Influence of Skewed Anisotropy Upon Exit Gradients

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ABSTRACT

The generalization of DARCY's law in 2.diemnsions has led to equations which describe permeability of the soil as a symmetric tensor of second rank (1, 2). In seepage problems it is common to define the GLOBAL coordinate axis to coincide with the principal directions of the permeability tensor, therefore, the cross product (off – diagonal) components (K_{xy} , K_{yx}) of the permeability tensor will be zero. It is therefore the purpose of this research to show the influence of the off- diagonal terms of the permeability tensor on the determined exit gradients behind a vertical sheet pile embedded in an anisotropic homogenous soil of finite depth. Also of interest is the derivation of an equation for the determination of exit gradients obtained from coordinate transformations of composite function.

Keywords : seepage, anisotropy, sheet pile, exit gradients.

تأثير الانحراف في الخواص غير الموحدة للترب على الانحدارات الهيدرولكية عند نقاط الخروج

الخلاصة

جرت دراسة اثر عدم التطابق أو الانحراف في الإحداثيات الشاملة أو المحلية لترب متجانسة غير موحدة الخواص على قيم الانحدارات الهيدروليكية عند نقاط الخروج. اعتمدت معادلة الجريان المستخدمة في البحث على القانون العام لدارسي مع استعمال ممتدة المستوصليات الهيدروليكية (permeability tensor). تمت مناقشة ومقارنة القيم للانحدارات الهيدروليكية بوجود هذا التأثير مع القيم التي تم الحصول عليها بافتراض تطابق الإحداثيات الشاملة والمحلية

INTRODUCTION

The problem of seepage around vertical sheet piles resting on a porous isotropic medium of finite depth was solved by pavlovsky , Muskat as presented by Liakopoulos (2) and polubarinova (3). Khosla solved the problem for the case of an isotropic porous medium of infinite depth.

Reddy, Mishra and seetharamiah (4) by using the principles of coordinate transformations and conformal mapping presented a solution for steady state confined flow around

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inclined sheet pile embedded in an isotropic and homogenous porous medium of finite depth.

Krizek and Anand (5) studied experimentally the flow at steady state condition around a vertical sheet pile embedded in an inclined stratified porous medium underlain by an impervious layer. They used coordinate transformation based upon consideration of the layered system as homogenous but anisotropic. By consideration of the porous medium as a semi- infinite – space and by using coordinate transformation, complex variable theory and conformal mapping techniques, they obtained approximations for the flow quantity and exit gradient variations. Their results showed that exit gradient approach infinity when the angles of inclination of the vertical sheet pile in the transformed domain are grater than (90) degrees, where as if the inclinations are less than (90) degrees exit gradients start from zero.

Mishra and Reddy (6) presented an analytical solution (which makes use of the calculated exit gradients of the corresponding fictitious domain) for the determination of the exit gradients under steady confined flow in an anisotropic porous medium. Results presented were for the distribution of exit gradients on the downstream side of vertical sheet pile embedded in an anisotropic homogenous flow domain of infinite depth.

Analysis of their results which are of interest to the present study indicates that if the angle of inclination (θ) of the major axis of the permeability tensor is less than (90) degrees, exit gradient at the pile is infinite and reduces gradually, whereas if (θ) is greater than (90) degrees, exit gradient at the sheet pile starts from zero reaches maximum and then decreases. Results of their analysis are given for a degree of anisotropy equal to (4) and for various angles (θ) of inclination of the major axis of the permeability tensor.

THEORY

Analyzing steady – state flow in an anisotropic homogenous porous medium requires the conversion of the flow domain to an equivalent isotropic domain by an appropriate coordinate transformation and scaling and then solving Laplace's equation in the fictitious domain by conformal mapping or by numerical methods.

Figure (1) shows the method of coordinate transformation for a section through a homogenous anisotropic flow region on the downstream side of a vertical sheet pile referenced by an $\overline{X}$ and $\overline{Y}$ coordinated axis system.

