Nonlinear Finite Element Analysis for Elastomeric Materials under Finite Strain

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ABSTRACT

In this paper the finite element methodis used as a numerical techniqueto investigatethe three-dimensional elastomeric materials (rubber or rubber-like materials) under finite, or large, strains analysis.The non-linear element equations for the displacement and pressure field parameters are formulated using the minimized variational approach. Essentially, approximate solutions for the displacement and pressure field parameters are obtained from the solutions of the two corresponding sets of non-linear simultaneous equations via the nonlinear Newton-Raphson iterative procedure. The basic iterative solution procedure convergence is further improved via breaking the applied load down into load incrementwith optimized incremental steps. Additionally,a complete finite element formulation is reported and detailed in this work,and the mathematical complexities conjoined with such kinds of analysis are simplified as possible.

Solving some numerical examples and comparing the results with that obtained from some available results and ANSYS 12.0 showed that the current formulation of the finite element methods is correctand the resulted program is capable for solving incompressible elastomeric materials under finite strain. The formulation used for the finite element derivations for large strain analysis gave satisfactory results as compared with that of available results.

Keywords: Nonlinear FEM, finite strain, large deformation, elastomers, rubber, & rubber-like materials.

التحليل اللاخطي بطريقة العناصر المحددة للمواد المطاطيـــة تحت تأثيرالانفعالات الكبيرة

الخلاصـــــة

استخدمت في هذا البحث طريقة العناصر المحددة كتقنية عددية لتحري المواد المطاطية عندتحليل الانفعالات الكبيرة. تم استخدام نهج تقليل التباين في صياغة معادلات العناصر اللاخطية لمعاملات الازاحة والضغط. تم في الاساس الحصول على الحلول التقريبية لمعاملات الازاحة والضغط بالاعتماد على حل مجموعتين من المعادلات الآنية اللاخطية عن طريق خوارزمية تكرارية اعتمادا على طريقة نيوتن-رافسن اللاخطية كون المسالة الحالية لاخطية بالكامل.تم تحسين الحل للاجراءات التكرارية من خلال تجزئة الحمل المسلط باجراءات تزايدية واختيار خطوات الزيادة بأمثل اسلوب.

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بالاضافة الى ان العمل الحالي يذكر سرد تفصيلي، نوعا ما، لكيفية الاشتقاق وطريقة التعامل ومحاولة لتذليل الصعوبات الرياضية المصاحبة لهذا النوع من المسائل.
اثبتت المقارنات عند حل بعض المسائل العددية ان الاشتقاقات الرياضية لطريقة العناصر المحددة االمستخدمة في هذا البحث صحيحة وممكن الاعتماد عليها لحل المسائل الخاصة بالمواد المطاطية. وكانت النتائج التي تم الحصول عليها مرضية عند مقارنتها مع النتائج المتوفرة او عند مقارنتها بالنتائج التي تم الحصول عليها من البرنامج الهندسية الخاصة بطريقة العناصر المحددة مثل الـــ ANSYS12.0.

Nomenclature

$\varepsilon_x, \varepsilon_y, \varepsilon_z$	Finite direct strains in *x, y* and *z* directions	m/m
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Finite shear Strains	
e_G	Green's or Lagrangian strain tensor	m/m
C	Right Cauchy-Green strain tensor	
f	Displacement derivative tensor	
$\mathbf{F}_e$, $\mathbf{F}_G$	Element and Global Nodal force vector, respectively	N
I	Identity matrix	
$\mathbf{K}_G$	Global stiffness matrix	N/m
K	Bulk modulus	
I_1, I_2, I_3	Stretch or strain invariants	
$\bar{I}_1, \bar{I}_2$	Modified stretch or strain invariants	
J	Determinant of deformation gradient	
J_D	Jacobain matrix	
N_i	Shape functions	
p	Hydrostatic pressure	N/mm^2
S	2nd Piola-Kirchhof stress	N/mm^2
S_{ij}	i,j=1,2,3components of2nd Piola-Kirchhof stress	N/mm^2
u, v, w	Displacements components	m
$\mathbf{u}_e$, $\mathbf{u}_G$	Element and Global nodal displacements tensor, respectively	m
W	Strain energy function	

INTRODUCTION

Successful analysis of elastomers, rubber or rubber-like materials, requires robust numerical methods and representative material models applicable to small/large strains and multiple deformation modes. Although the mathematical foundation of strain energy density function has been studied by many researchers, the application to engineering problems is not straightforward [1].

Two major challenges are encountered in the numerical analysis of rubber materials. The first is due to the material incompressibility of rubber. The finite element prediction is often much stiffer (locking) than analytical solution or experimental data resulting from the imposition of "constant volume constraint" in the numerical formulation. Locking usually accompanies with pressure oscillation that completely corrupts the numerical stress solution, even when the deformation is small [2]. The second difficulty is the mesh distortion caused by the large deformation nature in many elastomeric applications. Therefore it is necessary to modify the numerical techniques which may be used to help in getting improved numerical analysis[3].

The lack of inaccurate results of the conventional finite element method is due to Poisson's ratio values in elastomers, which ranges between 0.499 and 0.5. The elements used in FEA need to be reformulated to accommodate this high value of Poisson's ratio. This is usually accomplished by utilizing an approach developed by Herrmann [4], by introducing a new variational principle that includes another degree of freedom called the "mean pressure function."

