The Dynamics of Newton's Method on Complex Quartic Polynomial

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Key word: Newton's Method, Quartic Polynomial, Julia set.

Abstract:

Newton's method is used to calculate approximately the roots of any complex or real valued function consists iteration. For each complex polynomial $P(z)$, Newton's method defines a dynamical system on the complex Riemann sphere.

The our goal is to prove that any quartic polynomial with at least two distant roots topologically conjugate to N_μ and any two polynomials with roots has a similar quartic shape then the Newton function for each polynomial are conjugate.

Finally ,we study the symmetry of Newton's function with real and complex axis. The software MatLab will use to view the dynamics of quartic polynomial after the iterations of Newton's method. The graphical nature of the iterations gave us very nice properties that allow us to describe the behavior of the points in the plane.

1-Introduction

Newton's method is one of the preferred methods to find roots of differentiable function, Newton's method defines a dynamical system. If consists of iterating the function $f(z)$ then $N(z) = z - \frac{f(z)}{f'(z)}$

By starting with an initial approximation, z_0, sufficiently close to the root of $f(z)$,the sequence of iterates, $z_{n+1} = N(z_n)$, will converge to the root[1]. The study of iterated maps is the study of the dynamics of orbits of points under repeated composition of a function with itself[2].If the function $f(z)$ is polynomial ,then the iteration function $N(z)$ will be a rational function of the form,

$$N(z) = \frac{R(z)}{Q(z)}$$

Where $R(z)$ and $Q(z)$ are polynomials with real of complex coefficients [1]. The global study of Newton's method can now be analyzed using the theory of the complex dynamics of rational functions on the Riemann sphere ($\hat{C} = C \cup \{\infty\}$). Any complex analytical function will decompose the plane into two complementing sets, the stable set , where the dynamics are mostly tame, and the unstable set ,where the dynamics become chaotic and unpredictable [3]. The study of this idea was started by G. Julia and P. Fatou in the 1920[3],[4].

The orbit of a point z_0 is the set of iterates $\{z_0, z_1, z_2, ...\} = \{z_0, N(z_0), N(N(z_0)), ...\}$. The point z is a fixed point of $N(z)$ if $N(z) = z$, for Newton's method applied to a polynomial $P(z)$, each root of $P(z)$ will be a fixed point of $N(z)$,and these will be the only finite fixed points. If $P(z)$ is not degree one, then ∞ will also be a fixed point of $N(z)$ [1]. The point z is periodic point if $N^t(z) = z$, for some positive integer t. The least such integer t is called the period and the orbit of z is a t-cycle [1].

In the Newton's method ,we would like our initial point z_0 to converge to the fixed point that root. The certainly happens most of the time ,but other things can happen. The orbit of z_0 could converge to a t-cycle, or it could wander chaotically about the Riemann sphere [3]. If z is periodic point of period t, then the derivative $\lambda = (N^t)'(z)$ is called the eigenvalue of the periodic point z. It follows from the chain rule that λ is the product of the derivatives of $N(z)$ at each point on the orbit of z. A periodic point z classified as : superattracting if $\lambda = 0$; attracting if $|\lambda| < 1$; repelling if $|\lambda| > 1$; and neutral if $|\lambda| = 1$ [2].

The Julia set is the set of points whose orbits have unpredictable or chaotic behavior. We define the family of functions $\{f^n\}$ to be normal on U if every sequence of f^n has a subsequence that either converges uniformly on compact subsets of U or converges uniformly to ∞ on compact subsets of U. The Julia set, J, of f is defined to be the set of all points for which the family of iterates $\{f^n\}$ is not normal at z [5].equivalently, the Julia st of a rational map is equal to the closure of the set of repelling periodic points.. the Fatou set or stable set is the complement of the Julia set [4].

If we are interested in dynamics of $N(z)$ on the Riemann sphere, we can always conjugate $N(z)$ by invertible linear fractional (Mobius) transformation T, and the dynamics of the iterates of $N(z)$ ill be same as the iterates of $T \circ N \circ T^{-1}$ [1]. A Mobius transformation is a rational map of the form $T(z)=\dfrac{az+b}{cz+d},\quad ad-bc\neq 0$, where we have the usual convention that $T(\infty)=\dfrac{a}{c},\quad T(-\dfrac{d}{c})=\infty$,for more details see [4],[6].