The direction of K_{max} makes an angle (θ) with the $\overline{X}$ axis, u and v are orthogonal local axes chosen parallel to K_{max} and K_{min} respectively. The relation between the axes as given by simple resolution of vectors is:

$$u = \overline{X}\cos(\theta) + \overline{Y}\sin(\theta) \quad \dots (1)$$

$$v = -\overline{X}\sin(\theta) + \overline{Y}\cos(\theta) \quad \dots (2)$$

Transforming an anisotropic flow region into an equivalent isotropic domain requires either an expansion in the direction of (v) or contraction in the direction of (u) scaling. Electing the former, the ordinate in the (v) direction becomes:-

$$v = 1/\sqrt{kv/ku}\,(-\bar{X}\sin(\theta) + \bar{Y}\cos(\theta)) \quad \text{.... (3)}$$

The real downstream bed which is described by the equation $\bar{Y}$ =0, transforms to the straight line (ox) in the new isotropic domain.

The X and Y axes are chosen parallel and perpendicular respectively to line (ox) which is previously defined as the downstream bed in the factious domain (6).

Consequently, the inclination of the sheet pile in the transformed fictitious plane is a function of (θ) and (N), which is the ratio of (kv/ku). The inclination is given by:-

$$\alpha\pi = \tan^{-1}(\tan(\theta)/\sqrt{N}) + \tan^{-1}(\cot(\theta)/\sqrt{N}) \quad \text{... (4)}$$

The expression is derived from coordinate transformation given by polubarinova (3) and use throughout the analysis. The factor (α) will have the values between (0) and (1).

Figure (2) shows the transformed fictitious isotropic flow domain. In this context, it should be noted that the analysis presented herewith is for

$\bar{S}/\bar{T} = V/T = 0.5$

Mishra and Reddy (6) derived the following equation for the calculation of exit gradient at a point in an anisotropic domain:

$$I_E = I_{EF}\left(\sqrt{((1+\tan^2\delta/(N+\tan^2\delta))}\ \mathrm{cosec}\,(\tan\text{-}1\,((-\sqrt{N}\cot\delta - \tan(\theta))\,/\,(1-\sqrt{N}\cot\delta.\tan(\theta))))\right) \quad \text{... (5)}$$

I_{EF} ; is the calculated exit gradient; at a similar point in the fictitious domain.

NUMERICAL ANALYSIS

For an anisotropic porous medium, DARCY's law must be generalized. The proper generalization is in terms of the permeability tensor. The generalization requires that each component of the vector grad (h), therefore:-

$$q_X = -Kx\,(\partial h/\partial x) - Kxy\,(\partial h/\partial y) \quad \text{... (6a)}$$

$$q_y = -Kyx\,(\partial h/\partial x) - Ky\,(\partial h/\partial y) \quad \text{...(6b)}$$

Continuity equation for steady state conditions demands that:

$$(\partial q_X/\partial x) + (\partial q_y/\partial y) = 0 \quad \text{...(7)}$$

Therefore differentiating (6a) and (6b) w.r.t (x) and (y) respectively the following is obtained:-

$$-Kx(\partial^2 h/\partial x^2) - Kxy(\partial^2 h/\partial x\partial y) - Kyx(\partial^2 h/\partial x\,\partial y) - Ky(\partial^2 h/\partial y^2) = 0 \quad \text{...(8)}$$

Remembering that Kxy = Kyx globally, therefore eq. (8) can be written in its final form as:-

$$Kx(\partial^2 h/\partial x^2) + 2\,Kxy(\partial^2 h/\partial x\partial y) + Ky(\partial^2 h/\partial y^2) = 0 \quad \text{... (9)}$$

The coefficients Kx, Kxy and Ky are the global components of a second order symmetric tensor which needs to be defined for each nod in case (for any geological reason) the local and the components Kxy and Kyx (the cross products of the permeability tensor) vanish only when the \soil is isotropic (Kx = Ky) or if θ = 0, π, 2π….. etc (when u and v coordinates coincide with x and y). Under such condition eq. (9) reduces to a form which is widely used in seepage studies and that form is:-

$$Kx(\partial^2 h/\partial x^2) + Ky(\partial^2 h/\partial y^2) = 0 \qquad \dots (10)$$

