# Empirical Bayes estimators for Reliability Function of Inverse Rayleigh Distribution

# مقدرات بيز التجريبية لدالة المعولية لتوزيع معكوس رايلي

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**Abstract**

The Inverse Rayleigh distribution (IRD) used in various areas of statistics such as life testing and reliability and failure times of components tested, and healthy areas such as life times (in months) of bladder cancer patients. We used the classical method which is represented by the maximum likelihood estimation (MLE) and Bayes approach for the Reliability function for the Inverse Rayleigh distribution (IRD). We assumed that the scale parameter of the IRD is random variable, and we have two different informative prior distributions for the scale parameter of the IRD. So, we try to determine the suitable choice of informative prior distributions in bayes analysis for the Reliability function of the IRD at time $t = t_0$, which are represented by Inverse Gamma - Inverse Chi-square, Inverse Gamma - Standard Levey, Inverse Chi-square - Standard Levy distribution as double prior distributions. We derived the posterior density for the scale parameter of the IRD under three different double informative priors and we derived bayes' estimators for the Reliability function of IRD at time $t = t_0$ based on Squared Error Loss Function (SELF).A simulation study performed to obtain the empirical results of this study, we assumed different cases for the true values of the scale parameter of an IRD under different double prior distributions for different sample sizes(n). A comparison is made between the results based on the minimum Mean Square Error (MSE) criterion. We concluded that the best estimation for the Reliability function when the true value for scale parameter is equal 0.5, under the double prior is the Inverse Gamma and the Inverse Chi-square distribution with $(a = 0.5, b = 3, \nu = 10)$ at time $t_0 = 1.5, 2.5$ for sample sizes equal to 25, 50 and 100 and the maximum likelihood estimators (MLE's) when the true value for scale parameter is equal 1.5 and 2.5 at time $t_0 = 0.5, 1.5, 2.5, 3.5$ for small sample sizes. We recommend to use the double prior of the Inverse Gamma and the Inverse Chi-square distribution, to estimate reliability function using bayes analysis of the reliability function of IRD based on loss functions such as quadratic loss function and he general entropy loss function and the modified linear exponential (MLINEX) loss function to compare the accuracy of the different estimates.

**Keywords: Reliability Function, Maximum Likelihood, Informative Prior, Posterior Distribution, Squared Error Loss Function. Mean Square Error.**

**المستخلص**

توزيع معكوس رايلي استعمل في مجالات مختلفة من الإحصاء مثل اختبار الحياة والمعولية واوقات الفشل لمكونات مختبرة, المجالات الصحية مثل أوقات البقاء (بالاشهر) لمرضى سرطان المثانة. استعملت الطريقة الكلاسيكية المتمثلة بطريقة الإمكان الأعظم (MLE) وأسلوب بيز لدالة المعولية لتوزيع معكوس رايلي (IRD). افترضنا بان معلمة القياس متغير عشوائي, ولدينا توزيعان اوليان معلوماتية لمعلمة القياس لتوزيع معكوس رايلي. لذا نحاول تحديد التوزيعيين الأوليين المعلوماتية الملائمة في تحليل بيز لدالة المعولية لتوزيع معكوس رايلي عند وقت $(t = t_0)$, التي مثلت بتوزيع (معكوس كاما – معكوس مربع كاي) وتوزيع

(معكوس كاما –ليفي القياسي) وتوزيع( معكوس مربع كاي – ليفي القياسي) كتوزيعات أولية مضاعفة. فقد تم اشتقاق التوزيع اللاحق لمعلمة القياس لتوزيع IRD بافتراض ثلاث توزيعات أولية مضاعفة واشتقينا مقدرات بيز لدالة المعولية لتوزيع IRD بالوقت ( $t = t_0$ ) باستعمال دالة مربع الخسارة (SELF).

نفذت دراسة المحاكاة للحصول على النتائج التجريبية لهذه الدراسة, مفترضين حالات مختلفة للقيم الحقيقية لمعلمة القياس لتوزيع IRD ولنماذج مختلفة للتوزيعات الأولية المضاعفة ولاحجام مختلفة من العينات. وقد اعتمد معيار اقل متوسط لمربعات الخطأ (MSE) للمقارنة بين نتائج البحث. واستنتجنا بان افضل تقدير لدالة المعولية عندما تكون القيمة الحقيقية لمعلمة القياس مساوية لـ $0.5$, باستعمال مقدر بيز الأفضل باستعمال توزيع (معكوس كاما – معكوس مربع كاي) كتوزيعين اوليين مضاعفة عند القيم المفترضة للمعلمات الإضافية $(a = 0.5, b = 3, v = 10)$ عند الوقت $(t_0=1.5,2.5)$ لاحجام العينات $(n \le 100)$ . ومقدرات الإمكان الأعظم عندما تكون القيمة الحقيقية لمعلمة القياس مساوية لـ $1.5$ و $2.5$ عند الأوقات $(t_0=0.5,1.5,2.5, 3.5)$ لحجوم العينات الصغيرة.نوصي باستعمال التوزيعين الأوليين المضاعفين لتوزيع معكوس كاما وتوزيع معكوس مربع كاي لتقدير دالة المعولية لتوزيع معكوس رايلي , باستعمال تحليل بيز بافتراض دوال خسارة أخرى مثل دالة الخسارة التربيعية و دالة خسارة الانتربي العامة ودالة الخسارة الاسية الخطية المحورة (MLINEX) لمقارنة دقة التقديرات المختلفة .