It can be seen from the literature that there are many material models, all of which share certain features [1]. One of these being the requirement to calibrate the material constants from test data. This is alluded to by Boyce [1] who points out that although the Neo-Hookian and Mooney-Rivlin material models only require one and two calibrated constants, respectively, their ability to represent accurately even modestly large strains is poor for moderately very large deformation. Better models exist but require the evaluation of more material constants. Boyce [1] therefore investigates the Gent and Arruda-Boyce material models concluding that despite these two models only requiring two calibrated constants they nevertheless successfully model three dimensional finite strain behaviors.

Another common feature of the various material models is that their forms are either functions of the strain invariants or the principal extension ratios. Work by Davies *et.al.*[5], as well as Yeoh [6] indicates that for strain invariant based models the strain energy derivative with respect to the second strain invariant is negligible in comparison to the strain energy derivative with respect to the first strain invariant. Consequently, the second strain invariant is ignored in some material models.

Apparently with so many forms of material model the question arises as to which model is the most accurate and efficient? Charlton *et.al.*[7] state that for larger and more complex strains, higher order terms in the Rivlin polynomial need to be included. Ogden [8] also investigated the accuracy of three different nearly incompressible material models, these being the Mooney-Rivlin, Ogden and Valanis-Landel models, using Treloar uniaxial testdata for constant calibration purposes [9]. From the three models, the Valanis-Landel formulation provides the best correspondence between the theoretical and the experimental test data, for a variety of deformation modes.

The mostearlier example of a finite element formulation for modeling finite strains in elastomers was given by Lindley [10]. His work describes the use of triangular elements to discretize a rubber sheet enabling the total strain energy of the sheet to be evaluated by the finite element method. Subsequently an iterative procedure was employed to move all the model nodes so as to minimize the strain energy of the sheet. Initial displacement estimates for the iterative procedure were obtained from small strain linear elasticity. The boundary conditions were applied to the model by means of prescribed displacements.

Subsequent developments have resulted in three different approaches that have been evolved to deal with the hydrostatic pressure related to the volumetric part of the strain. The displacement method with reduced integration [11], the penalty type formulation [12] and the mixed displacement, pressure field parameter method [13]. The most popular and developed of these methods is the mixed field parameter approach which allows for full and near incompressibility. However the interpolations used are limited by instability in the mixed patch test[14]. The basic mixed field parameter finite element formulation is discussed by Canga *et.al.* [15] and Basar & Itskov [16] and consists of forming a set of non-linear simultaneous equations. These equations can be assembled in the usual finite element manner into a form compatible with the Newton-Raphson iterative solution procedure. For each iteration the tangent stiffness element equations are evaluated from the relevant constitutive model. For example, Chen et.al. [17] describe the tangent stiffness moduli matrix formulation based on the Rivlin polynomial material model. Holzapfel [18] and Basar & Itskov [16]utilize the Ogden material model to form the constitutive relationship. With the latter work describing how the Ogden model can be reformulated from a function in terms of principal stretches to one in terms of the strain invariants. Work by Kaliske & Rothert [19] suggests that generally constitutive material models in terms of strain invariants allow for simpler and more efficient formulations. By way of demonstration Kaliske & Rothert perform finite element analyses based on constitutive models utilizing the Neo-Hookian, Mooney-Rivlin, Swanson [20], Yeoh [6], Arruda-Boyce [21] and statistically derived Kilian material models. With regard to constitutive equations based on the Ogden model, Basar & Itskov state that the material model calibration is more complex as the calibration process itself requires an iterative non-linear solution procedure.

The main contribution of the present work is to give a detailed mathematical procedure for obtaining the element equations of three dimensional elastomeric problems using the variational principles. The material model used in this analysis will be Mooney-Rivlin. The solution procedure will be performed by using the nonlinear Newton-Raphson procedure. A try, as well, will be been given to clarify and simplify, as possible, the mathematical complexities conjoined with such a kind of problems.

FINITE ELEMENT APPLIED TO ELASTOMER

When forming the element equations for rubber or rubber-like materials, or generally elastomers, two sources of nonlinearity are introduced due to the ability of rubber to undergo finite (large) elastic deformations. These are geometric and material non-linearity. The presence of geometric non-linearity dictates that stress and strain measures should be used which are accurate for finite deformations. In present work the stress and strain measures used are the 2nd Piola-Kirchhof stress and Green's strain refer to undeformed or reference coordinates.

The material non-linearity requires the use of an appropriate material model to form the constitutive relationship between stress and strain. The constitutive relationship in the present analysis is based on Mooney-Rivlin strain energy expression which has the following form [1]:

$$W = C_1\left(\bar{I}_1 - 3\right) + C_2\left(\bar{I}_2 - 3\right) + \frac{1}{2}K(J-1)^2 \quad (1)$$

The first two terms in the above expression account for deviatoric strain energy and the third term accounts for the volumetric strain energy. The constants C_1, C_2 and K (bulk modulus) are material constants which can be evaluated from some experimental tests [23].

The 2nd Piola-Kirchhoff stress can be written as follows [24]:

$$S = \frac{\partial W}{\partial e_G} = 2\frac{\partial W}{\partial \mathbf{C}} \qquad (2)$$

The Green's strain tensor, $\boldsymbol{\varepsilon}_G$, and the Right Cauchy-Green strain tensor, $\mathbf{C}$, are related as: $\mathbf{C} = 2e_G + \mathbf{I}$.