The permeability tensor components transform with coordinate rotation through the following identity:

$$[K] = [R]^{-1}\,[\bar{K}]\,[R] \qquad \dots (11)$$

Where :-

$$[K] = \begin{vmatrix} Kx & Kxy \\ Kyx & Ky \end{vmatrix}$$

$$[\bar{K}] = \begin{vmatrix} Ku & 0 \\ 0 & Kv \end{vmatrix}$$

Where Ku and Kv are the maximum and minimum local permeabilities in the u and v directions respectively. Also,

$$[R] = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$[R]^{-1} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

Which can be simplified to the following equations:-

$$Kx = Ku\cos^2\theta + Kv\sin^2\theta \qquad (12a)$$

$$Ky = Kv\cos^2\theta + Ku\sin^2\theta \qquad (12b)$$

$$Kxy = Kyx = -(Ku - Kv)\sin\theta\cos\theta \qquad (12c)$$

Equation (9) and (10) are solved explicitly using a forward differences approxmiations which is described in various references on numerical analysis.

Solution of eqs. (9) and (10) in the fictitious domain provides the determination of the exit gradient (I_{EF}) in this domain. This value can be used in eq. (5) of Mishra and Reddy to obtain the corresponding value of the exit gradients in the anisotropic domain.

As an alternative approach to the use of eq. (5), an equation (in terms of coordinate transformation) is derived by the simple rule of differentiation of the composite function h ($\bar{x}, \bar{y}$) which provides the determination of exit gradients in an anisotropic domain. Full mathematical derivation was given in appendix (1).

RESULTS AND DISCUSSION

In order to determine the effect of skewed anisotropy on the exit gradients, a digital computer program is written for the solution of eqs. (9) and (10) respectively.

To develop the numerical solution, the data corresponding to this particular problem are :-

1- Depth ($\overline{T}$) of aquifer = 24m.
2- Depth ($\overline{S}$) of the sheet pile = 12m.
3- Upstream head = 10m.

The hydraulic heats obtained from a coarse network of (21×4) elements are reused in a network with one – half the mesh spacing i.e. (42×8) elements of the coarse grid which resulted in a better set of calculated hydraulic heads.

The method suggested by Rushton and Redshaw (7) regarding the representation of irregular boundaries and the determination of the coefficients of the finite difference equations for each node is used throughout the analysis.

Results are presented for a degree of anisotropy of (4) i.e. (Ku/Kv) = 4; and for various angles of skewness (θ) of the permeability tensor.

Results of the numerical solutions are presented in figures (3-6). Figures (3) and (4) represent the solutions of eq. (10) whereas figures (5) and (6) represent those of eq.(9).

Figures (3) and (4) for the case of ($\theta < 90^0$) show that exit gradients start from infinity and reduce gradually if eq. (10) is used but the reduction is steeper in the case of eq. (9) as show in figure (4).

Similarly figures (5) and (6) for the case of ($\theta > 90^0$) show that exit gradients start from zero reach maximum and then reduce gradually in the case of eq. (10), whereas if eq. (9) is used the reductions are steeper as shown in figure (6).

However, results obtained from eq. (10) as shown in figures (3) and (5), indicate similarity in trend and behavior if compared with those obtained by Mishra and Reddy (6) for the case of infinite soil.

Generally speaking, all figures show similarity in trend and large differences in the magnitudes of the calculated exit gradients at any given θ except 0, 90 and 180^0 . The reason is attributed to the influence of the cross – product terms of the permeability tensor especially, when this tensor does not coincide globally and locally.

In this context it should noticed that exit gradients calculated from equations (17) and (18) or from equation (5) of Mishra and Reddy give similar results.

CONCLUSIONS

A numerical solutions (based upon the generalization of DARCYs law) for the problem of seepage around vertical sheet pile embedded in an anisotropic domain of finite depth is given. The solutions clearly show the influence of the cross – product terms of the permeability tensor which, resulted in steeper descend and greater reductions of exit gradients (in case of $\theta < 90^0$) and smaller peaks of exit gradients (in case of $\theta > 90^0$). Only when θ is 0, 90, 180^0, both solutions gave identical results. The Kxy terms, dissipate the head in their direction and, reduces exit gradients at the downstream section of the sheet pile.