**الكلمات المفتاحية: دالة المعولية, الامكان الأعظم, المعلومة الأولية, التوزيع اللاحق, دالة الخسارة التربيعية.**

## 1. Introduction

The Inverse Rayleigh Distribution (IRD) used in many applications in the survival analysis and reliability problems studies. Many studies used different estimation methods of the unknown parameter and reliability function of the Inverse Rayleigh Distribution by assuming single prior distributions for comparison. We review some of these studies, such as Dey in 2005 [1] derived Bayes estimators of the parameter and reliability function for the inverse Rayleigh distribution. Also he used maximum likelihood method to estimate the parameter and reliability function. And he derived the Credible intervals and highest posterior density intervals to the parameter and the reliability function. He concluded that the relative error of MLE of the scale parameter and Reliability of Inverse Rayleigh distribution were greater than those of their Bayesian estimators, when the scale parameter is equals to 0.5 and 1 and t=1.Also, EI-Helbawy and Abd-EI-Monem in 2005 [2] studied Bayes estimation to estimate the parameter of the Inverse Rayleigh Distribution (IRD) they derived estimators of the parameter under four loss functions and study the Bayes prediction intervals. In 2010 Soliman et al. [3] used Bayesian and non-Bayesian estimation of the parameter of the Inverse Rayleigh Distribution. Also they derived Bayesian prediction based on a lower record values, under exponential distribution as prior distribution for the scale the parameter of the IRD. They noted that the best estimation of Bayesian prediction interval for the s=3, 4, 5, 6 future record when the values of the prior parameters equals to 0.5. And Dey in 2012 [4] studied Bayes estimators for the parameter and Reliability function of an Inverse Rayleigh Distribution under Non-informative prior distribution based on two different loss function. He noted that Bayes estimators under the LINEX loss function is best of Bayes estimators under the squared error loss function. In 2012 Shawkyn and Badr [5] used the maximum likelihood based on lower record values and Bayesian estimation to estimate the unknown parameter of the Inverse Rayleigh Distribution (IRD), the Reliability, cumulative failure rate function. They derived Bayes estimators under Gamma distribution prior based on two different loss function and prediction interval for the Inverse Rayleigh distribution. They concluded that the Bayes estimator for the Reliability best of the MLEs according to the smallest MSE. And Radha in 2020 [6] discussed Bayes estimation for the unknown parameter of Inverse Rayleigh Distribution under three different double prior as a combination of informative and non-informative prior which are Jeffrey-Gamma, Jeffrey-Chi-square, Gamma-Chi-square, single informative prior which is Gamma distribution. He depends on simulation study to obtain the results of this study. He compared with these informative and non-informative priors on the basis of posterior variances, BIC and AIC. He concluded that Bayes estimation for the scale parameter of Inverse Rayleigh under Gamma- Chi- Square priors best of the other priors according to the smallest values for the Akaike Information Criterion (AIC) and Bayesian information criterion (BIC). In 2021 Sabry et.al. [7] used classical methods and Bayesian methods to estimate the Reliability parameter for Inverse Rayleigh Distribution. Also, they used the maximum product of spacing and maximum likelihood estimation methods to estimate the Reliability parameter. They introduced the maximum product of spacing under fuzzy reliability of stress strength model to estimate the Reliability parameter. They used Monte Carlo simulation analysis to obtain Bayesian estimators of traditional and fuzzy reliability of stress strength. They concluded that the Maximum product of spacing estimation (MPS) method is best of the MLE and Bayesian methods for most cases. In 2022 Rani et.al. [8] study Reliability subjected to common stress using Inverse Rayleigh distribution if the variables be strengths of n- components under one stress. They used Bayesian estimation under Jeffery's prior and Exponential prior distribution. The results of this study obtained by Matlab. They conclude that Jeffrey's prior information is the best estimator than with Exponential prior. They concluded that the Bayes estimator for the Reliability under Jeffrey's prior information is the best estimator than with Exponential prior.

In Bayes analysis requires appropriate choice of informative prior distributions for the parameters. In 2006 Singpurwalla [9] find that it is useful to use the non-informative priors. However, many authors used the two different kind of information as double priors in Bayes analysis, such as Radha and Vekatesan in 2013 [10] used general uniform and inverse gamma distribution as double priors for the unknown parameter of Maxwell distribution. They proved that inverse gamma distribution is the posterior distribution. Also, Patel, R. and Patel, A. in 2017 [11] derived the posterior distribution under double priors which are represented by exponential-gamma distribution, gamma-chi-square distribution, chi-square-exponential distribution as double priors and Gamma distribution as single prior for the unknown parameter of the exponential distribution. They study exponential life time model under type -II censoring. And they obtained Bayes estimators of the parameter and Reliability at time t of Exponential life time model based on SELF. They noted that the Bayes estimator of parameter θ and length of its credible intervals under Exponential-Gamma as the double prior yields smallest values of MSE.

The aim of this study, we investigate the appropriate choice of informative prior distributions in Bayes analysis by using two different informative prior distributions of the Reliability function at time t=t0 for the Inverse Rayleigh distribution (IRD). We have assumed the Inverse Gamma and the Inverse Chi-square, the Inverse Gamma and the Standard Levey, the Inverse Chi-square and the Standard Levy distribution as double priors. And we derived the posterior density using the different double informative priors and bayes' estimators for the Reliability function at time $t = t_0$ of IRD based on SELF. Additionally, the classical estimator represented by the maximum likelihood estimator. We used simulation to obtain the results of this study and comparison is made between the results based on the Mean Square Error (MSE) criterion.

## 2. The Inverse Rayleigh Distribution (IRD)

The probability density function (pdf) of Inverse Rayleigh distribution (IRD) random variable t is given by [4, 8]:

$$g(t;\delta) = \frac{2}{\delta t^3}\exp\left(-\frac{1}{\delta t^2}\right) \quad , \quad t>0 \ , \ \delta>0 \qquad \dots (1)$$

Where $\delta > 0$ is scale parameter. And the cumulative distribution function (cdf) is given by:

$$G(t;\ \delta) = \exp\left(-\frac{1}{\delta t^2}\right) \quad , \quad t>0 \ , \ \delta>0 \qquad \dots (2)$$

And the reliability function is given by:

$$R(t) = \overline{G}(t) = 1 - \exp\left(-\frac{1}{\delta t^2}\right) \ , \quad t>0 \ , \ \delta>0 \qquad \dots (3)$$

### 3. Maximum Likelihood Estimation (MLE)

Let us assumed $(t_1, t_2, ..., t_n)$ be a random sample of size (n) of a IRD with the pdf as defined by equation (1) [8, 12, 13],then the likelihood function of IRD is:

$$\lambda(\underline{t} \setminus \delta) = \prod_{i=1}^{n} g(t_i ; \delta) = 2^n \delta^{-n} \prod_{i=1}^{n} \frac{1}{t_i^3} \exp(-\frac{1}{\delta} \sum_{i=1}^{n} \frac{1}{t_i^2}) \qquad ... (4)$$

