \text{So , } (b^{n+1}a)(b^{n+1}a) &= (b^{n+1}a)(bb^n a) = (b^{n+1}a)(b^n ab) \\ &= b^{n+1}(ab^n a)b = (b^{n+1}a)b . \end{aligned}$$

Now let $y = b^{n+1}a$, then $y \in I$.

$$\begin{aligned} \text{So , } ay^n a &= a(b^{n+1}a)a = a[(b^{n+1}a)(b^{n+1}a) \dots (b^{n+1}a)(b^{n+1}a)]a = \\ &= a[(b^{n+1}a)b \dots bb]a = a[(b^{n+1}a)b^{n-1}]a = a[(bb^n a)b^{n-1}]a = \\ &= a(b^n abb^{n-1})a = a(b^n ab^n)a = (ab^n a)(b^n a) = ab^n a = a . \end{aligned}$$

Therefore I is $\gamma(n)$ -regular . ■

2 :Strongly $\gamma(n)$ -regular rings :

In this section we introduce the definition of strongly $\gamma(n)$ -regular rings and we will show that this definition is equivalence to the definition of $\gamma(n)$ -regular rings where the rings are commutative , reduced rings , and so on .

Definition 2.1 :

A ring R is called strongly $\gamma(n)$ -regular ring (for some integer $n \geq 1$) if for every element $a \in R$, there exists $b \in R$ such that $a = a^2 b^n$.

Remark 2.2 :

- (1) In strongly $\gamma(n)$ -regular ring R , $a = a^2 b^n$ if and only if $a = b^n a^2$ [4] .
- (2) It is obvious that a homomorphic image of a strongly $\gamma(n)$ -regular ring is strongly $\gamma(n)$ -regular ring .

Theorem 2.3 :

For any ring R , the following statements are equivalent :

- (1) R is strongly $\gamma(n)$ -regular ring .
- (2) R is reduced $\gamma(n)$ -regular ring .

Proof :

1 \Rightarrow 2 . Since R is strongly $\gamma(n)$ -regular ring , then for every $a \in R$ there exists $b \in R$ such that $a = a^2 b^n = b^n a^2$. So $b^n a = b^n (a^2 b^n) = (b^n a^2) b^n = a b^n$. Hence $a b^n a = a^2 b^n = a$. That is R is $\gamma(n)$ -regular ring .

Now suppose that $c^t = 0$ for some $c \in R$ and t is a positive integer . Since $c = c^2 d^n$ for some $d \in R$. Then $c^{t-1} = c^{t-2} c = c^{t-}$

$c^2d^n = c^td^n = 0$. Similarly $c^{t-2}=0$ and so on , to have $c=0$. Hence R is a reduced ring .

$2 \Rightarrow 1$. Since R is $\gamma(n)$ -regular ring then for every $a \in R$ there exists $b \in R$ such that $a = ab^n a$. So , $(a - a^2 b^n)^2 = a^2 - a^3 b^n - a^2 b^n a + a^2 b^n a^2 b^n = a^2 - a^3 b^n - a^2 + a^3 b^n = 0$. Since R is reduced then $a - a^2 b^n = 0$, that is $a = a^2 b^n$. Similarly $(a - b^n a^2)^2 = 0$ implies $a = b^n a^2$. Therefore R is strongly $\gamma(n)$ -regular ring . ■

Corollary 2.4 :

Every reduced $\gamma(n)$ -regular ring is strongly regular ring .

Definition 2.5:

A ring R is called strongly $\pi(n)$ -regular ring (for some integer n) if for every element $a \in R$, there exists $b \in R$ such that $a^n = a^{2n} b$.

Proposition 2.6 :

Let R be strongly $\pi(n)$ -regular ring . Then a^n is strongly $\gamma(n)$ -regular element for every $a \in R$.

Proof :

Since R is a strongly $\pi(n)$ -regular ring , then for every $a \in R$ there exists $b \in R$ such that $a^n = a^{2n} b$. Hence $a^n = a^{2n} b = a^n a^n b = a^n (a^n b) b = a^{2n} b^2 = \dots = a^{2n} b^n$. Therefore a^n is strongly $\gamma(n)$ -regular element . ■

Theorem 2.7 :

Let I be an ideal of a strongly $\gamma(n)$ -regular ring R . Then R/I is also strongly $\gamma(n)$ -regular ring .