NOTATIONS

x, y = Global coordinates in the fictitious isotropic domain.
$\bar{x}$, $\bar{y}$ = Local coordinate in the anisotropic domain.
u, v = Coordinate axes parallel to the direction of maximum and minimum coefficients of permeability respectively in the anisotropic domain.
Θ = Angle of inclination of the major principal axis of the permeability tensor with the $\bar{x}$ axis.
δ = Angle between u and x axes.
Ku , Kv = Major and Minor principal coefficients of local permeabilities.
[K] = Global permeability tensor.
[$\bar{K}$] = Local permeability tensor.
h = Hydraulic head.
$\bar{T}$ = Aquifer thickness.
$\bar{S}$ = Depth of sheet pile.
T = Aquifer thickness of the transformed fictitious domain.
v = Vertical projection of the inclination of the fictitious domain.

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<u>Appendix (1)</u>

Figure (1) is used to derive an expression for the exit gradient in an anisotropic domain in terms of axes transformation:

<u>Case θ ≤90</u>

$(\partial h/\partial \bar{y}) = (\partial h/\partial u)\,(\partial h/\partial \bar{y}) + (\partial h/\partial v)\,(\partial v/\partial \bar{y})$ (13)
$(\partial h/\partial u) = (\partial h/\partial x)\,(\partial x/\partial u) + (\partial h/\partial y)\,(\partial y/\partial u)$ (14)
$(\partial h/\partial v) = (\partial h/\partial x)\,(\partial x/\partial v) + (\partial h/\partial y)\,(\partial y/\partial v)$ (15)

Knowing that:-
$x = u \cos\delta - v \sin\delta$

and ,

$y = u \sin\delta + v \cos\delta$

Therefore, $\partial x/\partial y = \cos\delta$, $\partial x/\partial v = -\sin\delta$

And $\partial y/\partial u = \sin\delta$, $\partial y/\partial u = \cos\delta$

Substitution leads to:-

$$(\partial h/\partial \bar{y}) = ((\partial h/\partial x)\cos\delta + (\partial h/\partial y)\sin\delta)(\partial u/\partial \bar{y}) + ((\partial h/\partial x)(-\sin\delta) + ((\partial h/\partial y)\cos\delta)(\partial v/\partial \bar{y}) \quad (16)$$

The relation between the u, v, $\bar{x}$, $\bar{y}$ axes and θ as given by polubarinova (3) is :-

$$u = \bar{x}\cos\theta + \bar{y}\sin\theta$$

$$v = 1/\sqrt{\frac{kv}{ku}}\,(-\bar{x}\sin\theta + \bar{y}\cos\theta)$$

Thus, $\partial u/\partial \bar{y} = \sin\theta$ and $\partial v/\partial \bar{y} = (1/\sqrt{\frac{kv}{ku}})\cos\theta$

Final substitution of the above identities will give an expression for the exit gradient in an anisotropic domain in a direction perpendicular to the downstream bed (6):-

$$(\partial h/\partial \bar{y}) = (\partial h/\partial x)\cos\delta\sin\theta + (\partial h/\partial y)\sin\delta\sin\theta - (\partial h/\partial x)(1/\sqrt{\frac{kv}{ku}})\sin\delta\cos\theta + (\partial h/\partial y)(1/\sqrt{\frac{kv}{ku}})\cos\delta\cos\theta \quad (17)$$

<u>Case θ> 90</u>

Using eq. (13), (14) and (15) with the proper axes transformation will give the following equation for the exit gradient in an anisotropic dimain acting also in a direction perpendicular to the downstream bed (6) :-

$$(\partial h/\partial \bar{y}) = (\partial h/\partial x)(-\cos\delta)\sin\theta + (\partial h/\partial y)\sin\delta\sin\theta\,(\partial h/\partial x)(1/\sqrt{\frac{kv}{ku}})\sin\delta\cos\theta + (\partial h/\partial y)(1/\sqrt{\frac{kv}{ku}})\cos\delta\cos\theta \quad (18)$$