And the log likelihood function for the scale parameter $(\delta)$ is given by,

$$L = \log(2^n) - n\log(\delta) + \log(\prod_{i=1}^{n} \frac{1}{t_i^3}) - \frac{1}{\delta} \sum_{i=1}^{n} \frac{1}{t_i^2} \qquad ... (5)$$

By solving the $(\frac{\partial L}{\partial \delta} = 0)$, We obtained the maximum likelihood estimator (MLE) for $\delta$ is given by:

$$\hat{\delta}_{MLE} = \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2})}{n} \qquad ... (6)$$

By using this estimation for the scale parameter $(\hat{\delta}_{MLE})$ the maximum likelihood estimator of the reliability function of the IRD is:

$$\hat{R}_{MLE}(t) = 1 - \exp(-\frac{1}{\hat{\delta}_{MLE}\, t^2}) \ , \quad t > 0 \ , \quad \delta > 0 \qquad ... (7)$$

### 4. Bayesian Estimation

To obtain the approximate Bayes estimates of the Reliability function of an IRD under the informative double priors which are considered as: Inverse Gamma [14, 15]- Inverse Chi-square distribution [15], and Inverse Gamma - Standard Levy distribution [18, 19], and Inverse Chi-square - Standard Levy distribution. as double prior distributions. based on the Squared Error Loss Function(SELF). We should derive the posterior distribution of the scale parameter $(\delta)$as shown in the following sections.

### 4.1 Posterior Distribution

We can derived the posterior distribution of the scale parameter $(\delta)$ of corresponding the data obtained, by applying Bayes' theorem as stated in the following equation [12, 13]:

$$w(\delta / t) = \frac{\lambda(\underline{t} \setminus \delta) k(\delta)}{\int_{\delta} \lambda(\underline{t} \setminus \delta) k(\delta) d\delta} \qquad ... (8)$$

Where $k(\delta)$ and $\lambda(\delta / \underline{t})$ are the double prior distribution and the Likelihood function respectively. By defining the double prior distributions for the scale parameter $(\delta)$ of the IRD with hyper parameters. And the posterior distribution of $\delta$ can be derived bellows :

**I. When the Inverse Gamma-the Inverse Chi-square as double prior distributions.**

Let us defined the pdf of the Inverse Gamma distribution with hyper parameters $(a, b)$ is given by [14, 15]:

$$k_1(\delta; a, b) = \frac{a^b}{\Gamma(b)} \delta^{-(b+1)} \exp(-\frac{a}{\delta}) \qquad \text{with } \delta, a \,\&\, b > 0 \qquad ... (9)$$

Where a>0 is the scale parameter and b>0 is the shape parameter. And the pdf of the Inverse Chi-square distribution with (v) degrees of freedom is given by [15] :

$$k_2(\delta;\nu)=\frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)}\delta^{-\frac{\nu}{2}-1}\exp(-\frac{\nu}{2\delta})\quad \text{with } \delta \text{ and } \nu>0 \qquad \text{... (10)}$$

The double prior distribution is given by:

$$k_{12}(\delta;a,b,\nu)=k_1(\delta;a,b)\times k_2(\delta;\nu)$$

$$k_{12}(\delta;\ a,b,\nu)=Q_{12}\delta^{-(b+1)-1-(\frac{\nu}{2})}\exp(-\frac{1}{\delta}(a+\frac{\nu}{2}))\quad \text{with } \delta,\ a,\ b \text{ and } \nu>0 \qquad \text{... (11)}$$

Where is $Q_{12}=(\frac{a^b}{\Gamma(b)})(\frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)})$. The posterior distribution of $\delta$ can be obtained by substituting equation (4) and equation (11) in equation (8) as follows [12, 13]:

$$w_{12}(\delta/t)=\frac{\delta^{-((b+0.5\nu+n+1)+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu))}{\int_{\delta=0}^{\infty}\delta^{-((b+0.5\nu+n+1)+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu))\,d\delta}\quad \text{with } \delta,a,b \text{ and } \nu>0 \qquad \text{... (12)}$$

Multiplying the integral in equation (12) by the quantity:

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu)^{(b+0.5\nu+n+1)}}{\Gamma(b+0.5\nu+n+1)}\right)\left(\frac{\Gamma(b+0.5\nu+n+1)}{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu)^{(b+0.5\nu+n+1)}}\right)$$

,where $\Gamma(.)$ is a Gamma function, we obtain :

$$w_{12}(\delta/t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu)^{(b+0.5\nu+n+1)}}{\Gamma(b+0.5\nu+n+1)h1(t;\delta)}\delta^{-((b+0.5\nu+n+1)+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu))$$

$$\text{with } \delta,\ a,b \text{ and } \nu>0 \qquad \text{... (13)}$$

Where $h1(t;\delta)=\int_0^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu)^{(b+0.5\nu+n+1)}}{\Gamma(b+0.5\nu+n+1)}\delta^{-((b+0.5\nu+n+1)+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu))d\delta=1$ ,

be the integral of the pdf of the Inverse Gamma Distribution [14, 15]. We can define the posterior distribution of $\delta$ as the Inverse Gamma Distribution with the pdf is given by:

$$w_{12}(\delta/t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu)^{(b+0.5\nu+n+1)}}{\Gamma(b+0.5\nu+n+1)}\delta^{-((b+0.5\nu+n+1)+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu))$$

$$\text{with } \delta>0,\ a,b,\nu>0 \qquad \text{... (14)}$$

When the shape parameter $(b+0.5\nu+n+1)$ and the scale parameter $(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu)$.