Proof :

Let $a + I \in R/I$, then $a \in R$. Since R is a strongly $\gamma(n)$ -regular ring, then there exists $b \in R$ such that $a = a^2 b^n$. So, $a + I = a^2 b^n + I = (a + I)^2 (b + I)^n$. Therefore R/I is strongly $\gamma(n)$ -regular ring. ■

Lemma 2.8 : [9]

Let a be a none zero element in a reduced ring R . Then $l(a) = R(a)$ and $l(a) = l(a^2)$.

Theorem 2.9 :

Let R be a reduced ring and $r/l(a)$ is $\gamma(n)$ -regular ring for every $a \in R$. Then R is strongly $\gamma(n)$ -regular ring.

Proof :

Let $a \in R$, then $R/l(a)$ is $\gamma(n)$ -regular ring. So there exists $b + l(a) \in R/l(a)$ such that $a + l(a) = (a + l(a))(b + l(a))^n(a + l(a))$. Hence $a + l(a) = (ab^n a) + l(a)$. So $a - ab^n a \in l(a)$. that is $(a - ab^n a)a = 0$ and $(1 - ab^n)a^2 = 0$. Hence $1 - ab^n \in l(a^2) = r(a^2) = r(a) \Rightarrow a(1 - ab^n) = 0 \Rightarrow a = a^2 b^n$. Similarly $a = b^n a^2$. Therefore R is a strongly $\gamma(n)$ -regular ring. ■

Note that R is also $\gamma(n)$ -regular ring, that is since R is strongly $\gamma(n)$ -regular ring.

3 : Semi-commutative rings :

In this section we deal with the relation between strongly $\gamma(n)$ -regular rings and $\gamma(n)$ -regular rings which they are semi-commutative rings.

Definition 3.1 : [12]

A ring R is said to be a semi-commutative ring if every idempotent element is central .

Hence every reduced ring is semi-commutative ring .[12]

Theorem 3.2 :

For any ring R , the following statements are equivalent :

- (1) R is strongly $\gamma(n)$ -regular ring .
- (2) R is a semi-commutative $\gamma(n)$ -regular ring .

Proof :

(1) \Rightarrow (2) . From corollary 2.4 .

(2) \Rightarrow (1) . Let $0 \neq a \in R$, then there exists $b \in R$ such that $a = ab^n a$. Since ab^n is an idempotent element and R is semi-commutative ring , then ab^n is central . So $(ab^n)a = a(ab^n)$. That is $a = ab^n a = a^2 b^n$. Similarly , $a = b^n a^2$ [Since $b^n a$ is an idempotent element] . Therefore R is a strongly $\gamma(n)$ -regular ring . ■

Proposition 3.3:

Let R be a semi-commutative $\gamma(n)$ -regular ring . Then R is a unit regular ring .

Proof :

Let $a \in R$, then there exists $b \in R$ such that $a = ab^n a$. Hence ab^n and $b^n a$ are idempotent elements . So they are central elements .

Now $ab^n = a(b^n a)b^n = a(b^n(b^n a)) = (ab^n)(b^n a) = b^n(ab^n)a = b^n a$.

Let $u = a + ab^n - 1$ and $v = ab^n + ab^{2n} - 1$. Since $ab^n = b^n a$ and $a = ab^n a$, we have :

$$\begin{aligned} uv &= (a + ab^n - 1)(ab^n + ab^{2n} - 1) \\ &= a^2 b^n + a^2 b^{2n} - a + ab^n ab^n + ab^n ab^{2n} - ab^n - ab^n - ab^{2n} + 1 \end{aligned}$$

$$= a + ab^n - a + ab^n + ab^n b^n - ab^n - ab^n - ab^n b^n + 1 = 1 .$$

Similarly , $vu = 1$ and $ava = a(ab^n + ab^{2n} - 1)a = a$. Therefore R is a unit regular ring . ■

Corollary 3.4 :

Every strongly $\gamma(n)$ -regular ring is a unit regular ring .

Proof :

Let R be a strongly $\gamma(n)$ -regular ring , then it is reduced $\gamma(n)$ -regular ring [Corollary 2.4] . Since every reduced ring is semi-commutative ring , then by [proposition 3.3] R is a unit regular ring . ■

Definition 3.5 :

A ring R is called clean if each element can be expressed as the sum of a unit and idempotent .