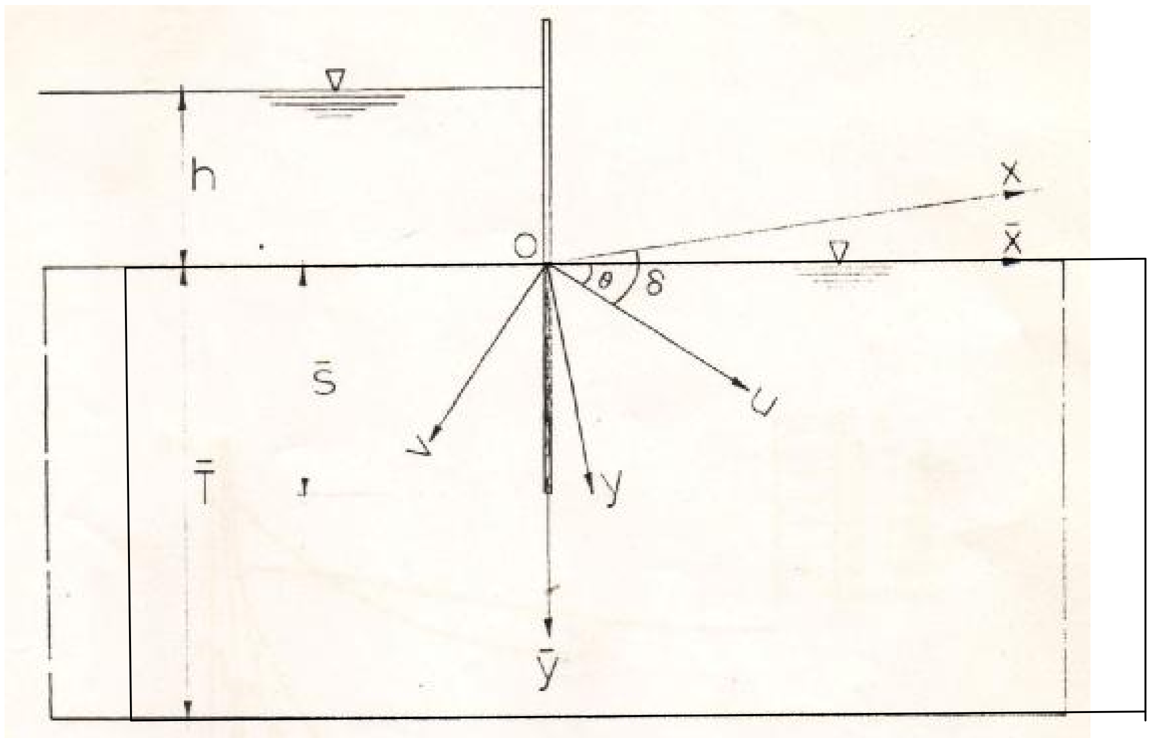

Figure (1)