Differentiating equation (1) with respect to Right Cauchy-Green strain tensor,

$$\frac{\partial W}{\partial \mathbf{C}} = C_1\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + C_2\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + K(J-1)\frac{\partial J}{\partial \mathbf{C}} \qquad (3)$$

Simplifying and rearrange equation (3), see Appendix A, leads to:

$$\frac{\partial W}{\partial \mathbf{C}} = \left(C_1 I_3^{-\frac{1}{3}} + C_2 I_3^{-\frac{2}{3}} I_1\right)\mathbf{I} - C_1 I_3^{-\frac{2}{3}}\mathbf{C} - \left(\frac{1}{3}C_1 I_1 I_3^{-\frac{1}{3}} + \frac{2}{3}C_2 I_2 I_3^{-\frac{2}{3}}\right)\mathbf{C}^{-1} + \frac{1}{2}KJ(J-1)\mathbf{C}^{-1} \qquad (4)$$

Hence, the 2nd Piola-Kirchhoff stress may be written as[23]:

$$S = D_1\mathbf{I} - D_2\mathbf{C} - D_3\mathbf{C}^{-1} + K\,J(J-1)\mathbf{C}^{-1} \qquad (5)$$

where:

$$D_1 = 2\left(C_1 I_3^{-\frac{1}{3}} + C_2 I_3^{-\frac{2}{3}} I_1\right), \quad D_2 = 2C_2 I_3^{-\frac{2}{3}} \quad \text{and} \quad D_3 = 2\left(\frac{1}{3}C_1 I_1 I_3^{-\frac{1}{3}} + \frac{2}{3}C_2 I_2 I_3^{-\frac{2}{3}}\right)$$

Equation (5) can further simplify by using the definition of the bulk modulus *K* as:

$$K = -\frac{p}{e_V} \quad \text{and} \quad e_V = \frac{dV - dV_o}{dV_o} = J - 1$$

Where *p* is the hydrostatic pressure which defined as [22]:

$$p = -\frac{1}{3}\left(s_x + s_y + s_z\right)$$

Therefore, equation (5) can now be rewritten as:

$$S = D_1\mathbf{I} - D_2\mathbf{C} - D_3\mathbf{C}^{-1} - p\,J\,\mathbf{C}^{-1} \qquad (6)$$

FINITE ELEMENT FORMULATION

From theory of elasticity the strain vector for three-dimensional, large strains, Green's strain-displacement equations, are of the form:

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right)$$
$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) \qquad (7a)$$
$$e_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right)$$

$$g_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$
$$g_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \qquad (7b)$$
$$g_{yz} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)$$

These equations can be written as the sum of small and large strain vectors as:

$$\boldsymbol{\varepsilon}_G = \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ g_{xy} \\ g_{xz} \\ g_{yz} \end{bmatrix} = \boldsymbol{\varepsilon}_{small} + \boldsymbol{\varepsilon}_{large} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \\ 2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\ 2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\ 2\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \end{bmatrix} = \mathbf{B}\,\mathbf{u} \qquad (8)$$

Where **u** is the nodal displacements tensor, and matrix **B** represents a strain-displacement transformation of the element. The strain vector can be expressed as a sum of linear and non-linear terms, depending upon displacements **u**. Therefore matrix **B** may be written as a sum:

$$\mathbf{B} = \mathbf{B}_l + \mathbf{B}_{nl}$$

The small strain vector $\boldsymbol{\varepsilon}_{small}$ isrelated to the nodal displacements by using $\mathbf{B}_l$ matrix:

$$\mathbf{B}_l = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \ldots & \frac{\partial N_n}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \ldots & 0 & \frac{\partial N_n}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial z} & \ldots & 0 & 0 & \frac{\partial N_n}{\partial z} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \ldots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} & 0 \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \ldots & \frac{\partial N_n}{\partial z} & 0 & \frac{\partial N_n}{\partial x} \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \ldots & 0 & \frac{\partial N_n}{\partial z} & \frac{\partial N_n}{\partial y} \end{bmatrix} \qquad (9)$$

The Cartesian shape function derivatives are obtained from intrinsic shape function derivatives as follows:

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \mathbf{J}_D^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial h} \\ \frac{\partial N_i}{\partial z} \end{bmatrix}, \text{ where } \mathbf{J}_D = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial x}{\partial h} & \frac{\partial y}{\partial h} & \frac{\partial z}{\partial h} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix}$$

The above relationship enables the arbitrary form of the discretized elements in Cartesian space to have generic shape function expressions in intrinsic space.

The large strain vector $\boldsymbol{\varepsilon}_{large}$ can be written as follows:

$$\boldsymbol{\varepsilon}_{large} = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \\ 2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\ 2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\ 2\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} & 0 & 0 & 0 \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} & 0 & 0 & 0 & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ 0 & 0 & 0 & \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \frac{1}{2}\mathbf{A\Theta} \qquad (10)$$

The Q vector can be derived in terms of nodal displacement as:

$$\boldsymbol{\Theta}=\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial x}\\ \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial y}\\ \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial z}\end{bmatrix}=\sum_{i=1}^{n}\begin{bmatrix}u_i\frac{\partial N_i}{\partial x}\\ v_i\frac{\partial N_i}{\partial x}\\ w_i\frac{\partial N_i}{\partial x}\\ u_i\frac{\partial N_i}{\partial y}\\ v_i\frac{\partial N_i}{\partial y}\\ w_i\frac{\partial N_i}{\partial y}\\ u_i\frac{\partial N_i}{\partial z}\\ v_i\frac{\partial N_i}{\partial z}\\ w_i\frac{\partial N_i}{\partial z}\end{bmatrix}=\begin{bmatrix}\frac{\partial N_1}{\partial x} & 0 & 0 & \mathbf{L} & \frac{\partial N_n}{\partial x} & 0 & 0\\ 0 & \frac{\partial N_1}{\partial x} & 0 & \mathbf{L} & 0 & \frac{\partial N_n}{\partial x} & 0\\ 0 & 0 & \frac{\partial N_1}{\partial x} & \mathbf{L} & 0 & 0 & \frac{\partial N_n}{\partial x}\\ \frac{\partial N_1}{\partial y} & 0 & 0 & \mathbf{L} & \frac{\partial N_n}{\partial y} & 0 & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & \mathbf{L} & 0 & \frac{\partial N_n}{\partial y} & 0\\ 0 & 0 & \frac{\partial N_1}{\partial y} & \mathbf{L} & 0 & 0 & \frac{\partial N_n}{\partial y}\\ \frac{\partial N_1}{\partial z} & 0 & 0 & \mathbf{L} & \frac{\partial N_n}{\partial z} & 0 & 0\\ 0 & \frac{\partial N_1}{\partial z} & 0 & \mathbf{L} & 0 & \frac{\partial N_n}{\partial z} & 0\\ 0 & 0 & \frac{\partial N_1}{\partial z} & \mathbf{L} & 0 & 0 & \frac{\partial N_n}{\partial z}\end{bmatrix}\begin{bmatrix}u_1\\ v_1\\ w_1\\ \vdots\\ u_n\\ v_n\\ w_n\end{bmatrix}=\boldsymbol{\Phi}\mathbf{u}_e \qquad (11)$$

Therefore, equation (8) can now be written as:

$$\boldsymbol{\varepsilon}_G=\boldsymbol{\varepsilon}_{small}+\boldsymbol{\varepsilon}_{large}=\left(\boldsymbol{\Psi}+\frac{1}{2}\mathbf{A}\right)\boldsymbol{\Theta}$$

where

$$\boldsymbol{\Psi}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1\\0&1&0&1&0&0&0&0&0\\0&0&1&0&0&1&0&0&0\\0&0&0&0&0&1&0&1&0\end{bmatrix}$$

The variation in Green's strain can now be written as:

$$d\boldsymbol{\varepsilon}_G = d\boldsymbol{\varepsilon}_{small} + \frac{1}{2} d\mathbf{A}\,\boldsymbol{\Theta} + \frac{1}{2}\mathbf{A}\,d\boldsymbol{\Theta} \tag{12}$$

It can be easily prove that $d\mathbf{A}\,\boldsymbol{\Theta} = \mathbf{A}\,d\boldsymbol{\Theta}$, therefore, equation (12) will now be rewritten as:

$$d\boldsymbol{\varepsilon}_G = d\boldsymbol{\varepsilon}_{small} + \mathbf{A}\,d\boldsymbol{\Theta} = (\boldsymbol{\Psi} + \mathbf{A})\boldsymbol{\Phi}\,d\mathbf{u} = \mathbf{B}d\mathbf{u} = (\mathbf{B}_l + \mathbf{B}_{nl})d\mathbf{u} \tag{13}$$

where $\mathbf{B}_{nl} = \mathbf{A}\,\boldsymbol{\Phi}$, equation (13) represents the Green's strain tensor in terms of nodal displacements.

ELEMENT EQUATION FORMULATION

Starting from the strain energy:

$$\text{W} = \iiint_V d\boldsymbol{\varepsilon}_G^T \;\text{S}\; dV \tag{14a}$$

Using equation (13) into equation (14) and the principle of virtual work $(d\text{W} = d(\text{work done}))$, in terms of nodal displacements:

$$d\mathbf{u}^T \iiint_V \mathbf{B}^T \;\text{S}\; dV - d\mathbf{u}^T\mathbf{F}_{ext} = d\mathbf{u}^T\mathbf{F}$$

The above equation represents an imbalance between internal and external virtual energy. Since $d\mathbf{u}$ represents a vector of arbitrary virtual infinitesimal nodal displacement:

$$\iiint_V \mathbf{B}^T \;\text{S}\; dV - \mathbf{F}_{ext} = \mathbf{F} \tag{14b}$$

Assumingexact equilibrium, equation (14b), after rearranging and simplification,can be written as:

$$d\mathbf{F} = \iiint_V \mathbf{B}^T \,d\,\text{S}\; dV + \iiint_V d\mathbf{B}^T \;\text{S}\; dV \tag{15}$$

To evaluate the terms of equation (15), an equivalent variational form of equation (14) may be written as:

$$d\text{W} = \iiint_V d\boldsymbol{\varepsilon}_G : d\,\text{S}\; dV + \iiint_V d\left(d\boldsymbol{\varepsilon}_G\right) : \text{S}\; dV \tag{16}$$

To evaluate the second term of equation (16), starting from Green strainwhich can be expressed as [25]:

$$\boldsymbol{\varepsilon}_G = \frac{1}{2}\left[\left(\mathbf{f}+\mathbf{f}^T\right)+\left(\mathbf{f}^T\ \mathbf{f}\right)\right]$$

where $\mathbf{f}$ is the *displacement derivative tensor* defined as:

$$\mathbf{f} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

The differentiation of the above leads to:

$$\begin{aligned} d\boldsymbol{\varepsilon}_G &= \frac{1}{2}\left[\left(d\ \mathbf{f}+d\ \mathbf{f}^T\right)+\left(d\ \mathbf{f}^T\ \mathbf{f}\right)+\left(\mathbf{f}^T\ d\ \mathbf{f}\right)\right] \\ &= \frac{1}{2}\left[\left(\mathbf{I}+\mathbf{f}^T\right)d\ \mathbf{f}+d\ \mathbf{f}^T\left(\mathbf{f}+\mathbf{I}\right)\right] \end{aligned} \qquad (17)$$

