Block and Bool Methods for Solving Special Type of n^{th} Order Nonlinear Delay Volterra Integro-Differential Equations

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Abstract

A proposed method is presented to solve a special type of n^{th} order nonlinear delay Volterra integro-differential equations (DVIDE's) numerically using fourth-order six stages block and Bool methods. An algorithm with the aid of Matlab language is derived to treat numerically three types (retarded, neutral and mixed) n^{th} order nonlinear DVIDE's using block and Bool methods. Comparison between numerical and exact results has been given via three numerical examples with illustrative graphing which were sketched by Matlab for conciliated the accuracy of results of the proposed method.

Keywords: Nonlinear Delay Volterra Integro-Differential Equation, Block method, Bool method and Algorithm.

1- Introduction

One of the most important and applicable subjects of applied mathematics, and in developing modern mathematics is the integral equations. The names of many modern mathematicians notably, Volterra, Fredholm, Cauchy and others are associated with this topic [1].

The name integral equation was introduced by Bois-Reymond in 1888 [2]. However, the nonlinear integral equation which is Volterra equation, was introduced by Volterra in 1884 and in 1959 Volterra's book " Theory of Functional and of Integral and Integro-Differential Equations" appeared [2].

The integral and integro-differential equations formulation of physical problems are more elegant and compact than the differential equation formulation, since the boundary conditions can be satisfied and embedded in the integral or integro-differential equation. Also the form of the solution to an integro-differential equation is often more stable for today's extremely fast machine computation. Delay integro-differential equation has been developed over twenty years ago where one of its types widely is used in control systems and digital communication systems [1,3].

A brief review of some background on the nonlinear delay integro-differential equation and their types are given in the following section.

2- Delay Integro-Differential Equation (DIDE):

The integro-differential equation is an equation involving one (or more) unknown function $u(x)$ together with both differential and integral operations on u. It means that it is an equation containing derivative of the unknown function $u(x)$, which appears outside the integral sign [1,4].

The delay integro-differential equation is a delay differential equation in which the unknown function $u(x)$ can appear under an integral sign [5]. The main difference between delay differential equation and ordinary differential equation is the kind of initial condition that should be used in delay differential equation differs from ordinary differential equation, so that one should specify in delay differential equations an initial functions on some intervals say $[x_0-\tau, x_0]$ and then try to find the solution for all $x \ge x_0$ [6,7].

Consider the following form of special type of n^th order nonlinear delay Volterra integro-differential equation (DVIDE) [7,8]:

$$\sum_{i=0}^{n} p_i(x)\frac{d^i u(x)}{dx^i}+\sum_{i=1}^{n} q_i(x)\frac{d^i u(x-\tau_i)}{dx^i}+\sum_{i=0}^{n} r_i(x)u(x-\tau_i)= g(x)+\lambda\int_a^x k(x,t)\left[u(t-\tau)\right]^w dt \quad x\ge a, w>1 \qquad \ldots (1)$$

with initial functions:
$$\left.\begin{array}{l} u(x)=\phi(x)\\ u'(x)=\phi'(x)\\ \quad\vdots\\ u^{(n-1)}(x)=\phi^{(n-1)}(x)\end{array}\right\} \quad x\le x_0\ , \quad i=0,1,\ldots,n$$

where $g(x), p_i(x), q_i(x), k(x,t)$ are known functions of x, $u(x)$ is the unknown function, λ is a scalar parameter (in this work λ=1), a is the limit of the integral where $a \le x$ and $\tau, \tau_0, \tau_1, ..., \tau_n$ and w are positive numbers.

Delay integro-differential equations are classified into three types [7,9,10] :-

1^st type:- Equation (1) is called Retarded type if the derivatives of unknown function appear without difference argument (i.e. the delay comes in u only) and the delay appears in the integrand unknown function (i.e. $\tau \ne 0$).

2^nd type: Equation (1) is called a neutral type if the highest-order derivative of unknown function appears both with and without difference argument and the delay does not appear in the integrand function (i.e. $\tau = 0$).

3^rd type:- All other DIDE's in eq.(1) are called mixed types, which are combination of the previous two types.

3- Block and Bool Methods

In this research block method was employed together with Bool method for finding the numerical solution for three types of nonlinear delay Volterra integro-differential equations.

- **Bool Method:**

Bool method is one of basic formula of quadrature approximation methods for integration. It approximates the function on the interval $[t_0, t_4]$ by a curve that possesses through five points. When it is applied over the interval [*a*,*b*], the composite Bool rule is obtained as [1,3] :

$$\int_a^b f(t)dt = \frac{2H}{45}\begin{bmatrix} 7f_0 + 32f_1 + 12f_2 + 32f_3 + 14f_4 + 32f_5 + 12f_6 + 32f_7 \\ + \cdots + 14f_{N-4} + 32f_{N-3} + 12f_{N-2} + 32f_{N-1} + f_N \end{bmatrix} \quad \ldots (2)$$

where *a*, *b* are the limits of the integral, $H = \frac{(b-a)}{N}$, *N* is the number of intervals $([t_0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N])$ which is the multiple of (4), $f_i = f(t_i)$ $t_0 = a$, $t_N = b$ and $t_i = a + iH$ are called the integration nodes which are lying in the interval [*a*,*b*] where $i = 0, 1, \ldots, N$.