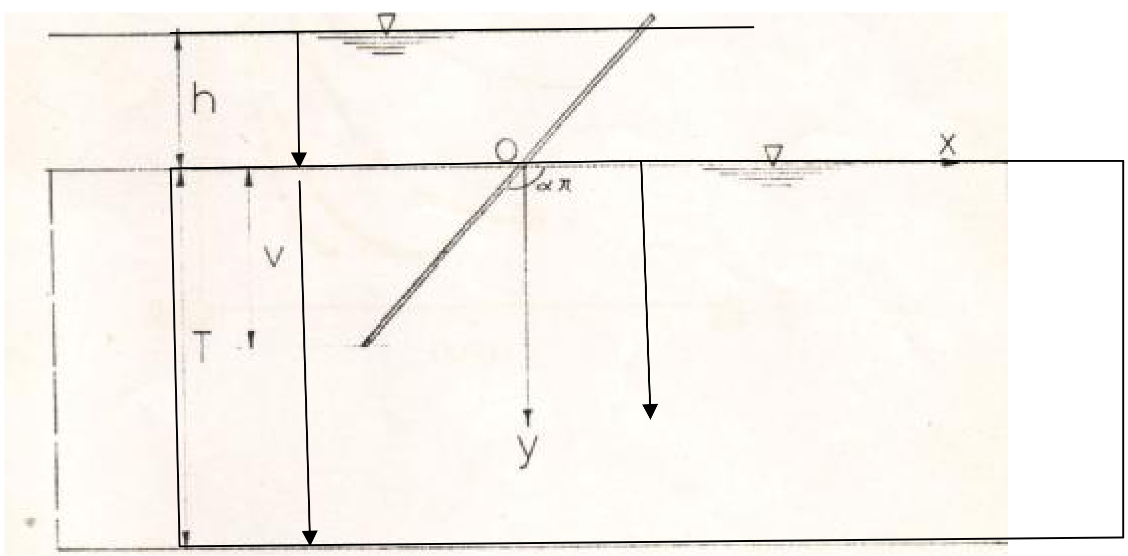

Figure (2)

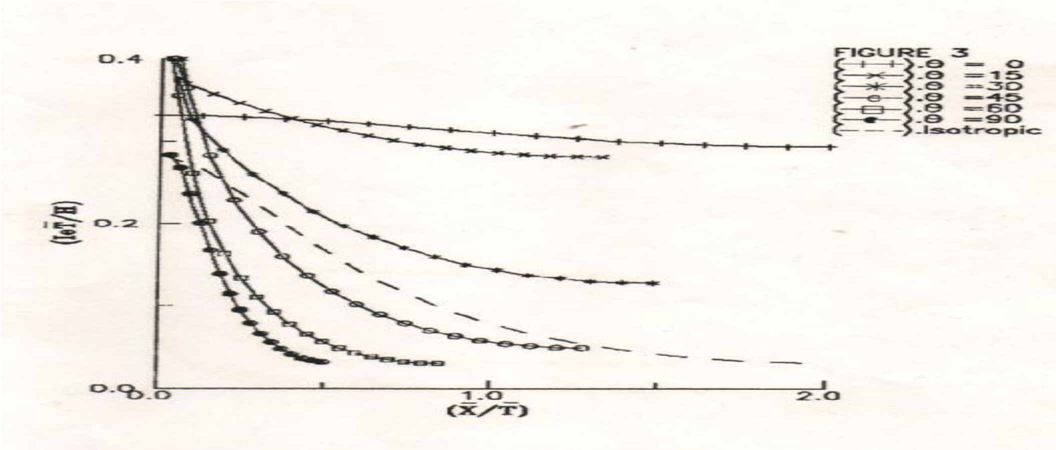

Figure (3)