## II. When the Inverse Gamma -the Standard Levy as Double Prior Distributions

As defined the pdf of the Inverse Gamma distribution with hyper parameters $(a,b)$ in equation (9) as follows :

$$k_1(\delta\delta a,b)=\frac{a^b}{\Gamma(b)}\delta^{-(b+1)}\exp(-\frac{a}{\delta})\quad \text{with } \delta,a \text{ and } b>0$$

And the pdf of the Standard Levy distribution with scale parameter( $\psi$ ) is given by [16, 17] :

$$k_3(\delta;\psi) = (\frac{\psi}{2\pi})^{\frac{1}{2}}\delta^{-(\frac{3}{2})}\exp(-\frac{\psi}{2\delta}) \quad \text{with } \delta \text{ and } \psi > 0 \qquad \dots (15)$$

Then the double prior distribution is given by:

$$k_{13}(\delta; a, b, \nu) = k_1(\delta; a, b) \times k_3(\delta;\psi)$$

$$k_{13}(\delta; a, b, \psi) = Q_{13}\,\delta^{-(b+1)-\frac{3}{2}}\exp(-\frac{1}{\delta}(a+\frac{\psi}{2})) \quad \text{with } \delta, a, b \text{ and } \psi > 0 \qquad \dots (16)$$

Where is $Q_{13} = (\frac{a^b}{\Gamma(b)})(\frac{\psi}{2\pi})^{\frac{1}{2}}$. The posterior distribution of $\delta$ can be obtained by substituting equation (4) and equation (16) in equation (8) as follows [12, 13] :

$$w_{13}(\delta/t) = \frac{\delta^{-((n+b+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+\frac{\psi}{2}))]}{\int_{\delta=0}^{\infty}\delta^{-((n+b+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+\frac{\psi}{2}))\,d\delta} \qquad \dots (17)$$

Multiplying the integral in equation (17) by the quantity:

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi)^{(n+b+1.5)}}{\Gamma(n+b+1.5)}\right)\left(\frac{\Gamma(n+b+1.5)}{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi)^{(n+b+1.5)}}\right),$$ where $\Gamma(.)$ is a Gamma function,

we have:

$$w_{13}(\delta/t) = \frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi)^{(n+b+1.5)}}{\Gamma(n+b+1.5)h2(t;\delta)}\delta^{-((n+b+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+\frac{\psi}{2}))$$

$$\text{with } \delta,\ a, b \text{ and } \psi > 0 \qquad \dots (18)$$

where:

$h2(t;\delta) = \int_{0}^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi)^{(n+b+1.5)}}{\Gamma(n+b+1.5)}\delta^{-((n+b+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+\frac{\psi}{2}))d\delta = 1$, be the integral of the pdf of the Inverse Gamma Distribution [14, 15]. We can define the posterior distribution of $\delta$ as the Inverse Gamma Distribution with the pdf is given by:

$$w_{13}(\delta/t) = \frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi)^{(n+b+1.5)}}{\Gamma(n+b+1.5)}\delta^{-((n+b+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi))$$

$$\text{with } \delta,\ a, b \text{ and } \psi > 0 \qquad \dots (19)$$

When the shape parameter $(n+b+1.5)$ and the scale parameter $(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi)$ .

## III. When the Inverse Chi-square -the Standard Levy Distribution as Double Prior Distributions

As defined the pdf of the Inverse Chi-square distribution with (v) degrees of freedom in equation (10) as follows :

$$k_2(\delta;\nu)=\frac{(\nu\ /2)^{\frac{\nu}{2}}}{\Gamma(\ \nu/2)}\delta^{-1-(\frac{\nu}{2})}\exp(-\frac{\nu}{2\delta})\quad \text{with } \delta \text{ and } \nu>0$$

Also, the pdf of the Standard Levy distribution with scale parameter( $\psi$ ) as defined in equation (15) as follows :

$$k_3(\delta\ ;\psi\ )=(\frac{\psi}{2\pi})^{\frac{1}{2}}\delta^{-(\frac{3}{2})}\exp(-\frac{\psi}{2\delta})\quad \text{with } \delta \text{ and } \psi>0$$

The double prior distribution is given by:

$$k_{23}(\delta;\nu,\psi)=k_2(\delta;\nu)\times k_3(\delta\ ;\psi\ )$$

$$k_{23}(\delta;\nu,\ \psi)=Q_{23}\delta^{-1-(\frac{\nu}{2})-\frac{3}{2}}\exp(-\frac{1}{\delta}(\frac{\nu}{2}+\frac{\psi}{2}))\quad \text{with } \delta,\nu \text{ and } \psi>0 \qquad \dots (20)$$

Where is $Q_{23}=\frac{(\nu\ /2)^{\frac{\nu}{2}}}{\Gamma(\ \nu/2)}(\frac{\psi}{2\pi})^{\frac{1}{2}}$.The posterior distribution of $\delta$ can be obtained by substituting equation (4) and equation (20) in equation (8) as follows [12, 13]:

$$w_{23}(\delta/t)=\frac{\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}))}{\int_{\delta=0}^{\infty}\delta^{-n-1-(\frac{\nu}{2})-\frac{3}{2}}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}))d\delta}\quad \text{with } \delta,\nu \text{ and } \psi>0 \qquad \dots (21)$$

Multiplying the integral in equation (21) by the quantity:

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2})^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}\right)\left(\frac{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2})^{(n+(\frac{\nu}{2})+\frac{3}{2})}}\right)$$

, where $\Gamma(.)$ is a Gamma function , we have :

$$w_{23}(\delta/t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2})^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})\ \ h3(t;\delta)}\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}))$$

$$\text{with } \delta,\nu \text{ and } \psi>0 \qquad \dots (22)$$

Where:

$$h3(t;\delta)=\int_{0}^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2})^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp(-\frac{1}{\delta}(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}))d\delta=1$$

, be the integral of the pdf of the Inverse Gamma Distribution [14, 15]. We can define the posterior distribution of $\delta$ as the Inverse Gamma Distribution with the pdf is given by:

$$w_{23}(\delta/t)=\frac{\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp\left(-\frac{1}{\delta}\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)\right)$$

with $\delta,\nu$ and $\psi>0$ ... (23)

With the shape parameter $(n+(\frac{\nu}{2})+\frac{3}{2})$ and the scale parameter $(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2})$.