From the properties of the Fatou and Julia sets we saw that they were compliments of one another of one another ,and the Julia set was the boundary for the points that converge that those that do not [3],[7] . Since the dynamics of quartic polynomials are more complicated than that quadratics and cubic polynomials which study in [1],[3] .we shall be relying more on computer graphics to illustrate their behavior and help give us a better understanding of what is actually happening. The conclusions for quadratics were somewhat easily obtained, and would be ideal if the same approach could be taken for quartic, along with getting the same sort of results. However ,the global conditions of the dynamics of N_μ seem to be much more complex .we will assign a coloring scheme to the points on the dynamical plane and use it for part of our investigation. we do not want to get ahead of ourselves thus we are going to develop idea like those in [3] for quartic polynomial.

The main results of this paper are prove the following propositions.

Proposition(1-1) :

For any complex quartic polynomial P with at least two distinct roots, N_p is to topologically conjugate to N_μ for some $\mu\in C$.

Proposition (1-2):

If $p(z)$ and $q(z)$ are quartic polynomials whose roots similar quadrate shape S_p and S_q, respectively then N_p is conjugate to N_q via some affine map.

Proposition (1-3):

If $\mu\in R$, then N_μ is symmetric with respect to the real axis.

We also will be studying the types of graphs one would get if they looked at these same Newton's function in the complex plan.

2-Newton's Method on Complex Quartic Polynomial

Let $P(z)$ be a fourth degree complex polynomial (**Quartic Polynomial**) $P(z)=a_4z^4+a_3z^3+a_2z^2+a_1z+a_0$ with four roots .it is also known that we can view this same polynomial in terms of its roots ,if we first factor out a_4 ,we now have $P(z)=a_4(z-a)(z-b)(z-c)(z-d)$ where a,b,c,d are roots of $P(z)$ [3].Newton's method on any general quartic polynomial $P:\hat{C}\to\hat{C}$ that has at least two distinct root is $N_p(z)=z-\dfrac{P(z)}{P'(z)}$

To simplify the understanding of the dynamical properties of Newton's Method on quartic polynomials, we will utilize the one-parameter family $p_{\mu}(z)=(z-\mu)(z+\mu)(z-1)(z+1)$ rather than $P(z)$ itself. This still contains quartic with at least two distinct roots, and we will refer to the function created after applying Newton's method as N_{μ}. This is defined by $N_{\mu}(z)=\dfrac{3z^4-2\mu^2z^2-\mu^2}{4z^3-4\mu^2 z}$

We must introduce the notion of a cross ratio to find a conjugacy between N_{μ} and N_p . We are allowed to use this category of maps, since they are the class that is analytical and differentiable on the Riemann sphere .

We us the cross-ratio of five distinct point z_0,z_1,z_2,z_3,z_4 as the Mobius transformation [6];[8] .

$$(z_0,z_1,z_2,z_3,z_4)=\frac{(z_0-z_2)(z_1-z_2)(z_3-z_4)}{(z_0-z_3)(z_1-z_3)(z_2-z_4)}$$

which brings us to the following proposition

Proof of Proposition (1-1):

Let us assume, we have a polynomial $P(z)$ with at least two distinct roots ,so we will have to consider two cases for this.

First case:

when a,b,c,d are all distinct, let's choose a complex quartic polynomial

$$P(z)=a_4(z-a)(z-b)(z-c)(z-d).$$

Considering the cross ratio of the roots of $P(z)$ and the roots of

$$p_{\mu}(z)=(z-\mu)(z+\mu)(z-1)(z+1),$$

we can derive a Mobius transformation ,call it T ,which conjugates N_{μ} and N_p. We find T by setting

$$(z,a,b,c,d)=(w,\mu,-\mu,1,-1)$$

$$\frac{(z-b)(a-b)(c-d)}{(z-c)(a-c)(b-d)}=\frac{(w+\mu)(\mu+\mu)(1+1)}{(w-1)(\mu-1)(-\mu+1)}$$

$$\frac{z(ac-bc-ad+bd)-b(ac-bc-ad+bd)}{z(ab-cb-ad+cd)-c(ab-cb-ad+cd)}=\frac{4\mu w+4\mu^2}{(-\mu^2+2\mu-1)(w-1)}$$ Cross multiplying and solving for w yields:

$$w=T(z)=\frac{a_1z+b_1}{c_1z+d_1}$$

Where

$$a_1=(ac-bc-ad+bd)(-\mu^2+2\mu-1)+4\mu^2(ab-cd-ad+cd)$$

$b_1 = b(ac-bc-ad+bd)(-\mu^2+2\mu-1)-4\mu^2 c(ab-cd-ad+cd)$

$$c_1 = (ac-bc-ad+bd)(-\mu^2+2\mu-1)-4\mu(ab-cd-ad+cd)$$

$d_1 = b(ac-bc-ad+bd)(-\mu^2+2\mu-1)-4\mu c(ab-cd-ad+cd)$

This is somewhat tedious to work with, so let's simplify by find a value of μ for which the map T is affine map. That is, make the function a linear mapping, we set $c_1 = 0$ we get the following transformation

$$T(z) = \frac{a_1}{d_1} z + \frac{b_1}{d_1}$$

And solve for μ to give us

$$\mu = \frac{-(2ac+2bc+2ad+2bd-4ab-4cd) \pm \sqrt{(2ac+2bc+2ad+2bd-4ab-4cd)^2 - 4(ac-bc-ad+bd)^2}}{2(ac-bc-ad+bd)}$$

So ,we have two μ:

$$\mu_1 = \frac{-(2ac+2bc+2ad+2bd-4ab-4cd) + \sqrt{(2ac+2bc+2ad+2bd-4ab-4cd)^2 - 4(ac-bc-ad+bd)^2}}{2(ac-bc-ad+bd)}$$

and

$$\mu_2 = \frac{-(2ac+2bc+2ad+2bd-4ab-4cd) - \sqrt{(2ac+2bc+2ad+2bd-4ab-4cd)^2 - 4(ac-bc-ad+bd)^2}}{2(ac-bc-ad+bd)}$$

It immediately follows that $N_p \sim N_{\mu_1}$ and $N_p \sim N_{\mu_2}$.

Second case :

let $P(z)$ have only two distinct roots ,b and c. By letting $c = a$ in our original equation for c_1 ,then we have $\mu = 1$, $N_p \sim N_1$ where $P(z) = a_4(z-c)^2(z-b)(z-d)$ □

Let us define a quadrate shape created by the roots of N_μ when $\mu \in C - R$. We will denote this quadrate S_μ where each side of S_μ is determined by any pair of distinct root. Note that if $\mu \in R$, then all the roots will lie on the real axis and thus will be collinear[3].we will be able to infer some things about N_μ just by looking at this quadrate shape. we can even generalize the following property to generic quartic polynomial

Remark[1],[4]:

Any Mobius transformation can be represented as the composition of a finite number of inversions , created by the function $v(z) = \frac{1}{z}$, magnifications, created by the function $m(z) = Az$ where $A \in R^+$,rotations , created by the function $r(z) = e^{i\theta} z$,where $\theta \in R$, and translations, created by the function $t(z) = z + B$ where $B \in C$.

Now, we prove the proposition (1-2).

Proof of Proposition (1-2) :

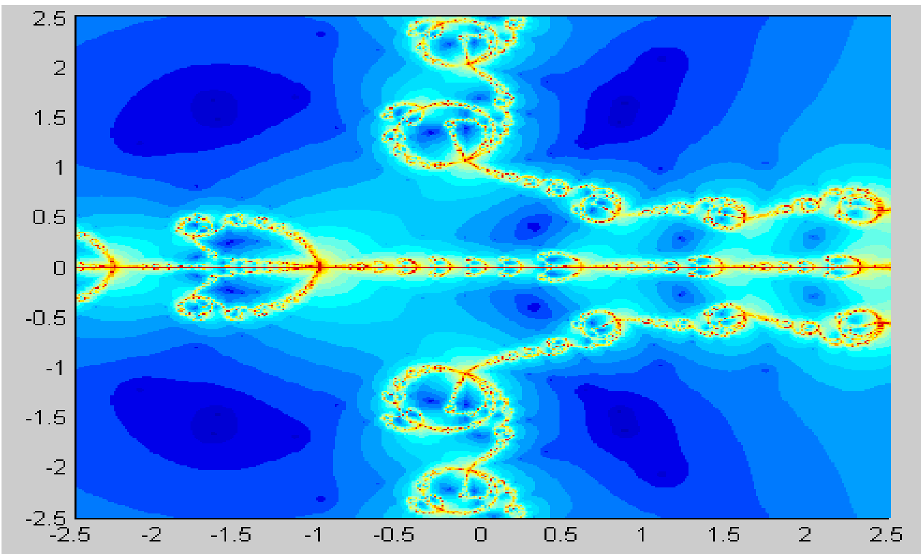
Consider the geometric representation of the two quadrate shape S_p and S_q. We only need to make . we only need to make three manipulations to achieve our goal. Recall a linear transformation is the form $g(z) = az + b$,where $a,b \in C$ and $a \neq 0$.

