

On Generalized π - Regular Rings

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Abstract:

The purpose of this paper is to study right generalized π -regular rings and give some of its properties. Also, we proved:

- 1- Let R be a right generalized π -regular ring without zero divisors element. Then R is a division ring.
- 2- Let R be a ring with $l(a^n) \subseteq r(a^n)$, for every $a \in R$ and $n \in \mathbb{Z}^+$. Then R is regular. If and only if R is a right generalized π -regular ring.

π حول الحلقات المنتظمة المعممة من النمط

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ملخص البحث :

- الغرض من هذا البحث هو دراسة الحلقات المنتظمة المعممة من النمط π وإعطاء بعض من هذه الخواص. كذلك برهنا:
1. لتكن R حلقة منتظمة معممة من النمط π يميني ولا تحتوي على عنصر قاسم للصفر عند ذلك تكون R حلقة القسمة.
 2. لتكن R حلقة و $l(a^n) \subseteq r(a^n)$ لكل $a \in R$ و $n \in \mathbb{Z}^+$ عندها تكون R حلقة منتظمة اذا فقط اذا كانت R حلقة منتظمة معممة من النمط π يميني.

1. Introduction:

Throughout this paper, R denote an associative rings with identity. The right singular ideal and the Jacobson radical of a ring R are denoted by $Y(R)$ and $J(R)$, respectively. We say that R is right duo ring if all right ideal are ideal of R [6]. We say that ring R is regular if for all $a \in R$, there exists $b \in R$ such that $a=aba$. This concept was first introduced by von Neumann[8] and Chen [4]. As a generalization of this concept Azumaya[3] introduced π -regular rings as a ring R which π -regular if for every $a \in R$, $\exists n \in \mathbb{Z}^+$ and $b \in R$ such that $a^n = a^n b a^n$. An ideal I of the ring R is said to be regular if for all $a \in I$, there exists $b \in I$ such that $a=aba$.

2. Generalized π -regular ring:

Definition 2.1: [4]

Let $0 \neq a \in R$, we say that a is a right (left)generalized π -regular, if there exists a positive integer n such that $a^n = aba^n (a^n b a^n)$, for some, $b \in R$.

We call to the ring R is said to be a right (left) generalized π -regular if and only if every element in R is a right (left) generalized π -regular. If R is right and left generalized π -regular, we called R to be generalized π -regular ring.

Every regular and π -regular is generalized π -regular.

Examples:

1- $\mathbb{Z}_6, \mathbb{Z}_{10}, \mathbb{Z}_{14}$ and \mathbb{Z}_{15} is a generalized π -regular ring.

2- Let \mathbb{Z}_2 be a ring of integers modulo 2, and let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Therefore

R is a generalized π -regular ring.

Definition 2.2:

A right ideal I of a ring R is said to be right (left) generalized π -regular ideal if and only if for all $b \in I$, b is right (left) generalized π -regular element.

Proposition 2.3 :

Let R be a right generalized π -regular ring, then every two sided ideal I of R is a right generalized π -regular.

Proof :

Let I be an ideal of ring R . For any $x \in I$, since R is right generalized π -regular, there exists a positive integer n and $y \in R$ such that $x^n = xyx^n$, $x^n = xyxyx^n$, we set $z = yxy$, then we have $x^n = xzx^n$ with $z \in I$ (since $x \in I$ and I is an ideal). Hence I is a right generalized π -regular .

Proposition 2.4:

Let R be a ring if every principal right ideal is a right generalized π -regular, then R is a right generalized π -regular.

Proof :

It is clear.

Theorem 2.5:

Let R be a right generalized π -regular ring and I is a right ideal. Then R/I is a right generalized π -regular .

Proof :

Let $a + I \in R/I$, where $a \in R$. Since R is a generalized π -regular ring, then there exists a positive integer n such that $a^n = aba^n$ for some $b \in R$, then $a^n + I = aba^n + I = (a + I)(b + I)(a^n + I)$. Therefore, R/I is a right generalized π -regular ring .

Theorem 2.6:

Let I be an ideal of a ring R . If R/I is a right generalized π -regular ring and I is regular, then R is a right generalized π -regular ring.