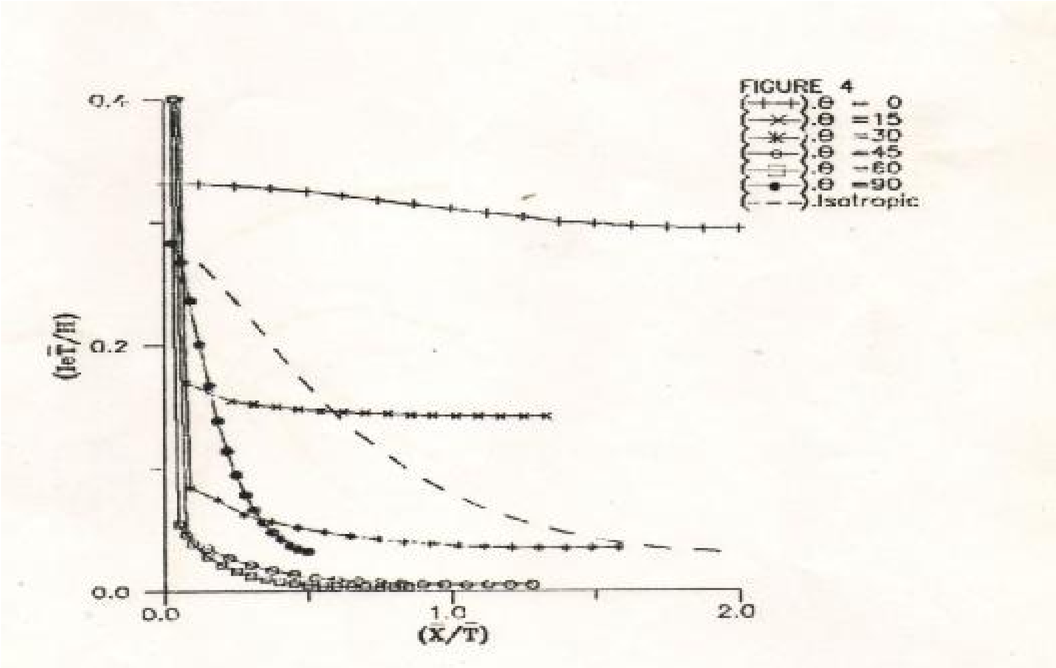

Figure (4)

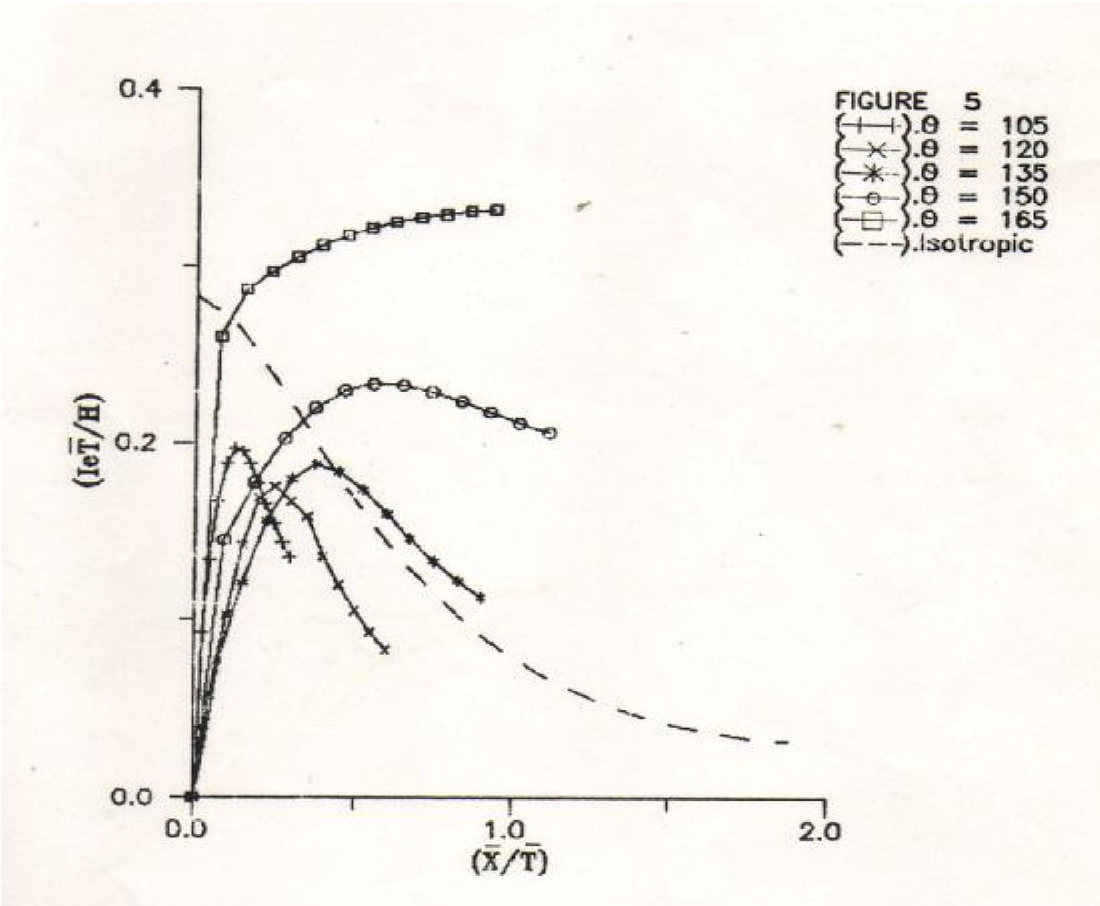
FIGURE 5
.θ = 105
.θ = 120
.θ = 135
.θ = 150
.θ = 165
.Isotropic
0.4
0.2
0.0
(IeT/H)
0.0
1.0
2.0
(X/T)