Corollary 3.6 :

Let R be a strongly $\gamma(n)$ -regular ring . Then a^n is a unit $\gamma(n)$ -regular element for each $a \in R$.

Proof :

Since R is a strongly $\gamma(n)$ -regular ring , then R is a semi-commutative $\gamma(n)$ -regular ring . Hence a is a unit regular element [Corollary 3.4] .

Since R is $\gamma(n)$ -regular ring , then for $a \in R$ there exists $b \in R$ such that $a = ab^n a$. So ab^n and $b^n a$ are central idempotent elements . That is $a = ab^n a = a^2 b^n$.

Let $u = ab^{2n} - ab^n + 1$ and $v = a - ab^n + 1$, then $uv = vu = 1$ and $u^n = ab^n b^n - ab^n + 1$. Hence $a^n = a^n u^n a^n$. Therefore a^n is a unit $\gamma(n)$ -regular element . ■

Corollary 3.7 :

Let R be a reduced $\gamma(n)$ -regular ring . Then for each $a \in R$:

- (1) a^n is a unit $\gamma(n)$ -regular element .
- (2) $a = eu$ for some idempotent element $e \in R$ and unit element $u \in R$.
- (3) a is a clean element .

Proof :

- (1) follows from [Corollary 3.6] .
- (2) and (3) follows from [12] . ■

In the following ; $N(R)$ is the set of the nilpotent elements of R .

Proposition 3.8 :

Let R be a semi-commutative $\gamma(n)$ -regular ring . Then for each $a \in R$, $a^t = eu + x$ for some idempotent element e , unit element u and nilpotent element x .

Proof :

Since R is a semi-commutative $\gamma(n)$ -regular ring , then $R/N(R)$ is strongly regular [12] . Hence $R/N(R)$ is reduced , that is , semi-commutative ring . So for each $a+N(R) \in R/N(R)$, $a^t+N(R)$ is a unit $\gamma(n)$ -regular element [Corollary 3.6] . Hence there exists a unit $u+N(R)$ such that $a^t+N(R) = (a^t+N(R))(u^t+N(R))^n(a^t+N(R))$. That is u^n is a unit element in R [12] . Now since $a^t u^n + N(R)$ is an idempotent in $R/N(R)$ then $(a^t u^n + N(R))^2 = (a^t u^n a^t u^n) + N(R) = a^t u^n + N(R)$. Then there exists an idempotent element $e \in R$ such that $a^t u^n + N(R) = e + N(R)$ [6] . So , $a^t + N(R) = (a^t u^n + N(R))((u^n)^{-1} + N(R)) = e(u^n)^{-1} + N(R)$. Hence , $a^t - e(u^n)^{-1} \in N(R)$. Therefore $a^t = e(u^n)^{-1} + x$ for some $x \in N(R)$. ■

4 : Quasi-commutative rings :

The definition of quasi-commutative rings was introduced by Kandasamy [15] , that is a ring R with $ab = b^na$ for every pair $a, b \in R$ and for some positive integer n depending on a and b .

We now turn to the case where n is a fixed integer greater than one .

Definition 4.1 :

A ring R is said to be $q(n)$ -commutative , where n is an integer greater than one , if for every $1 \neq a \in R$ and $b \in R$, $ab = b^na$.

Proposition 4.2 :

Let R be a quasi-commutative $\gamma(n)$ -regular ring , then R is strongly regular ring .

Proof :

Let $a \in R$, then there exists $b \in R$ such that $a = ab^na$. Since R is a quasi-commutative ring , then $ab^n = (b^n)^ta = b^{nt}a$ for some positive integer t . Hence $a = ab^na = b^{nt}aa = b^{nt}a^2$. Also we have $b^na = a^mb^n$ for some positive integer m .

Then $a = ab^na = aa^mb^n = a^2(a^{m-1}b^n)$.

Therefore R is strongly regular ring . ■

Theorem 4.3 :

Let R be a $q(n)$ -commutative ring . Then the following statements are equivalent :

- (1) R is $\gamma(n)$ -regular ring .
- (2) R is strongly regular ring .