And,

$$d\left(d\boldsymbol{\varepsilon}_G\right) = d\ \mathbf{f}^T d\ \mathbf{f} \qquad (18)$$

To evaluate the first term of equation (16), the differential $d\,\mathrm{S}$ must be evaluated, this is may be achieved via the use of equation (5),

$$\begin{aligned} d\,\mathrm{S} &= d\mathrm{D}_1\mathbf{I} - \left(d\mathrm{D}_2\ \mathbf{C} + \mathrm{D}_2\ d\mathbf{C}\right) - \left(d\mathrm{D}_3\ \mathbf{C}^{-1} + \mathrm{D}_3\ d\mathbf{C}^{-1}\right) \\ &\quad - \left[dp\left(J\ \mathbf{C}^{-1}\right) + p\,d\left(J\ \mathbf{C}^{-1}\right)\right] \end{aligned} \qquad (19)$$

Evaluating each term of equation (19) separately leads to:

$$
\begin{aligned}
d\mathrm{D}_1 &= \left[-\frac{2}{3} C_1 I_3^{-\frac{1}{3}} \mathbf{C}^{-1} + 2C_2 \left(-\frac{2}{3} I_3^{-\frac{2}{3}} I_1 \mathbf{C}^{-1} + I_3 \left(\mathbf{I} - \frac{1}{3} I_1 \mathbf{C}^{-1} \right) \right) \right] : d\mathbf{C} \\
d\mathrm{D}_2 &= \left[-\frac{4}{3} C_2 I_3^{-\frac{2}{3}} \mathbf{C}^{-1} \right] : d\mathbf{C} \\
d\mathrm{D}_3 &= \left[\frac{2}{3} C_1 \left(I_3^{-\frac{2}{3}} \left(\mathbf{I} - \frac{1}{3} I_1 \mathbf{C}^{-1} \right) - \frac{1}{3} I_1 I_3^{-\frac{1}{3}} \mathbf{C}^{-1} \right) \right. \\
&\left. + \frac{4}{3} C_2 \left(I_3^{-\frac{4}{3}} \left(I_1 \mathbf{I} - \mathbf{C} - \frac{2}{3} I_2 \mathbf{C}^{-1} \right) - \frac{2}{3} I_2 I_3^{-\frac{2}{3}} \mathbf{C}^{-1} \right) \right] : d\mathbf{C} \\
d\mathbf{C}^{-1} &= -\mathbf{C}^{-1} d\mathbf{C}\, \mathbf{C}^{-1} \\
dJ &= \frac{1}{2} I_3^{\frac{1}{2}} \mathbf{C}^{-1} : d\mathbf{C}
\end{aligned}
\tag{20}
$$

The second term at the right of equation (16) can now be written as:

$$
\iiint_V d\left(d\boldsymbol{\varepsilon}_G\right) : \mathrm{S} \;\; dV = \iiint_V \left(d\;\mathbf{f}^T d\;\mathbf{f}\right) : \mathrm{S} \;\; dV \tag{21}
$$

The above expression may be written after some mathematical manipulation as:

$$
\iiint_V d\left(d\boldsymbol{\varepsilon}_G\right) : \mathrm{S}\, dV = \iiint_V
\begin{bmatrix}
d\left(\frac{\partial u}{\partial x}\right) \\
d\left(\frac{\partial v}{\partial x}\right) \\
d\left(\frac{\partial w}{\partial x}\right) \\
d\left(\frac{\partial u}{\partial y}\right) \\
d\left(\frac{\partial v}{\partial y}\right) \\
d\left(\frac{\partial w}{\partial y}\right) \\
d\left(\frac{\partial u}{\partial z}\right) \\
d\left(\frac{\partial v}{\partial z}\right) \\
d\left(\frac{\partial w}{\partial z}\right)
\end{bmatrix}^T
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
S_{21} & S_{22} & S_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
S_{31} & S_{32} & S_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S_{11} & S_{12} & S_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & S_{21} & S_{22} & S_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & S_{31} & S_{32} & S_{33}
\end{bmatrix}
\begin{bmatrix}
d\left(\frac{\partial u}{\partial x}\right) \\
d\left(\frac{\partial v}{\partial x}\right) \\
d\left(\frac{\partial w}{\partial x}\right) \\
d\left(\frac{\partial u}{\partial y}\right) \\
d\left(\frac{\partial v}{\partial y}\right) \\
d\left(\frac{\partial w}{\partial y}\right) \\
d\left(\frac{\partial u}{\partial z}\right) \\
d\left(\frac{\partial v}{\partial z}\right) \\
d\left(\frac{\partial w}{\partial z}\right)
\end{bmatrix} dV
$$

$$
= \iiint_V d\boldsymbol{\Theta}^T \hat{\mathrm{S}}\; d\boldsymbol{\Theta}\, dV \tag{22}
$$

Using equation (11), equation (22) can now be written as:

$$\iiint_V d\left(d\boldsymbol{\varepsilon}_G\right) : \mathrm{S}\ dV = d\mathbf{u}_e^T\left[\iiint_V \boldsymbol{\Phi}^T\ \hat{\mathrm{S}}\ \boldsymbol{\Phi}\, dV\right] d\mathbf{u}_e \qquad (23)$$