Figure (5)

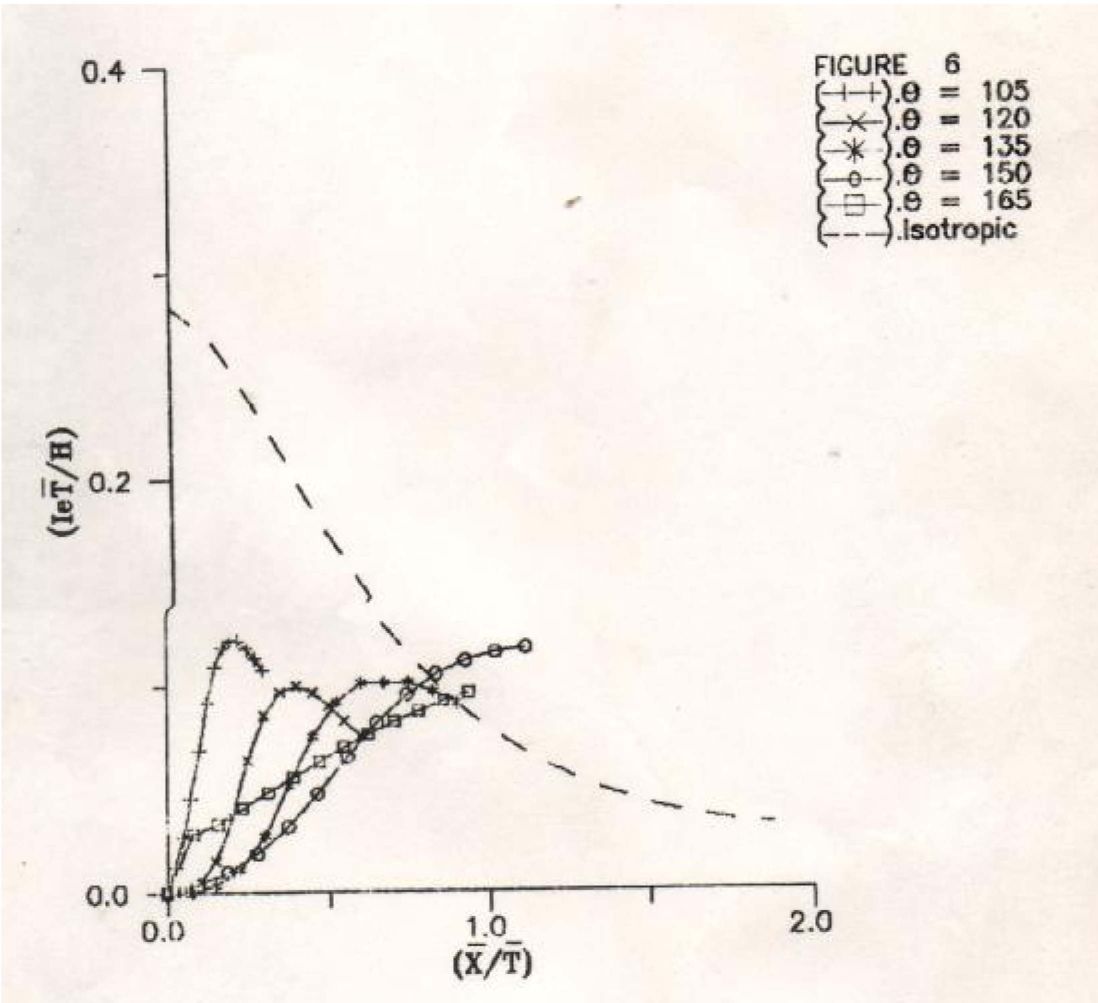
FIGURE 6
(—+—).θ = 105
(—×—).θ = 120
(—✱—).θ = 135
(—o—).θ = 150
(—□—).θ = 165
(- - -).Isotropic
0.4
0.2
0.0
(IeT̄/H̄)
0.0
1.0
2.0
(X̄/T̄)

Figure (6)