Proof :

(1) \Rightarrow (2) . Since R is a $q(n)$ -commutative ring , then R is quasi-commutative ring . Hence by [proposition 4.2], R is strongly regular ring.

(2) \Rightarrow (1) . Since R is strongly regular ring , then for every $a \in R$ there exists $b \in R$ such that $a = a^2b$.

Now since R is $q(n)$ -commutative ring , then $ab = b^na$. Hence $a = aab = ab^na$. Therefore R is $\gamma(n)$ -regular ring . ■

Corollary 4.4 :

Let R be a $q(n)$ -commutative reduced ring . Then the following statements are equivalent :

(1) R is $\gamma(n)$ -regular ring .

(2) R is regular ring .

Proof :

(1) \Rightarrow (2) . The proof is straight forward check which we omit .

(2) \Rightarrow (1) . Since R is reduced and regular , then R is strongly regular ring . Therefore R is $\gamma(n)$ -regular ring [Theorem 4.3] . ■

Corollary 4.5 :

The center of every reduced $q(n)$ -commutative $\gamma(n)$ -regular ring is $\gamma(n)$ -regular ring .

Proof :

Let R be a reduced $q(n)$ -commutative $\gamma(n)$ -regular ring . Then R is regular ring . So $\text{cent } R$ is regular [6] . Therefore $\text{cent } R$ is $\gamma(n)$ -regular ring [Corollary 4.4] . ■

Corollary 4.6 :

Let R be a $q(n)$ -commutative reduced ring such that every ideal of R is a maximal right ideal, then R is $\gamma(n)$ -regular ring.

Proof :

Since R is reduced and every prime ideal is maximal right ideal, then R is regular ring [7].

Now since R is $q(n)$ -commutative ring and reduced then R is $\gamma(n)$ -regular ring [Corollary 4.4]. ■

Corollary 4.7 :

Let R be a $q(n)$ -commutative reduced ring such that R/P is $\gamma(n)$ -regular ring for every prime ideal P in R . Then R is $\gamma(n)$ -regular ring.

Proof :

Let P be a prime ideal in R . Then R/P is division ring, because R/P is a $\gamma(n)$ -regular ring and with nonzero divisors. Therefore P is a maximal right(left) ideal in R and R is $\gamma(n)$ -regular ring [Corollary 4.6]. ■

Proposition 4.8 :

Let R be a $q(n)$ -commutative ring. Then the following statements are equivalent :

- (1) R is $\gamma(n)$ -regular ring.
- (2) $r(a)$ is direct summand for every $a \in R$.

Proof :

(1) \Rightarrow (2). From proposition 1.3.

(2) \Rightarrow (1). Let $a \in R$. Since $r(a)$ is direct summand, then $R = aR + r(a)$. Since $1 \in R$, then $1 = ar + d$ for some $r \in R$ and $d \in r(a)$. So,

$a = a^2r + ad$. That is $a = a^2r$. Since R is $q(n)$ -commutative ring, then $a = a(ar) = ar^na$. Therefore R is $\gamma(n)$ -regular ring. ■

Theorem 4.9 :

Let R be a $q(n)$ -commutative reduced ring. Then the following statements are equivalent :

- (1) R is $\gamma(n)$ -regular ring.
- (2) Every element $a \in R$ can be written as $a = ue$ for some idempotent $e \in R$ and unit $u \in R$.

Proof :

(1) \Rightarrow (2). From Corollary 3.7.

(2) \Rightarrow (1). Let $a = eu$, where e is idempotent and u is unit, then $e = au^{-1}$. Now $ea = au^{-1}a$, but $ea = eeu = e^2u = eu = a$. So $a = au^{-1}a$ which is regular element.

Therefore by [Corollary 4.4] R is $\gamma(n)$ -regular ring. ■

Theorem 4.10 :

Let R be a $q(n)$ -commutative ring. Then the following statements are equivalent :

- (1) R is $\gamma(n)$ -regular ring.
- (2) For every right ideal I and left ideal J in R , $IJ = I \cap J$.

Proof :

(1) \Rightarrow (2). Since R is $\gamma(n)$ -regular ring, then it is regular. Hence (2) holds by [9; Theorem 1.1.7].