Therefore, equation (15) can now be written as:

$$d\mathbf{F}_e = \left[\iiint_V \mathbf{B}^{\mathrm{T}}\ \mathrm{D}\ \mathbf{B}\, dV + \iiint_V \boldsymbol{\Phi}^T\ \hat{\mathrm{S}}\ \boldsymbol{\Phi}\, dV\right] d\mathbf{u}_e + \left[\iiint_V \mathbf{B}^{\mathrm{T}}\ \mathrm{H}\ dV\right] dp \qquad (24a)$$

or,

$$d\mathbf{F}_e = \mathrm{K}_e\, d\mathbf{u}_e + \mathrm{P}_e dp \qquad (24b)$$

where $\mathrm{K}_e = \left[\iiint_V \mathbf{B}^{\mathrm{T}}\ \mathrm{D}\ \mathbf{B}\, dV + \iiint_V \boldsymbol{\Phi}^T\ \hat{\mathrm{S}}\ \boldsymbol{\Phi}\, dV\right]$, $\mathrm{P}_e = \left[\iiint_V \mathbf{B}^{\mathrm{T}}\ \mathrm{H}\ dV\right]$,

$\mathrm{H} = -\sqrt{I_3}\ \mathbf{C}^{-1}\ \mathrm{N}_p^{\mathrm{T}}$, and N_p is the hydrostatic pressure shape function.

Equation (24) represents the element equation of the current problem. The subsequent analysis includes standard finite element procedure, which consists of discretization, assembling all elements matrices, … etc.,tillgetting the global system of equations.

The resulted global system of equations is highly nonlinear due to:the nonlinear behavior of the material, the presence of hydrostatic pressure related to the volumetric part of the strain, and geometrical nonlinearity results from large deformation. Consequently an iterative solution procedure based on nonlinear Newton-Raphson algorithm is required.

SOLUTION ALGORITHM

Equation (24) represents non-linear set of equations,the non-linearity source, as mentioned above consists of material and geometrical terms,which are functions of the displacement derivatives and hydrostatic pressure. Thus a solution to equation (24) must involve an iterative solution procedure.As with most iterative solution procedures the first step is to solve the equivalent linear problem to obtain an initial estimate of the solution. The solution vector to this equation can then be used to evaluate the displacement derivatives and hydrostatic pressures to use in the full non-linear system of equations including the non-linear terms.Obviously this first solution vector, and subsequent solution vectors before convergence, will not fully satisfy the full non-linear set equations and a residual or error value will result if the solution vector is substituted back into equation (24). This residual can be reduced if a new set of hydrostatic pressures and stresses are calculated based on the previous iteration results. These values can then be used to compute new non-linear terms to augment the standard linear force vector. The solution to this set of equations will result in an improved vector of total nodal displacements. The difference between the new solution vector and the old one gives the iterative improvement in the solution vector.

This process is repeated until the ratio of the square of the solution vector increment over the square of the total solution vector results in a value less than a given permissible.

The basic iterative solution procedure convergence is further improved via breaking the applied load down into load increments. Therefore it is essential to develop an incremental version of the basic iterative solution algorithm. This can be achieved with the following algorithm shown in Figure 1. The significant difference however with the incremental algorithm is that for each iteration the 'old' and 'new' values of the solution vector and the non-linear stresses are stored. Therefore the total solution vector is formed from the total cumulative sum of the change in displacements at each iteration of each increment. The change in non-linear stress, or the difference between the current and previous iteration values, is used in the non-linear domain term. This means that for each increment the non-linear domain term reduces for each iteration until convergence occurs. Therefore, this algorithm is pseudo incremental because at each iteration of every increment the system of equations are still solved for the total displacements and hydrostatic pressure.

The non-linearity of the element equations is noticeable for the reason that the matrices in equation (24) are functions of the displacements and hydrostatic pressures. The standard Newton-Raphson algorithm is used based on the Taylor's series to linearize the element equations, higher order terms is omitted.The load is split up into incremental steps which are optimized by choosing steps according to the feeding back errors. This basically means that a separate Newton-Raphson solution is run for each load step.

