Theorems on Certain Fractional Function and Derivative

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ABSTRACT

The aim of this paper is to prove the correctness of the relation

$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}\ \overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\,\mathrm{h}_a(x) = \mathrm{h}_a(x)$ for all $x \in (a,b]$, **$0<a\leq 1$, after we prove the continuity of the fractional function**

$\mathrm{h}_a(\mathrm{x}) = \dfrac{m(\mathrm{x}-\mathrm{a})^{1-a}}{\Gamma(a)} + \dfrac{1}{\Gamma(a)}\int (x-t)^{a-1} f(g(t))dt$ **in (a,∞), where $|h_a(x)| \leq M$ for all $x \in (a,\infty)$ and $(M \in R+, M>0)$.**

Keywords: Fractional function, derivative.

نظريات في الدوال الكسرية الخاصة مع المشتقات

الخلاصة

الهدف الرئيسي في هذا البحث هو برهان صحة العلاقة

$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}\ \overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\,\mathrm{h}_a(x) = \mathrm{h}_a(x)$ for all $x \in (a,b], 0 < a \leq 1$ **بعد ان نبرهن استمرارية الدالة**

$\mathrm{h}_a(\mathrm{x}) = \dfrac{m(\mathrm{x}-\mathrm{a})^{1-a}}{\Gamma(a)} + \dfrac{1}{\Gamma(a)}\int (x-t)^{a-1} f(g(t))dt$ **في الفترة (a,∞) حيث ان**

$|h_\alpha(x)| \leq M$ و$(M\in R+, M>0)$ لكل $x\in (a,\infty)$.

INTRODUCTION

Fractional Calculus is three Centuries old as the conventional calculus, but not very popular among science and engineering community. This subject was with mathematicians only; in last few years this was pulled to several applied fields of science, economies and engineering. The advantage of fractional derivative apparent in mechanics and stability analysis of fractional control of robotic time delays systems by [1], [2] and in nuclear energy science by [3], [4]

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also in the other physical and chemistry fields by [5], [6] and in computer hard disc by control [7]. In this paper we shall prove that:

$$\underset{a}{\overset{x}{\mathrm{I}}}{}^{-a}\,\underset{a}{\overset{x}{\mathrm{I}}}{}^{a}\,\mathrm{h}_a(x)=\mathrm{h}_a(x),$$

$$\frac{d^a f}{dx^a}=f^a(x)=\underset{a}{\overset{x}{\mathrm{I}}}{}^{-a} f \quad \text{for all } x\in(a,b],$$

$0<\alpha\leq 1$, after we prove the continuity of the function

$$\mathrm{h}_a(x)=\frac{m(x-a)^{1-a}}{\Gamma(a)}+\frac{1}{\Gamma(a)}\int_a^x (x-t)^{a-1} f(g(t))dt$$

on (a,∞) for all $x>a$, $0<\alpha\leq 1$, where μ is constant, f, g are continuous functions on (a,∞).

THEOREM OF CONTINUITY

Let $0<\alpha\leq 1$ and f, g be continuous functions on (a,∞), where $a\in R$ and such that $\sup\{|f(g(x))|: x\in(a,\infty)\}=M<\infty$.

Define $\mathrm{h}_a(x)=g_1(x)+\dfrac{1}{\Gamma(a)}\displaystyle\int_a^x (x-t)^{a-1} f(g(t))dt$ for all $x>a$, where $g_1=\dfrac{m(x-a)^{1-a}}{\Gamma(a)}$, μ is constant. Then $h_\alpha(x)$ is continuous on (a,∞).

Proof

Clearly $g_1(x)$ is continuous for all $x>a$.

$$\text{Let } y(x)=\int_a^x (x-1)^{a-1} f(g(t))dt \quad \text{for all } x\geq a \qquad \ldots(2.1)$$

and if $x_1, x_2\in[a,\infty)$ such that $x_2>x_1$

[The proof is similar for the case $x_1>x_2$]

Then from (2.1) we have

$$|y(x_2)-y(x_1)|=\left|\int_a^{x_2}(x_2-t)^{a-1} f(g(t))dt-\int_a^{x_1}(x_1-t)^{a-1} f(g(t))dt\right|$$

$$=\left|\int_a^{x_1}(x_2-t)^{a-1} f(g(t))dt+\lim_{d\to 0}\int_{x_1}^{x_2}(x_2-d-t)^{a-1} f(g(t))dt-\lim_{d\to 0}\int_a^{x_1-d}(x_1-d-t)^{a-1}dt\right|$$