(2) \Rightarrow (1). Let $a \in R$. Since $a \in aR \cap Ra = aRa$. So $a = aba$ for some $b \in R$. Since R is $q(n)$ -commutative ring, then $a = (ab)a = b^na^2$. Hence R is strongly $\gamma(n)$ -regular ring. Therefore R is $\gamma(n)$ -regular ring. ■

Theorem 4.11 :

Let R be a $q(n)$ -commutative reduced ring such that every principal left ideal of R is left annihilator, then R is $\gamma(n)$ -regular ring.

Proof :

Since R is reduced and every principal left ideal of R is a left annihilator, then R is strongly regular ring [9]. Since R is $q(n)$ -commutative, then by [Theorem 4.3] R is $\gamma(n)$ -regular ring. ■

Proposition 4.12 :

Let R be $q(n)$ -commutative ring. Then the following statements are equivalent :

- (1) R is $\gamma(n)$ -regular ring.
- (2) Every principal ideal is a direct summand.

Proof :

(1) \Rightarrow (2). Clearly from [10].

(2) \Rightarrow (1). Let $R = aR \oplus K$ for some ideal K of R . Since $1 \in R$, then $1 = ar + k$ for some $r \in R$ and $k \in K$.

Since R is $q(n)$ -commutative ring, then $ar = r^n a$.

So, $1 = r^n a + k$. That is $a = ar^n a + ak$.

Hence $ak = a - ar^n a \in aR \cap K = \{0\}$. So $a = ar^n a$. Therefore R is $\gamma(n)$ -regular ring. ■

Theorem 4.13 :

Let R be a $q(n)$ -commutative reduced ring such that every maximal ideal of it is a right annihilator, then R is $\gamma(n)$ -regular ring.

Proof :

Let $a \in R$. Suppose that $aR + r(a) \neq R$, then there exists a maximal right ideal M containing $aR + r(a)$.

So $M = r(b)$ for some $b \in R$.

Hence $b \in l(aR + r(a)) \subseteq l(a) = r(a)$ [9 ;Theorem 1.3.10].

So $b \in M = r(b)$. Then $b^2 = 0$. Since R reduced, then $b = 0$, a contradiction. Hence $aR + r(a) = R$. Now $1 \in R$ implies $1 = ax + y$ for some $x \in R$ and $y \in r(a)$. So $a = a^2x + ay$, that is $a = a^2x$. Hence R is strongly regular ring.

Therefore R is $\gamma(n)$ -regular ring [Theorem 4.3]. ■

Theorem 4.14 :

Let R be a $q(n)$ -commutative ring. Then R is $\gamma(n)$ -regular ring if and only if each ideal in R is a radical ideal.

Proof :

Suppose that R is $\gamma(n)$ -regular ring. Then R is regular ring. Hence every ideal is radical ideal [11].

Conversely : Let $I = \sqrt{I}$ for each ideal I in R . Take $I = a^2R = \sqrt{a^2R}$. Then $a^2 \in a^2R$. So $a \in \sqrt{a^2R} = a^2R$. That is $a = a^2b$ for some $b \in R$. Hence R is strongly regular ring. Therefore by [Theorem 4.3] R is $\gamma(n)$ -regular ring. ■

Corollary 4.15 :

Let R be a $q(n)$ -commutative ring. Then R is $\gamma(n)$ -regular ring if and only if each ideal in R is semi-prime.

Theorem 4.16 :

Let R be a $q(n)$ -commutative regular ring, then R is strongly $\gamma(n)$ -regular ring.

Proof :

Let $a \in R$, then there exists $b \in R$ such that $a = aba$. Since R is $q(n)$ -commutative ring, then $ab = b^na$. So $a = a^2b^n$. Therefore R is strongly $\gamma(n)$ -regular ring. ■

Theorem 4.17 :

Let R be $q(n)$ -commutative ring, then the following statements are equivalent :

- (1) R is strongly $\gamma(n)$ -regular ring .
- (2) R is strongly regular ring .

Proof :

(1) \Rightarrow (2) . The proof is straight forward check which we omit .

(2) \Rightarrow (1) . Since R is strongly regular ring, then there exists $b \in R$ such that $a = a^2b$. Since R is $q(n)$ -commutative ring then $ab = b^na$. Hence $a = ab^na$. Since R is strongly regular ring, then R is reduced. So $a = a^2b^n$. Therefore R is strongly $\gamma(n)$ -regular ring. ■

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