Now ,let us consider

$$\begin{aligned} g(z) &= (m \circ r \circ t)(z) \\ &= m(r(z+c)) \\ &= m(e^{i\theta}(z+c)) \\ &= A(e^{i\theta}(z+c)) \\ &= Ae^{i\theta}z + Ae^{i\theta}c)) \\ &= A_1 z + B_1 \end{aligned}$$

Where $A_1 = Ae^{i\theta}$ and $B_1 = Ae^{i\theta}c$. Each of these mappings are non-fractional, linear mappings, thus $g(z)$ is conformal for all points in complex plane. With a few calculations, one can see that the angles of S_p are preserved under each mapping. This is independent from our choice of c,θ and A. Therefore, we can choose these such that $g(z)$ will conjugate S_p and S_q. Thus implying that $p(z)$ conjugates to $q(z)$,and from our previous knowledge, making N_p conjugate to N_q through this affine map.

The following pictures illustrates the rate of convergence for each point in the plane with a different color .we can see that the points form a right quadrate shape in the plane. Thus, any roots from another polynomial that a right quadrate shape will be conjugate to this polynomial.



Figure(1):Newton's method on quartic polynomial $p(z) = z^4 + z^3 + z^2 + z + 1$

Not only can the function created with the Newton's method of our particular family of quartic polynomial be examined by quadrate shape created with the roots, even when we do not see a quadrate shape, N_μ exhibits some very nice properties. In fact, what we realize is that the roots of the quadrate shape have become collinear and form a line the plane.

This exemplifies the following proposition .

Proof of Proposition (1-3):

we need to show is that for the two complex roots of the function $\mathrm{Re}(N_\mu(z)) = \mathrm{Re}(N_\mu(\bar{z}))$ and $\mathrm{Im}(N_\mu(z)) = -\mathrm{Im}(N_\mu(\bar{z}))$

$\mathrm{Re}(N_\mu(x+iy)) = \mathrm{Re}(N_\mu(x-iy))$ and $\mathrm{Im}(N_\mu(x+iy)) = -\mathrm{Im}(N_\mu(x-iy))$

We call our function N_μ since we are assuming that we have $\mu \in R$. Refer back to our original Newton's function

$$N_\mu(z) = \frac{3z^4 - 2\mu^2 z^2 - \mu^2}{4z^3 - 4\mu^2 z}$$

Now ,if we substitute $z = x + iy$ and simplify we get

$$N_\mu(x+iy) = \frac{3(x+iy)^4 - 2\mu^2(x+iy)^2 - \mu^2}{4(x+iy)^3 - 4\mu^2(x+iy)}$$

$$N_\mu(z) = \frac{3(x^2-y^2)^2 - 12x^2y^2 - 2\mu(x^2-y^2) - \mu^2 + i(12(xy(x^2-y^2)) + 4\mu^2 xy)}{4(x^3-3xy^2) - 4\mu^2 x + i(4(3x^2y - y^3) - 4\mu^2 y)}$$

With the same approach, let us examine $\bar{z} = x - iy$

$$N_\mu(x-iy) = \frac{3(x-iy)^4 - 2\mu^2(x-iy)^2 - \mu^2}{4(x-iy)^3 - 4\mu^2(x-iy)}$$

$$N_\mu(\bar{z}) = \frac{3(x^2-y^2)^2 - 12x^2y^2 - 2\mu(x^2-y^2) - \mu^2 - i(12(xy(x^2-y^2)) + 4\mu^2 xy)}{4(x^3-3xy^2) - 4\mu^2 x - i(4(3x^2y - y^3) - 4\mu^2 y)}$$

Consider the following substitutions,

$$r_1 = 3(x^2-y^2)^2 - 12x^2y^2 - 2\mu(x^2-y^2) - \mu^2$$
$$s_1 = 12(xy(x^2-y^2)) + 4\mu^2 xy$$
$$r_2 = 4(x^3 - 3xy^2) - 4\mu^2 x$$
$$s_2 = 4(3x^2y - y^3) - 4\mu^2 y$$