Proof :

Let $a \in R$, then $a + I \in R/I$, if $a + I = I$ we have $a \in I$, since I is a regular, there exists $d \in I$ such that $a = ada$, for any positive integer m , $a^m = ada^m$ and done for all $a \in I$, now if $a + I \neq I$, so there exists a positive integer n ($1 \neq n$) such that

$(a + I)^n = (a + I)(b + I)(a + I)^n$ for some $b \in R$, $a^n + I = aba^n + I$. Therefore, $a^n - aba^n \in I$, since I is regular, then there exist $c \in I$ such that

$$\begin{aligned} a^n - aba^n &= (a^n - aba^n)c(a^n - aba^n) \\ &= (a^n c - aba^n c)(a^n - aba^n) \\ &= a^n c a^n - a^n c ab a^n - a b a^n c a^n + a b a^n c a b a^n \\ a^n &= a^n c a^n - a^n c a b a^n - aba^n c a^n + a b a^n c a b a^n + a b a^n \\ a^n &= a(a^{n-1}c - a^{n-1}cab - ba^n c + ba^n cab + b) a^n \end{aligned}$$

set $h = a^{n-1}c - a^{n-1}cab - ba^n c + ba^n cab + b \in R$

So $a^n = a h a^n$. Therefore, R is a right generalized π -regular ring .

Proposition 2.7:

Let R be a right generalized π -regular ring. Then :

- (1) $J(R)$ is nil ideal.
- (2) $Y(R)$ is nil ideal.

Proof (1) :

Let $a \in J(R)$. Since R is a right generalized π -regular ring, then there exists, a positive integer n such that $a^n = a b a^n$ for some $b \in R$, $(1 - ab)a^n = 0$, Since $a \in J(R)$ then $1 - ab$ is invertible. Therefore must $a^n = 0$ for all $a \in J(R)$. So $J(R)$ is nil ideal.

Proof (2):

Let $a \in Y(R)$. Since R is a right generalized π -regular ring, then there exists a positive integer n such that $a^n = aba^n$ for some $b \in R$. Since $a \in Y(R)$ then $r(ab)$ is essential right ideal of R . So $r(ab) \cap a^n R \neq 0$. There exists $0 \neq x \in r(ab) \cap a^n R$, $abx = 0$ and $x = a^n r = aba^n r = abx = 0$, but $r(ab)$ is essential. Therefore $a^n R = 0$ which lead us to $a^n = 0$ for all $a \in Y(R)$. So $Y(R)$ is a nil ideal.

A ring R is called reversible if $ab = 0$ implies $ba = 0$ for every $a, b \in R$ [5].

Theorem 2.8:

Let R be a reversible ring and $a \in R$. If $r(a)$ is right generalized π -regular ring then for any positive integer $m > 1$ and $x \in r(a^m)$, $a^{m-1}x$ is a nilpotent element.

Proof :

Let $x \in r(a^m)$. Then $a^m x = 0$, $aa^{m-1}x = 0$, and hence $a^{m-1}x \in r(a)$. Since $r(a)$ is generalized π -regular, there exists a positive integer n with $(1 \neq n)$ and $y \in r(a)$ such that $(a^{m-1}x)^n = a^{m-1}x y (a^{m-1}x)^n$ we have $y a = 0$ (since $y \in r(a)$, $ay = 0 = y a$, R is reversible ring) so $(a^{m-1}x)^n = 0$, it is a nilpotent element.

3. The connection between generalized π -regular rings and other rings:

Proposition 3.1:

Let R be a reversible ring. Then R is a right generalized π -regular ring if and only if $aR + r(a^n) = R$ for all $a \in R$ and $n \in \mathbb{Z}^+$.

Proof :

Let R be a right generalized π -regular, and $a \in R$, then there exists a positive integer n such that $a^n = aba^n$, for some $b \in R$. If $aR + r(a^n) \neq R$, then there exists a maximal right ideal M of R such that $aR + r(a^n) \subseteq M$. Since

$a^n = aba^n$, $(1-ab)a^n = 0$, since R is reversible $a^n(1-ab) = 0$, $1-ab \in r(a^n) \subseteq M$. So $1 \in M$, which is a contradiction therefore $aR + r(a^n) = R$ for all $a \in R$.