Since f is bounded by M and $0<\alpha\leq 1$, $x_2>x_1$ it follows that

$$|y(x_2)-y(x_1)|=\left|\lim_{d\to 0}\int_a^{x_1-d}\{(x_2-t)^{a-1}-(x_1-d-t)^{a-1}\} f(g(t))dt+\lim_{d\to 0}\int_{x_1}^{x_2-d}(x_2-d-t) f(g(t))dt\right|$$

$$=\lim_{d\to 0}\int_{0}^{x_1-d}\{(x_1-d-t)^{a-1}-(x_2-t)^{a-1}\}dt$$

$$\leq \lim_{d\to 0}\int_{a}^{x_1-d} M\left|\{(x_2-t)^{a-1}-(x_1-d-t)^{a-1}\}\right|dt + M\lim_{d\to 0}\int_{x_1}^{x_2-d}(x_2-d-t)^{a-1}dt$$

$$+\lim_{d\to 0}\int_{x_1}^{x_2-d} M(x_2-d-t)^{a-1}dt$$

$$\leq M\lim_{d\to 0}\left[\left.\frac{-(x_1-d-t)^{a-1}}{a}\right\}_{0}^{x_1-d}-\left.\frac{-(x_2-d)^{a}}{a}\right\}_{a}^{x_1-d}\right]=M\frac{(x_2-a)^a}{a}+M\frac{(x_2-x_1)^a}{a}-$$

$$+M\lim_{d\to 0}\left.\frac{-(x_2-d-t)^{a}}{a}\right\}_{x_1}^{x_2-d} \qquad M\frac{(x_2-a)^a}{a}+M\frac{(x_2-x_1)^a}{a}$$

$$\leq 2M\frac{(x_2-x_1)^a}{a}, \text{ because}$$

$$M\frac{(x_1-a)^a}{a}<M\frac{(x_2-a)^a}{a}$$

Let δ be a positive number such that $x_2-x_1<\delta$, it follows

$$|y(x_2)-y(x_1)|<\frac{2Md^a}{a}$$

Now let ε be any positive number and choose δ such that

$d=\left(\dfrac{ae}{2M}\right)^{\frac{1}{a}}$ Then we have $|y(x_2)-y(x_1)|<e$ when ever $|x_2-x_1|<d$

This shows that y(x) is continuous on $[a,\infty)$.

Since $h_\alpha(x)=g_1(x)+y(x),(x\in(a,\infty))$ therefore h_α is continuous on (a,∞).

THEOREM OF DERIVATIVE

If $0<\alpha\leq 1$ and $h_\alpha(x)$ is continuous on (a,b],

$|h_\alpha(x)|\leq M$ for all $x\in(a,b]$, where ($M\in R+$, $M>0$).

Then

$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}\ \overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\ \mathrm{h}_a(x)=\mathrm{h}_a(x)$ for all $x\in(a,b]$, where $\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}$ and $\overset{x}{\underset{a}{\mathrm{I}}}{}^{a}$ are define by [8].

Proof

Since $h_\alpha(x)$ is continuous and bounded by (theorem2).

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\ \mathrm{h}_a(x) \text{ exists for } a\leq t\leq x\leq b.$$

Now since $0<\alpha\leq 1$ take n=1 then by [8] p(11) we have

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}\left(\overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\ \mathrm{h}_a\right)=\overset{1}{\underset{x}{D}}\ \overset{x}{\underset{a}{\mathrm{I}}}{}^{1-a}\left(\overset{t}{\underset{a}{\mathrm{I}}}{}^{a}\ \mathrm{h}_a\right)$$ and from [8] p(17) we get

$$\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}\ \overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\ \mathrm{h}_a=\overset{1}{\underset{x}{D}}\overset{x}{\underset{a}{\mathrm{I}}}{}^{1-a+a}\mathrm{h}_a$$

$$=\overset{1}{\underset{x}{D}}\overset{x}{\underset{a}{\mathrm{I}}}{}^{1}\ \mathrm{h}_a(\mathrm{x})=\overset{1}{\underset{a}{D}}\int_a^x \mathrm{h}_a(\mathrm{x})\mathrm{dt}=\mathrm{h}_a(\mathrm{x})$$ for all $\mathrm{x}\in(\mathrm{a,b}]$

Hence $\overset{x}{\underset{a}{\mathrm{I}}}{}^{-a}\ \overset{x}{\underset{a}{\mathrm{I}}}{}^{a}\ \mathrm{h}_a(x)=\mathrm{h}_a(x)$ for all $\mathrm{x}\in(\mathrm{a,b}]$.

CONCLUSIONS

In this paper we proved (theorem1) of continuity and (theorem2) of derivative. In the future we will use them to find the solution of certain kind of fractional differential equation.

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