So ,we have that

$$N_\mu(z) = \frac{r_1 + is_1}{r_2 + is_2}$$

$$N_\mu(\bar{z}) = \frac{r_1 - is_1}{r_2 - is_2}$$

Multiplying the top and the bottom by complex conjugate of each denominator, we have $N_{\mu}(z)=\frac{r_1r_2+s_1s_2}{r_2^2+s_2^2}+i\frac{(-r_1s_2+s_1r_2)}{r_2^2+s_2^2}$

$$N_{\mu}(\bar{z})=\frac{r_1r_2+s_1s_2}{r_2^2+s_2^2}+i\frac{(r_1s_2-s_1r_2)}{r_2^2+s_2^2}$$

With this representation it is clearly that the real parts of $N_{\mu}(z)$ and $N_{\mu}(\bar{z})$ are in fact equal, and the corresponding imaginary parts are complex conjugates.

3-Graphical analysis

In proposition (1-1),we prove that any complex quartic polynomial with at least two distinct root is topologically conjugate to

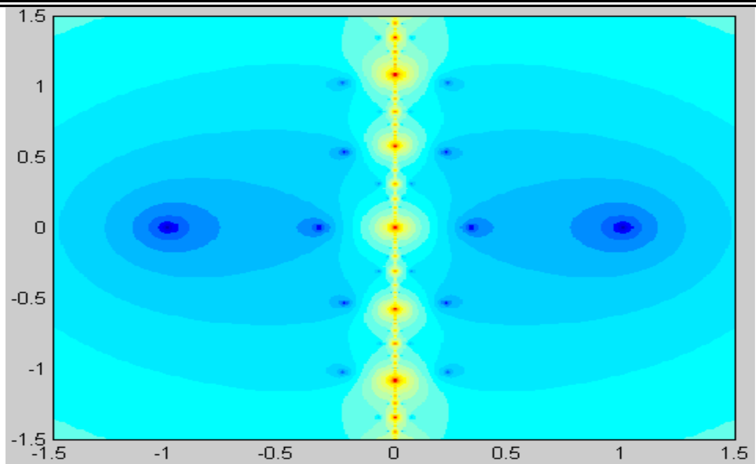
$$N_{\mu}(z)=\frac{3z^4-2\mu^2z^2-\mu^2}{4z^3-4\mu^2z}$$

where

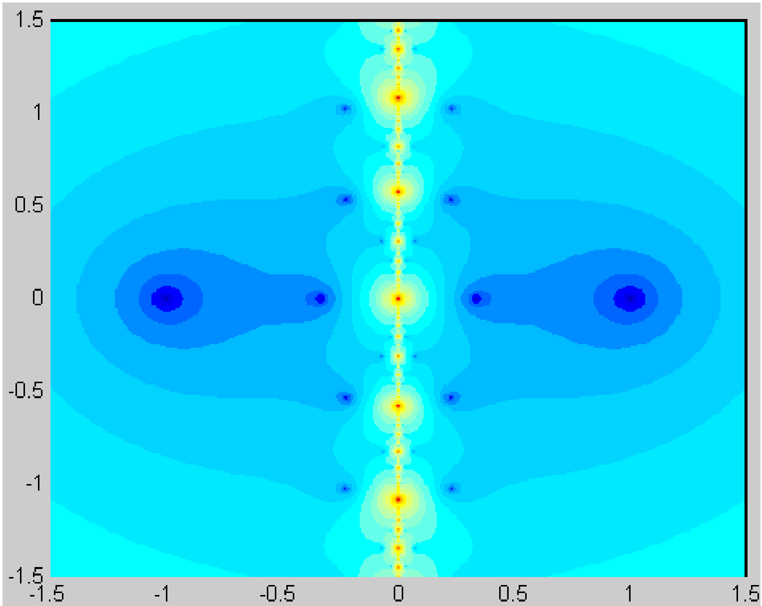
$$\mu=\frac{-(2ac+2bc+2ad+2bd-4ab-4cd)\pm\sqrt{(2ac+2bc+2ad+2bd-4ab-4cd)^2-4(ac-bc-ad+bd)^2}}{2(ac-bc-ad+bd)}$$

is only parameter with a,b,c,d as the coefficients of our quartic polynomial. This Newton's function, $N_{\mu}(z)$ is that in which we will be referring to for the remainder of the discussion. we want to utilize the ability of computer graphics created with the program MatLab which will enable us to describe and visualize the dynamics of quartic polynomials. Let us focus on the behavior of a particular point in the plane. The Julia set of a complex function is the set of all points on the boundary between the set of points that escape to infinity and the set of points that do not escape to infinity. We see this with the following example.