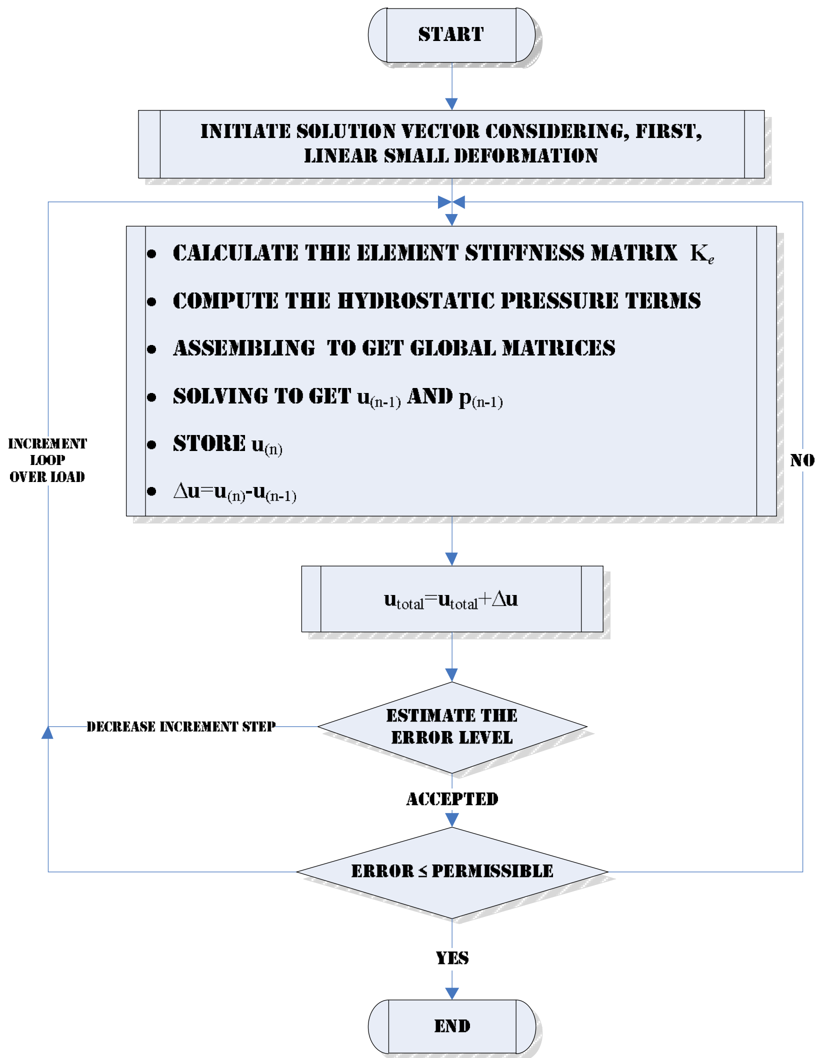

Figure (1): Incremental procedure withan iterative solution

NUMERICAL EXAMPLES

This section is concerned with the numerical evaluation of the constructed program which based on the nonlinear finite element methods derived in this paper.This was achieved via solving two numerical examples. The first one is hyperelastic circular plate subjected to water pressure, whereas, the second is concern with hyperelastic elastomeric cylinder.

Hyperelastic Circular Plate

In this example a flat circular membrane made of a rubber material is subjected to uniform water pressure, as shown in Figure (2). The edges of the membrane are fixed. The response of the membrane will be studied as pressure is increased from zero to 165kPa.

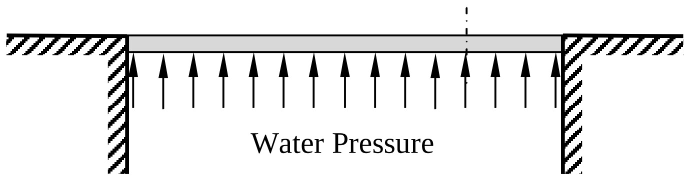

Figure (2): Flat circular rubber membrane

The dimensions and loading of the current problem is listed as:

Radius (mm)	Thickness (mm)	Water Pressure (kPa)
190	12.7	165

The results of the current problem are drawn in Figure 3, which shows the vertical displacements along the radial distance under different pressure values. The results are drawn, for both, ANSYS 12.0 and the corresponding ones of the current analysis. In ANSYS the problem is considered as a thin plate and the element type used in this simulation is chosen to be SHELL181, a finite strain layered shell, the model used for the material modeling, for both ANSYS and the current analysis,is Mooney-Rivlin.

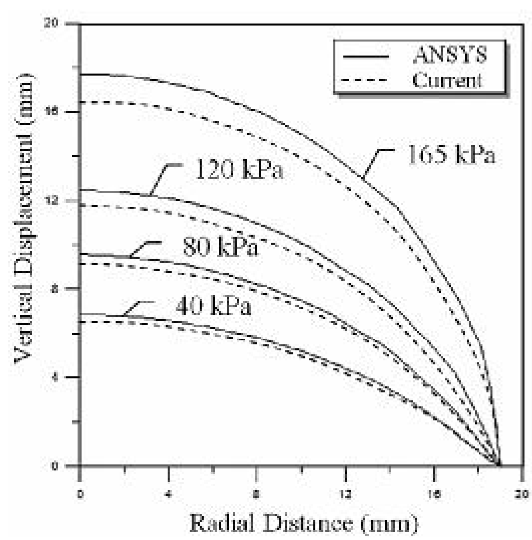

Figure (3): Vertical displacement of flat circular rubber Membrane under different pressure values.

While, for the current study, the problem is considered as three dimensional solid and the element type is taken to be eight nodes brick element. The plate is discretized using a mesh of 24 8-nodes brick elements for 3-D modeling of this problem.It is evident from these curves that there are relatively good agreements. And as the pressure values increase, which results larger deformation, the difference seems to be larger between the current results as compared with ANSYS. The maximum percentage error is 6% at the center of the plate for the highest pressure. This error is decreases as the pressure value decreases, also the error decreases as the deformation decreases, i.e. near the edges of the plate.

Hyperelastic Elastomeric Cylinder

In this example a hyperelastic elastomeric cylinder, Figure 4, subjected to internal pressure is considered, the geometric and loading properties of the current problem is chosen to coincide with Shi Shouxia and Yang Jialing [26], while the material properties is considered as Mooney-Rivlin and listed as:

Mooney-Rivlin Coefficients		**Inner Diameter (mm)**	**Outer Diameter (mm)**	**Length (mm)**	**Pressure (kPa)**
c_{10}	c_{01}				
1.37890	**0.324855**	**140**	**372**	**400**	**4000**

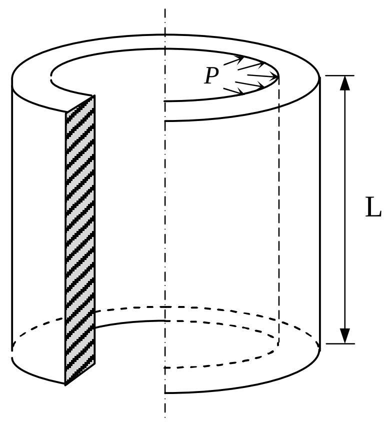

Figure (4):Hyperelastic elastomeric cylinder

The cylinder is discretized using a mesh of 96(8-nodes) brick elements for 3-D modeling of this problem. While in ANYSY, SOLID185 is used for 3-D modeling of hyperelastic solid structures. It also has mixed formulation capability for simulating deformations of fully incompressible hyperelastic materials. The number of elements is kept 96 to make a comparative study.