## 4.2 Bayes estimators of the Reliability Function based on Squared Error Loss Function

We use squared error loss function (SELF) to obtain Bayes estimators of the reliability function, which is used SELF to measure the difference between the estimated and true values squared. The SELF can be defined as follows [18, 19]:

$$L(\hat{R},R)=(\hat{R}-R)^2$$

$$L(\hat{R},R)=E(\hat{R}-R)^2=\hat{R}^2-2\hat{R}E(R/t)+E(R^2/t)$$

Where $\hat{R}$ is the Bayesian estimator of the Reliability Function $(R(t))$ the based on SELF, that can be achieved by minimizing the he expectations of the loss function obtained as follows:

$$\frac{\partial}{\partial\hat{R}}L(\hat{R},R)=2\hat{R}-2E(R/t)=0 \Rightarrow \hat{R}_{Bayes}=E(R/t)$$, it means that

$$\hat{R}_{Bayesi}(t)=\left[\int_{\delta=0}^{\infty}R(t)w_{ij}(\delta/t)d\delta\right] \text{ with } i,j=1,2,3 \quad ... (24)$$

So, the $\hat{R}_{Bayes}$ of the Reliability Function $(R(t))$ can be derived under different double priors based on the SELF as follows:

### I. Under the Inverse Gamma and the Inverse Chi-square as Double Priors.

By Substituting equation (14) in equation (24) we obtain:

$$\hat{R}_{Bayes1}(t)=\int_{\delta=0}^{\infty}\left[1-\exp\left(-\frac{1}{\delta t_0^2}\right)\right]\left[\frac{\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu\right)^{(b+0.5\nu+n+1)}}{\Gamma(b+0.5\nu+n+1)}\delta^{-((b+0.5\nu+n+1)+1)}\exp\left(-\frac{1}{\delta}\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu\right)\right)\right]d\delta$$

$$\hat{R}_{Bayes1}(t)=1-\int_{\delta=0}^{\infty}\frac{\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu\right)^{(b+0.5\nu+n+1)}}{\Gamma(b+0.5\nu+n+1)}\delta^{-((b+0.5\nu+n+1)+1)}\exp\left(-\frac{1}{\delta}\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu\right)\right)d\delta \quad ... (25)$$

Multiplying the integral in equation (25) by the quantity which equal:

$$\left(\frac{\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu\right)^{(b+0.5\nu+n+1)}}{\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\nu\right)^{(b+0.5\nu+n+1)}}\right)$$. After simplification, we have:

$$\hat{R}_{Bayes1}(t) = 1 - [(\frac{\sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\nu}{\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\nu})^{(b+0.5\nu+n+1)} A1(\delta \setminus t)]$$ , where:

$$A1(\delta / t) = \int_{\delta=0}^{\infty} \frac{(\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\nu)^{(b+0.5\nu+n+1)}}{\Gamma(b + 0.5\nu + n + 1)} \delta^{-((b+0.5\nu+n+1)+1)} \exp(-\frac{1}{\delta}(\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\nu))d\delta = 1$$

be the integral of the pdf of the Inverse Gamma Distribution[14, 15]. The Bayes estimator for $(R(t))$ is given by:

$$\hat{R}_{Bayes1}(t) = 1 - (\frac{\sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\nu}{\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\nu})^{(b+0.5\nu+n+1)} \quad \text{with } t_0,\ a, b,\ \nu \text{ and } n > 0 \qquad \dots (26)$$

**II. Under the Inverse Gamma and the Standard Levy Distribution as Double Prior**

By Substituting equation (19) in equation (24) we obtain:

$$\hat{R}_{Bayes2}(t) = \int_{\delta=0}^{\infty} [1 - \exp(-\frac{1}{\delta t_0^2})][\frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\psi)^{(n+b+1.5)}}{\Gamma(n + b + 1.5)} \delta^{-((n+b+\frac{3}{2})+1)} \exp(-\frac{1}{\delta}(\sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\psi))]d\delta$$

$$\hat{R}_{Bayes2}(t) = 1 - \int_{\delta=0}^{\infty} \frac{(\sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\psi)^{(n+b+\frac{3}{2})}}{\Gamma(n + b + 1.5)} \delta^{-((n+b+\frac{3}{2})+1)} \exp(-\frac{1}{\delta}(\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\psi))d\delta \qquad \dots (27)$$

Multiplying the integral in equation (27) by the quantity which equal:

$$(\frac{(\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\psi)^{(n+b+1.5)}}{(\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\psi)^{(n+b+1.5)}})$$ . After simplification, we have:

$$\hat{R}_{Bayes2}(t) = 1 - [(\frac{\sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\psi}{\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + a + 0.5\,\psi})^{(n+b+\frac{3}{2})} A2(\delta \setminus t)]$$ , where:

$$A2(\delta/t)=\int_{\delta=0}^{\infty}\frac{\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\,\psi\right)^{(n+b+\frac{3}{2})}}{\Gamma(n+b+1.5)}\,\delta^{-((n+b+\frac{3}{2})+1)}\exp\left(-\frac{1}{\delta}\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\psi\right)\right)d\delta=1$$

be the integral of the pdf of the Inverse Gamma Distribution [14, 15] .And the Bayes estimator for $(R(t))$ is given by:

$$\hat{R}_{Bayes2}(t)=1-\left(\frac{\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\,\psi}{\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+a+0.5\,\psi}\right)^{(n+b+\frac{3}{2})}\quad \text{with } t_0,\ a, b, \psi \text{ and } n>0 \qquad \dots (28)$$

**III. Under the Inverse Chi-square and the Standard Levy Distribution as Double Prior**

By Substituting equation (23) in equation (24) we obtain:

$$\hat{R}_{Bayes3}(t)=\int_{\delta=0}^{\infty}\left[1-\exp\left(-\frac{1}{\delta t_0^2}\right)\right]\left[\frac{\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}\,\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp\left(-\frac{1}{\delta}\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)\right)\right]d\delta$$

$$\hat{R}_{Bayes3}(t)=1-\int_{\delta=0}^{\infty}\frac{\left(\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}\,\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp\left(-\frac{1}{\delta}\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)\right)d\delta \qquad \dots (29)$$

Multiplying the integral in equation (29) by the quantity which equal:

$$\left(\frac{\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}}\right)$$

. And simplification, we have:

$$\hat{R}_{Bayes3}(t)=1-\left[\left(\frac{\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}}{\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}A3(\delta\backslash t)\right]$$

, where:

$$A3(\delta/t)=\int_{\delta=0}^{\infty}\frac{\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)^{(n+(\frac{\nu}{2})+\frac{3}{2})}}{\Gamma(n+(\frac{\nu}{2})+\frac{3}{2})}\,\delta^{-((n+(\frac{\nu}{2})+\frac{3}{2})+1)}\exp\left(-\frac{1}{\delta}\left(\frac{1}{t_0^2}+\sum_{i=1}^{n}\frac{1}{t_i^2}+\frac{\nu}{2}+\frac{\psi}{2}\right)\right)d\delta=1$$

be the integral of the pdf of the Inverse Gamma Distribution [14, 15]. And the Bayes estimator for $(R(t))$ will be as follows:

$$\hat{R}_{Bayes3}(t) = 1 - \left( \frac{\sum_{i=1}^{n} \frac{1}{t_i^2} + \frac{\nu}{2} + \frac{\psi}{2}}{\frac{1}{t_0^2} + \sum_{i=1}^{n} \frac{1}{t_i^2} + \frac{\nu}{2} + \frac{\psi}{2}} \right)^{(n + (\frac{\nu}{2}) + \frac{3}{2})} \quad \text{with } t_0,\ \nu,\ \psi \text{ and } n > 0 \qquad \ldots (30)$$