Consider $p(z)=z^4+z^3+z^2+z+1$, we calculate the values of μ from proposition(1-1) $\mu_1=-37.9526+20.4559\text{i}$,which $\mu_2=18.6806-16.2022\text{i}$. Let's look at the convergence of the points under Newton's method ,for deferent value of μ .we see this in Figure (2) and Figure(3).

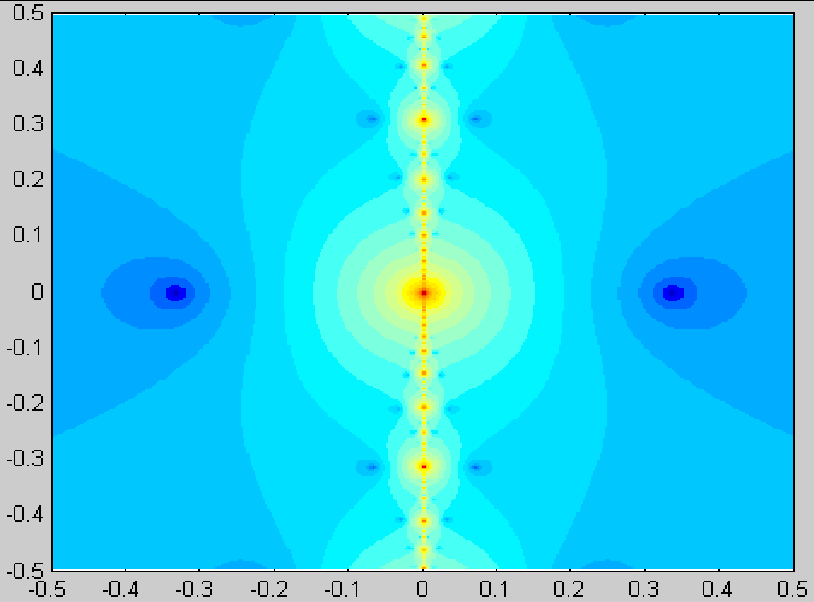


Figure(2): N_μ for $\mu_1 = -37.9526 + 20.4559i$ from -1.5 to 1.5



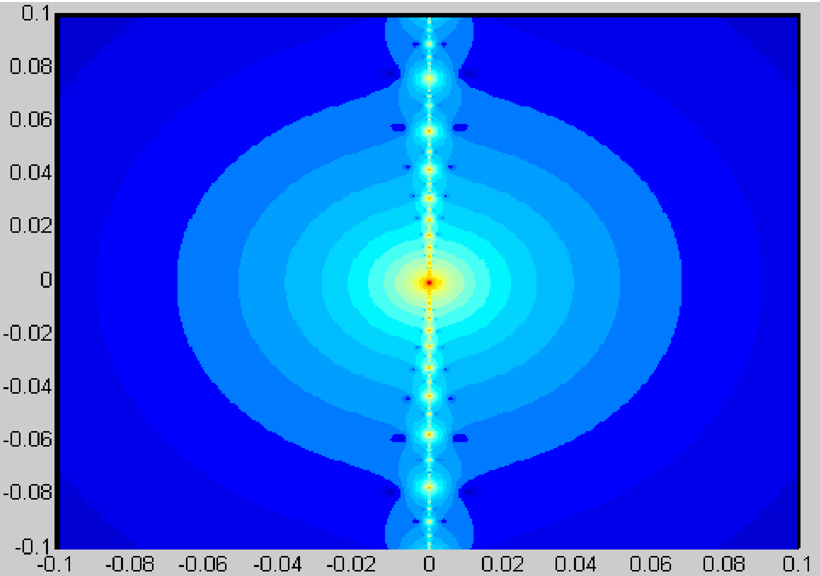
Figure(3): N_μ for $\mu_2 = 18.6806 - 16.2022i$ from -1.5 to 1.5

Now, we zoom in on the portion in the center that looks like it is glowing. As we zoom in, we see this part taking shape. This is our Julia set for the particular polynomial created with $\mu_1 = -37.9526 + 20.4559i$



Figure(4): N_μ for $\mu_1 = -37.9526 + 20.4559i$ from -0.5 to 0.5

The is not only figure that we can see with a particular value of μ .we also gets pictures that illustrate parts of nature .