Similar to the first example, the problem is solved by using ANSYS 12.0 and compared with the current analysis. The material model used in ANSYS is Mooney-Rivlin.Table 1shows the results of the radial displacement at the inner surface under different internal pressure values. The results showed a good agreement between the current work and ANSYS 12.0 especially when the pressure value is low, and as the value of the pressure increases the difference between the results is slightly increases.

Table (1): Radial Displacement At The Inner Surface For Hyperelastic Elastomeric Cylinder under Different Internal Pressure Values.

	Radiad Displacements (mm)		
Internal Pressure (kPa)	**Current FEM**	**ANSYS 12.0**	**% Discrepancy**
1000	**14.9**	**15.2**	**2.01**
2000	**39.6**	**42.3**	**6.81**
2500	**68.1**	**72.7**	**6.75**
3000	**114.3**	**126.5**	**10.67**

The results of hoop stresses at the inner surface for different internal pressure values are shown in Table 2, which shows a very good agreement between the current work with that of ANSYS.

Table (2): Hoop stresses at the inner surface for hyperelastic elastomeric cylinder under different internal pressure values.

	Hoop Stresses (MPa)		
Internal Pressure (kPa)	**Current FEM**	**ANSYS 12.0**	**% Discrepancy**
1000	**1.70**	**1.71**	**0.58**
2000	**3.02**	**3.09**	**2.31**
3000	**5.10**	**5.32**	**4.31**
3500	**9.62**	**10.5**	**9.14**
4000	**20.8**	**22.5**	**8.17**

CONCLUSIONS

Proper use of nonlinear finite element analysis can make good predictions for the behavior of elastomeric materials components for the design purposes. Recently, this subject received excessive attention by many researchers to model such kinds of problems, but to the author knowledge, there is no full analysis available in the literature, all authors concentrated on specific part of the problem but not as a whole. Therefore, the main contribution of the present work is to give a detailed mathematical procedure for obtaining the element equation of this kind of the analysis. Afterwards, the numerical results if further improved via the use ofoptimized incremental stepsfor solving the nonlinear Newton-Raphson. This study assumes that the material obeysthe Mooney-Rivlin constitutive model with moderately large deformation up to approximately 100% extra streches.

Comparing the obtained results with that of ANSYS 12.0verified that the current formulation and the resulted computer program are valid and correct.There after,the obtained results are satisfactory enough for designing these kinds of problems.

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APPENDIX A

DIFFERENTIATION OF STRAIN INVARIANTS

A.1 Differentiation of I_1

The first modified strain invariants is defined as [22]:

$$\bar{I}_1 = I_3^{-\frac{1}{3}} I_1 = \left[|\mathbf{C}|\right]^{-\frac{1}{3}} Trace(\mathbf{C})$$

Hence,

$$d\bar{I}_1 = -\frac{1}{3}\left[|\mathbf{C}|\right]^{-\frac{1}{3}} Trace(\mathbf{C})\,\mathbf{C}^{-1} : \mathrm{d}\mathbf{C} + \left[|\mathbf{C}|\right]^{-\frac{1}{3}} : \mathrm{d}\mathbf{C}$$

Or:

$$\frac{\partial \bar{I}_1}{\partial \mathbf{C}} = \left[|\mathbf{C}|\right]^{-\frac{1}{3}} - \frac{1}{3}\left[|\mathbf{C}|\right]^{-\frac{1}{3}} Trace(\mathbf{C})\,\mathbf{C}^{-1}$$
$$= I_3^{-\frac{1}{3}}\left(\mathbf{I} - \frac{1}{3} I_1 \mathbf{C}^{-1}\right)$$

A.2 Differentiation of I_2

The second modified strain invariants is defined as [22]:

$$\bar{I}_2 = I_3^{-\frac{2}{3}} I_2$$

Where,

$$I_2 = \frac{1}{2}\left(I_1^2 - Trace\left(\mathbf{C}^2\right)\right)$$

Hence,

$$d\bar{I}_2 = -\frac{2}{3} I_3^{-\frac{2}{3}} I_2 \mathbf{C}^{-1} : \mathrm{d}\mathbf{C} + I_3^{-\frac{2}{3}}\left(I_1 \mathbf{I} - \mathbf{C}\right) : \mathrm{d}\mathbf{C}$$

Or:

$$\frac{\partial \bar{I}_2}{\partial \mathbf{C}} = I_3^{-\frac{2}{3}}\left(I_1 \mathbf{I} - \mathbf{C} - \frac{2}{3} I_2 \mathbf{C}^{-1}\right)$$

A.3 Differentiation of I_3

The third strain invariants is defined as [22]:

$$I_3 = \left|\mathbf{C}\right|$$

Hence,

$$dI_3 = I_3 \mathbf{C}^{-1} : d\mathbf{C}$$

Or:

$$\frac{\partial I_3}{\partial \mathbf{C}} = I_3 \mathbf{C}^{-1}$$

Using $J = \sqrt{I_3}$, then [22]:

$$dJ = \frac{1}{2} I_3^{-\frac{1}{2}} dI_3$$

$$\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2} I_3^{\frac{1}{2}} \mathbf{C}^{-1}$$