## 5. Simulation and Discussion

To compare the accuracy of the different estimates for the Reliability function at time t $(t = t_0)$ of IRD using maximum likelihood estimation and Bayes' estimation under the different double informative priors which are represented by the Inverse Gamma-Inverse Chi-square, the Inverse Gamma-Standard Levy and the Inverse Chi-square -Standard Levy as the double priors based on SELF. We depend on Simulation study to obtain the results of this paper, using program was written in matlabR2018b. Sample of sizes $(n = 25, 50, 100, 150)$ with the true values of the scale parameter $\delta$ of an Inverse Rayleigh Distribution using several values of the true value parameter $(\delta = 0.5, 1.5, 2.5)$.

The data of Inverse Rayleigh Distribution were generated through inverse transformation for different samples sizes, using the quantile function from equation (2) as $t_i = \left(\frac{-1}{\delta \ \ln(G_i)}\right)^{1/2}$, where $G_i = U_i$ is a Uniform distribution with (0,1). Using different combinations of double priors, with the hyper parameters $(a, b, \nu, \psi)$ which have been selected arbitrarily as shown in Table 1:

**Table 1. The values for the hyper parameters of the double priors.**

| No. | Double priors | Values for the parameter |
|---|---|---|
| 1. | Inverse Gamma - Inverse Chi-square | $(a = 0.5, b = 3, \nu = 10)$ |
| 2. | Inverse Gamma -Standard Levy | $(a = 0.5, b = 3, \psi = 0.5)$ |
| 3. | Inverse Chi-square - standard Levy | $(\nu = 10, \psi = 0.5)$ |

In order to compute the Reliability function at time $(t = t_0)$, we assumed different values for $t = t_0 = 0.5, 1.5, 2.5, 3.5$ ,with replications number of the experiments (r=5000).We depend on the Mean Square Error (MSE) criterion.

$$MSE = \frac{1}{5000} \sum_{r=1}^{5000} (\hat{R}_r(t) - R(t))^2 \qquad \ldots (31)$$

The empirical results for Reliability function of IRD, by using the Maximum Likelihood Estimation (MLE) and Bayesian estimation under different double priors based on SELF and the true the Reliability function at time $(t = t_0)$ $(R(t))$ are summarized and tabulated in Tables (2-4). And the estimates with the minimum MSE's will be the best.

**Table 2. The estimated values for the Reliability Function and MSE's of the different estimates for the Inverse Rayleigh distribution when $(\delta = 0.5)$ and r=5000.**

| n | $t=t_0$ | R(t) | MLE | | Bayes1 Inverse Gamma - Inverse Chi-square $(a = 0.5, b = 3, \nu = 10)$ | | Bayes2 Inverse Gamma - Standard Levy $(a = 0.5, b = 3, \psi = 0.5)$ | | Bayes3 Inverse Chi-square - standard Levy $(\nu = 10, \psi = 0.5)$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\hat{R}_{MLE}(t)$ | MSE | $\hat{R}_{Bayes1}(t)$ | MSE | $\hat{R}_{Bayes2}(t)$ | MSE | $\hat{R}_{Bayes3}(t)$ | MSE |
| 25 | 0.5 | 0.9997 | 0.9993 | 0 | 0.9986 | 0 | 0.9993 | 0 | 0.9980 | 0 |
| | 1.5 | 0.5889 | 0.5971 | 0.0054 | 0.5678 | 0.0029 | 0.6288 | 0.0062 | 0.5455 | 0.0043 |
| | 2.5 | 0.2739 | 0.2820 | 0.0024 | 0.2636 | 0.0011 | 0.3057 | 0.0033 | 0.2500 | 0.0015 |
| | 3.5 | 0.1506 | 0.1560 | 0.0009 | 0.1450 | 0.0004 | 0.1708 | 0.0013 | 0.1370 | 0.0005 |
| 50 | 0.5 | 0.9997 | 0.9995 | 0 | 0.9991 | 0 | 0.9995 | 0 | 0.9989 | 0 |
| | 1.5 | 0.5889 | 0.5923 | 0.0028 | 0.5764 | 0.0020 | 0.6095 | 0.0030 | 0.5637 | 0.0025 |
| | 2.5 | 0.2739 | 0.2775 | 0.0012 | 0.2679 | 0.0008 | 0.2900 | 0.0014 | 0.2600 | 0.0009 |
| | 3.5 | 0.1506 | 0.1531 | 0.0004 | 0.1474 | 0.0003 | 0.1608 | 0.0005 | 0.1428 | 0.0003 |
| 100 | 0.5 | 0.9997 | 0.9996 | 0 | 0.9994 | 0 | 0.9996 | 0 | 0.9993 | 0 |
| | 1.5 | 0.5889 | 0.5909 | 0.0013 | 0.5824 | 0.0011 | 0.5998 | 0.0014 | 0.5756 | 0.0012 |
| | 2.5 | 0.2739 | 0.2758 | 0.0005 | 0.2708 | 0.0004 | 0.2822 | 0.0006 | 0.2666 | 0.0005 |
| | 3.5 | 0.1506 | 0.1519 | 0.0002 | 0.1490 | 0.0002 | 0.1558 | 0.0002 | 0.1465 | 0.0002 |
| 150 | 0.5 | 0.9997 | 0.9996 | 0 | 0.9995 | 0 | 0.9996 | 0 | 0.9994 | 0 |
| | 1.5 | 0.5889 | 0.5900 | 0.0009 | 0.5843 | 0.0008 | 0.5961 | 0.0009 | 0.5797 | 0.0008 |
| | 2.5 | 0.2739 | 0.2750 | 0.0004 | 0.2717 | 0.0003 | 0.2793 | 0.0004 | 0.2688 | 0.0003 |
| | 3.5 | 0.1506 | 0.1514 | 0.0001 | 0.1495 | 0.0001 | 0.1541 | 0.0001 | 0.1477 | 0.0001 |