Conversely : Assume that $aR + r(a^n) = R$, for all $a \in R$ and $n \in \mathbb{Z}^+$. Hence $ab + d = 1$, $a \in R$ and $d \in r(a^n)$ so $aba^n + da^n = a^n$, since $d \in r(a^n)$, $a^n d = 0$, R is reversible $da^n = 0$, so $aba^n = a^n$ for all $a \in R$. Therefore R is a right generalized π -regular ring.

Theorem 3.2:

Let R be a right generalized π -regular ring without zero divisors elements, then R is a division ring.

Proof :

Let $0 \neq a \in R$. Since R is a right generalized π -regular ring, then there exists a positive integer n such that $a^n = aba^n$, for some $b \in R$, implies that $(1-ab)a^n = 0$. Since R without zero divisors then either $a^n = 0$ or $1-ab = 0$. If $a^n = 0$ then a is a zero divisors which a contradiction. Thus $1-ab = 0$, $1 = ab$, so a is a right invertible element. Let $b \in R$ and R is a right generalized π -regular ring, then there exists a positive integer $m(m \neq 1)$ such that $b^m = bcb^m$, for some $c \in R$, gives $1 = bc$. Now $a = a.1 = abc = 1c$, so $a = c$, therefore R is a division ring.

Theorem 3.3:

Let R be a ring, $l(a^n) = l(a)$ for all $a \in R$ and any positive integer n . Then the following condition are equivalent.

- 1- R is regular.
- 2- R is right generalized π -regular.

Proof :

1 \rightarrow 2 It is clear.

2 \rightarrow 1 Let $a \in R$, since R is a right generalized π -regular ring, then there exists a positive integer n such that $a^n = aba^n$, for some $b \in R$, $(1-$

$ab)a^n = 0$, $(1- b a) \in l(a^n) = l(a)$. So $(1-a b)a = 0$. $a=aba$ for all $a \in R$. Therefore, R is a regular ring.

A right ideal I of a ring R is said to be GP-ideal, if for all $a \in I$ there exists $b \in I$ such that $a^n = a^n b$ [7].

Theorem 3.4:

Let R be a right duo ring. Then R is right generalized π -regular if and only if every principal right ideal is left GP-ideal.

Proof :

Let R be a right generalized π -regular and $a \in R$. Let $x \in aR$, since R is a right generalized π -regular ring, then there exists a positive integer n such that $x^n = xbx^n$, for some $b \in R$, set $xb \notin aR$, so $d \in aR$, $x^n = dx^n$ for all $x \in aR$ then aR is a left GP-ideal.

conversely

Let every principal right ideal is left GP-ideal and $a \in R$, since $a \in aR$, there exists a positive integer n such that $a^n = ba^n$, for some $b \in aR$, now there exists $c \in R$ such that $b = ac$, therefore $a^n = aca^n$ that is hold for all $a \in R$. Then R is right generalized π -regular ring.

A ring R is said to be right (left) semi π -regular if for all $a \in R$, there exists a positive integer n and an element $b \in R$, such that $a^n = a^n b$ ($a^n = ba^n$) and $r(a^n) = r(b)$ ($l(a^n) = l(b)$) [2].

Theorem 3.5:

Let R be a right generalized π -regular ring and for any non zero element x, y in a ring R , $Rxy \cap l(a^n) = 0$ for any positive integer n . Then R is a left semi-regular ring.

Proof :

Let $a \in R$, since R is a right generalized π -regular, then there exists a positive integer n such that $a^n = aba^n$, for some $b \in R$. Set that $c = ab$, $x^n = cx^n$, now we to show that $l(a^n) = l(c)$. Let $x \in l(c)$, then $xc = 0$, $xab = 0$, $xaba^n = 0$, $xa^n = 0$, $x \in l(a^n)$, which yield to $l(c) \subseteq l(a^n)$ ------(1). Let $y \in l(a^n)$, $ya^n = 0$. (since $a^n = aba^n$), so $yaba^n = 0$ ($c = ab$), $yc a^n = 0$, $yc \in l(a^n)$ also $yab = yc \in Rab$, but $Rab \cap l(a^n) = 0$, must $yc = 0$, $y \in l(c)$ which yield to $l(a^n) \subseteq l(c)$ -----(2). From (1) and (2) we get that $l(a^n) = l(c)$. Therefore R is a left semi-regular ring.

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