Figure(5): N_μ for $\mu_1 = -37.9526 + 20.4559i$ from -0.1 to 0.1

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Appendix

A: Newton's Method on Complex Quartic Polynomial

```
%This program will plot convergence of the values for the
Newton function
%of quartic polynomial for any choice of coefficients.
function NMQ(a b c d e)
% default settings
min_re=-1.5;
max_re=1.5;
min_im=-1.5;
max_im=1.5;
n_re=400;
n_im=400;
tol=0.01;
coeff=[a b c d e];polyRoots=roots(coeff)
format compact;
max_steps=20;
%step size
delta_re=(max_re-min_re)/n_re;delta_im=(max_im-
min_im)/n_im;
x=min_re:delta_re:max_re;y=min_im:delta_im:max_im;
[X,Y]=meshgrid(x,y);Z=X+i*Y;
for j=1:n_im+1
    for k=1:n_re+1
        z=Z(j,k);
        if z==0
        z=tol;
    end
    m=0;
    flag=0;
```

```
    while(flag==0)
        %iteration
        z=z-
(a*z.^4+b*z.^3+c*z.^2+d^z+e)./(4*a*z.^3+3*b*z.^2+2*c*z+d)
        if norm(a*z.^4+b*z.^3+c*z.^2+d^z+e)<=tol
            flag=1;
        end
        if m>max_steps
            flag=1;
        end
        m=m+1;
    end
    %assign color according to number of steps
    Z(j,k)=m;
end
end
%plot the result
colormap(hot);colormap(jet(20));
brighten(0.5);
image(Z)
pcolor(X,Y,Z)
axis off;
shading flat;
```

B: Newton's Method on μ Quartic

```
%This program will plot the rate of convergence for value
of the Newton
% function of quartic polynomial simply change the values
for the mim and
% max of the x axis and y axis to zoom in or out .Figures
(2),(4)and (5)
```

% with $\mu_1 = -37.9526 + 20.4559i$ and Figures(3) with $\mu_2 = 18.6806 - 16.2022i$.

```
function NMmu(mu)
min_re=-1.5;
max_re=1.5;
min_im=-1.5;
max_im=1.5;
n_re=400;
n_im=400;
tol=0.01;
```

$mu = -37.9526 + 20.4559i$;

```
%forms x and y vectors of n points between min and max
default values
```

```
x=linspace(min_re,max_re,n_re);y=linspace(min_im,max_im,n
_im);
max_steps=50;
%step size
[X,Y]=meshgrid(x,y);Z=X+i*Y;
for j=1:n_im
   for k=1:n_re
   z=Z(j,k);
     if z==0
     z=tol;
     end
   m=0;
   flag=0;
   while(flag==0)
   %iteration
   z=z-(z.^4-((mu).^2+1)*z.^2+(mu).^2)./(4*z.^3-
4*((mu).^2*z));
     if norm(z.^4-((mu).^2+1)*z.^2+(mu).^2)<=tol
     flag=1;
     end
     if m>max_steps
     flag=1;
     end
  m=m+1;
  end
  %assign color according to number of steps
  Z(j,k)=m;
  end
end
%plot the result
colormap(hot);
colormap(jet(50));
brighten(0.5);image(Z)
pcolor(X,Y,Z)
shading flat;
```

دينمية طريقة نيوتن على متعددات الحدود من الدرجة الرابعة

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الخلاصة:

تستخدم طريقة نيوتن في حساب الجذور التقريبية للدوال الحقيقية والمعقدة وتتضمن تكرار. ان الهدف الأساسي لهذا البحث هو برهنة ان أي متعددة حدود من الدرجة الرابعة $P(z)$ وتحوي على الأقل جذريين مختلفين ترافقها دالة نيوتن N_{μ} و تعتمد على المعلمة μ والتي يمكن حسابها من $P(z)$ ، كذلك برهنا على ان أي متعددتين حدود لجذورهما نفس الشكل الرباعي يكونان مترافقين.

أخيراً تم دراسة خاصية التناظر حول المحور الحقيقي لدالة نيوتن وقد استخدمنا برنامج (MatLab) لعرض دينمية متعددات الحدود هذه بعد حلها بطريقة نيوتن.