**Table 3. The estimated values for the Reliability Function and MSE's of the different estimates for the inverse Rayleigh distribution when $(\delta = 1.5)$ and r=5000.**

| n | $t=t_0$ | R(t) | MLE | | Bayes1 Inverse Gamma - Inverse Chi-square $(a = 0.5, b = 3, \nu = 10)$ | | Bayes2 Inverse Gamma - Standard Levy $(a = 0.5, b = 3, \psi = 0.5)$ | | Bayes3 Inverse Chi-square - Standard Levy $(\nu = 10, \psi = 0.5)$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\hat{R}_{MLE}(t)$ | MSE | $\hat{R}_{Bayes1}(t)$ | MSE | $\hat{R}_{Bayes2}(t)$ | MSE | $\hat{R}_{Bayes3}(t)$ | MSE |
| 25 | 0.5 | 0.9305 | 0.9290 | 0.0013 | 0.9500 | 0.0010 | 0.9451 | 0.0010 | 0.9391 | 0.0008 |
| | 1.5 | 0.2564 | 0.2653 | 0.0023 | 0.3023 | 0.0041 | 0.2978 | 0.0042 | 0.2852 | 0.0026 |
| | 2.5 | 0.1012 | 0.1055 | 0.0005 | 0.1224 | 0.0009 | 0.1205 | 0.0009 | 0.1146 | 0.0006 |
| | 3.5 | 0.0530 | 0.0553 | 0.0001 | 0.0646 | 0.0003 | 0.0636 | 0.0003 | 0.0603 | 0.0002 |
| 50 | 0.5 | 0.9305 | 0.9287 | 0.0007 | 0.9412 | 0.0006 | 0.9377 | 0.0006 | 0.9344 | 0.0005 |
| | 1.5 | 0.2564 | 0.2595 | 0.0010 | 0.2800 | 0.0015 | 0.2760 | 0.0015 | 0.2707 | 0.0011 |
| | 2.5 | 0.1012 | 0.1027 | 0.0002 | 0.1119 | 0.0003 | 0.1102 | 0.0003 | 0.1078 | 0.0002 |
| | 3.5 | 0.0530 | 0.0538 | 0.0001 | 0.0588 | 0.0001 | 0.0579 | 0.0001 | 0.0566 | 0.0001 |
| 100 | 0.5 | 0.9305 | 0.9301 | 0.0003 | 0.9368 | 0.0003 | 0.9348 | 0.0003 | 0.9330 | 0.0003 |
| | 1.5 | 0.2564 | 0.2585 | 0.0005 | 0.2694 | 0.0007 | 0.2669 | 0.0006 | 0.2645 | 0.0005 |
| | 2.5 | 0.1012 | 0.1022 | 0.0001 | 0.1070 | 0.0001 | 0.1060 | 0.0001 | 0.1049 | 0.0001 |
| | 3.5 | 0.0530 | 0.0535 | 0 | 0.0561 | 0 | 0.0556 | 0 | 0.0550 | 0 |
| 150 | 0.5 | 0.9305 | 0.9304 | 0.0002 | 0.9350 | 0.0002 | 0.9335 | 0.0002 | 0.9323 | 0.0002 |
| | 1.5 | 0.2564 | 0.2580 | 0.0003 | 0.2653 | 0.0004 | 0.2636 | 0.0004 | 0.2620 | 0.0003 |
| | 2.5 | 0.1012 | 0.1019 | 0.0001 | 0.1052 | 0.0001 | 0.1044 | 0.0001 | 0.1037 | 0.0001 |
| | 3.5 | 0.0530 | 0.0534 | 0 | 0.0551 | 0 | 0.0547 | 0 | 0.0544 | 0 |

**Table 4. The estimated values for the Reliability Function and MSE's of the different estimates for the inverse Rayleigh distribution when $(\delta = 2.5)$ and r=5000.**

| n | $t=t_0$ | R(t) | MLE | | Bayes1 Inverse Gamma - Inverse Chi-square $(a = 0.5, b = 3, \nu = 10)$ | | Bayes2 Inverse Gamma - Standard Levy $(a = 0.5, b = 3, \psi = 0.5)$ | | Bayes3 Inverse Chi-square - Standard Levy $(\nu = 10, \psi = 0.5)$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\hat{R}_{MLE}(t)$ | MSE | $\hat{R}_{Bayes1}(t)$ | MSE | $\hat{R}_{Bayes2}(t)$ | MSE | $\hat{R}_{Bayes3}(t)$ | MSE |
| 25 | 0.5 | 0.7981 | 0.7986 | 0.0040 | 0.8555 | 0.0057 | 0.8354 | 0.0045 | 0.8351 | 0.0041 |
| | 1.5 | 0.1629 | 0.1677 | 0.0010 | 0.2032 | 0.0028 | 0.1917 | 0.0021 | 0.1905 | 0.0018 |
| | 2.5 | 0.0620 | 0.0641 | 0.0002 | 0.0789 | 0.0005 | 0.0741 | 0.0004 | 0.0736 | 0.0003 |
| | 3.5 | 0.0321 | 0.0333 | 0 | 0.0411 | 0.0001 | 0.0386 | 0.0001 | 0.0383 | 0.0001 |
| 50 | 0.5 | 0.7981 | 0.7995 | 0.0020 | 0.8323 | 0.0027 | 0.8195 | 0.0022 | 0.8199 | 0.0021 |
| | 1.5 | 0.1629 | 0.1657 | 0.0005 | 0.1846 | 0.0010 | 0.1779 | 0.0007 | 0.1778 | 0.0007 |
| | 2.5 | 0.0620 | 0.0632 | 0.0001 | 0.0710 | 0.0002 | 0.0682 | 0.0001 | 0.0682 | 0.0001 |
| | 3.5 | 0.0321 | 0.0328 | 0 | 0.0369 | 0 | 0.0354 | 0 | 0.0354 | 0 |
| 100 | 0.5 | 0.7981 | 0.7992 | 0.0010 | 0.8169 | 0.0012 | 0.8096 | 0.0011 | 0.8100 | 0.0010 |
| | 1.5 | 0.1629 | 0.1644 | 0.0002 | 0.1741 | 0.0004 | 0.1705 | 0.0003 | 0.1706 | 0.0003 |
| | 2.5 | 0.0620 | 0.0626 | 0 | 0.0666 | 0.0001 | 0.0652 | 0 | 0.0652 | 0 |
| | 3.5 | 0.0321 | 0.0325 | 0 | 0.0346 | 0 | 0.0338 | 0 | 0.0338 | 0 |
| 150 | 0.5 | 0.7981 | 0.7982 | 0.0007 | 0.8104 | 0.0008 | 0.8053 | 0.0007 | 0.8056 | 0.0007 |
| | 1.5 | 0.1629 | 0.1636 | 0.0002 | 0.1701 | 0.0002 | 0.1677 | 0.0002 | 0.1678 | 0.0002 |
| | 2.5 | 0.0620 | 0.0623 | 0 | 0.0650 | 0 | 0.0640 | 0 | 0.0640 | 0 |
| | 3.5 | 0.0321 | 0.0323 | 0 | 0.0337 | 0 | 0.0332 | 0 | 0.0332 | 0 |

In general, we observed that the true the reliability function at time t $(R(t))$ is:

- Decrease as t increase for fixed values of sample size (n).
- Fixed as sample size (n) increase for fixed t.

for all the assumed values of the true scale parameter $(\delta = 0.5, 1.5, 2.5)$ of IRD. From the empirical results for Reliability function of IRD, which are listed in tables (2 -4). We see that

- The Maximum Likelihood estimation (MLE) is given the minimum MSE's when the $\delta = 1.5, 2.5$ at all times $(t_0 = 0.5, 1.5, 2.5, 3.5)$ for the sample size n= 25, and at times $t_0 = 1.5$ for the sample size n= 50,100. See tables (3 - 4).
- The Bayes estimation under the double priors of the Inverse Gamma - the Inverse Chi-square distribution with $(a = 0.5, b = 3, \nu = 10)$ is given the smallest MSE's when $(\delta = 0.5)$ at all times $(t_0 = 1.5, 2.5, 3.5)$ for n=25, and at times $t_0 = 1.5, 2.5$ for n=50,100. See table (2).
- The MLE and the Bayes estimation under the all double priors are given the same value for the MSE when:
  - The $\delta = 0.5$ at time $t_0 = 0.5$ for n=25,50 and at time $t_0 = 0.5, 3.5$ for n=50,100.See table (2).
  - The $\delta = 1.5, 2.5$ at time $t_0 = 3.5$ for n=50,100 and at time $t_0 = 0.5, 3.5$ for n=50,100 and at time $t_0 = 1.5$ for n=150. (See tables (3-4)).
  - The $\delta = 1.5, 2.5$ at time $t_0 = 2.5, 3.5$ for n=150. (See tables (3-4)).
  - The $\delta = 2.5$ at time $t_0 = 1.5$ for n=150. (See table (4)).

- The MLE and the Bayes estimation under the double priors of the Inverse Gamma - the Inverse Chi-square distribution with $(a=0.5, b=3, \nu=10)$ and the Inverse Chi-square and standard Levy distribution with hyper parameters $(\nu=10, \psi=0.5)$ are given the same value for the MSE when $\delta=1.5, 2.5$ at time $t_0=2.5$ for n=50,100 .(See table (3-4)).
- The MLE and the Bayes estimation under the double priors of the Inverse Gamma - the Inverse Chi-square distribution with $(a=0.5, b=3, \nu=10)$ and the Inverse Chi-square and standard Levy distribution with hyper parameters $(\nu=10, \psi=0.5)$ are given the same value for the MSE when $\delta=1.5$ at time $t_0=0.5, 1.5$ for n=150 .and when $\delta=2.5$ at time $t_0=0.5$ for n=150 .
- The Bayes estimation under the double priors of the Inverse Gamma - the Inverse Chi-square distribution with $(a=0.5, b=3, \nu=10)$ and the Inverse Chi-square and standard Levy distribution with hyper parameters $(\nu=10, \psi=0.5)$ are given the same value for the MSE when $\delta=0.5$ at time $t_0=1.5, 2.5$ for n=150 .
- The MLE and the Bayes estimation under the double priors the Inverse Chi-square and standard Levy distribution with $(\nu=10, \psi=0.5)$ are given the minimum MSE when $\delta=1.5, 2.5$ at time $t_0=0.5, 1.5$ for n=100 .

## 6. Conclusion

In this paper, we discussed the classical method which is represented by the Maximum Likelihood Estimation (MLE) and Bayes estimation method for the Reliability function of the inverse Rayleigh distribution (IRD).We obtain Bayes estimators under three different informative double prior selection in Bayesian modeling which are the Inverse Gamma - the Inverse Chi-square distribution and the Inverse Gamma -the Standard Levy distribution and the Inverse Chi-square- the Standard Levy distribution as double prior distributions based on the Squared Error Loss Function(SELF).We observed that The MSE's decreases as n increases. Also the MSE's decreases as $\delta$ increases using all estimation methods, (See tables (2-4)).

And we see that

- The Bayes estimators under the double prior of the Inverse Gamma and the Inverse Chi-square distribution with $(a=0.5, b=3, \nu=10)$ based on SELF when $\delta=0.5$ for $t_0=1.5, 2.5$ and n=25, 50, 100 is superior to the other methods with respect to the minimum MSE measure.
- All estimation methods( MLE and Bayes estimation) have the same value with respect to the minimum MSE measure when $\delta=0.5$ at time $t_0=0.5$ and all (n), and for $t_0=1.5, 2.5$ at time $t_0=3.5$ and n=50, 100 and $t_0=1.5, 2.5$ at time $t_0=2.5, 3.5$ for n=150.
- The MLE's estimate perform better than the Bayes estimates when $\delta=1.5, 2.5$ at all times( $t_0$ ) for the sample size n=25, and at time $t_0=1.5, 2.5$ for n=50, and at time $t_0=1.5$ for n=50, and at time $t_0=1.5$ for n=100.

## 7. Recommendation

From the empirical results of bayes estimators for Reliability function of IRD, we recommend to use different loss functions such as quadratic loss function and he general entropy loss function and the modified linear exponential (MLINEX) loss function to compare the accuracy of the different estimates for bayes estimators of the reliability function, under the double prior of the Inverse Gamma and the Inverse Chi-square distribution.

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