In order to solve three types of nonlinear delay Volterra integro-differential equations numerically, Bool method is valid for the integral of DVIDE's.

- **Block Method:**

Block method is sufficient for the solution of non-stiff and stiff problems. The concept of block method is essentially an extrapolation procedure and has the advantage of being self-starting. Block method was described for differential equation by Milne and Young [12,13].

Consider the following first order differential equation :

$y' = f(t, y(t))$ with initial condition $y(t_0) = y_0$ ...(3)

A block method up to the fourth-order for eq.(3) is computed by:

$$y_{n+1} = y_n + hy'_n \qquad \textit{order } 2$$
$$y_{n+2} = y_n + 2hy'_{n+1} \qquad \textit{order } 2$$
$$y_{n+1} = y_n + (h/2)[y'_n + y'_{n+1}] \qquad \textit{order } 3$$
$$y_{n+2} = y_n + (h/2)[y'_n + y'_{n+2}] \qquad \textit{order } 3$$
$$y_{n+1} = y_n + (h/12)[5y'_n + 8y'_{n+1} - y'_{n+2}] \qquad \textit{order } 4$$
$$y_{n+2} = y_n + (h/3)[y'_n + 4y'_{n+1} + y'_{n+2}] \qquad \textit{order } 4$$

Block method of second and third order have been little used for ordinary differential equations, in general, and delay differential equations in particular because they required more evaluation of the function *f*.

However, the following fourth order block method, which is most popular and more efficient for dealing with differential equations.

Let

$$\left.\begin{aligned}
B_1 &= f(t_n, y(t_n)) \\
B_2 &= f(t_n + h, y(t_n) + hB_1) \\
B_3 &= f\left(t_n + h, y(t_n) + \frac{h}{2}B_1 + \frac{h}{2}B_2\right) \\
B_4 &= f(t_n + 2h, y(t_n) + 2hB_3) \\
B_5 &= f\left(t_n + h, y(t_n) + \frac{h}{12}(5B_1 + 8B_3 - B_4)\right) \\
B_6 &= f\left(t_n + 2h, y(t_n) + \frac{h}{3}(B_1 + B_4 + 4B_5)\right)
\end{aligned}\right\} \quad \ldots (4)$$

Then the fourth order-six stages block method may be written in the form:

$$y_{n+1} = y_n + \frac{h}{12}(5B_1 + 8B_3 - B_4) \quad \ldots (5)$$

$$y_{n+2} = y_n + \frac{h}{3}(B_1 + 4B_5 + B_6) \quad \ldots (6)$$

A fourth order six-stage block method is said to be stable, since (i) its region of absolute stability contains Ω_1 and Ω_2 and (ii) it is accurate for all $h \in \Omega_2$ when applied the scalar test equation $y' = \lambda y,\ y(x_0) = y_0$, with constant step size *h* and λ is a complex constant with $\mathrm{Re}\,\lambda < 0$, where

$$\Omega_1 = \{h\lambda \mid \mathrm{Re}\ h\lambda < -a\}$$

$$\Omega_2 = \{h\lambda \mid -a \le \mathrm{Re}\ h\lambda \le b, -c \le \mathrm{Im}\ h\lambda \le c\}$$

and *a*, *b* and *c* are positive constants [13,14]. The regions Ω_1 and Ω_2 of the complex $h\lambda$ – plane are shown in figure (1) [14]:

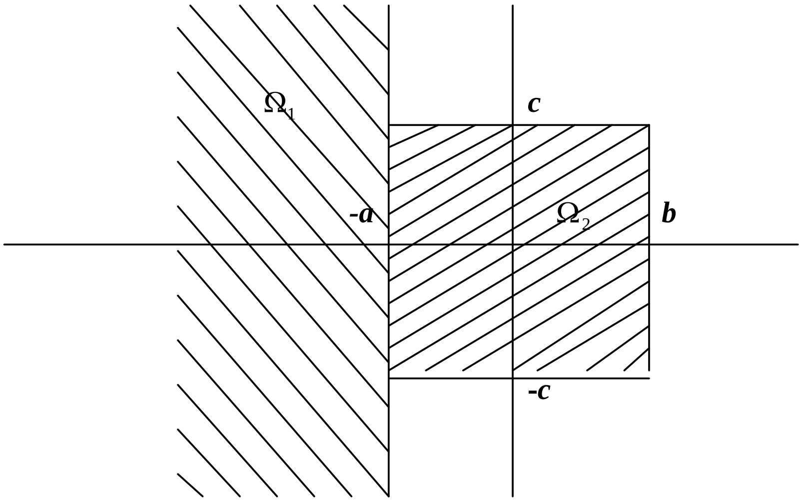

Figure (1) The regions Ω_1 and Ω_2 of the complex $h\lambda$ –plane.

4- The Solution of n^th^ Order Nonlinear DVIDE Using Block and Bool Methods:

The n^th^ order nonlinear DVIDE in eq.(1) can be written as:

$$f\begin{pmatrix} x, p_0(x)u(x), p_1(x)u'(x),\dots, p_{n-1}(x)u^{(n-1)}(x), p_n(x)u^{(n)}(x), q_1(x)u'(x-\tau_1), \\ \dots, q_n(x)u^{(n)}(x-\tau_n), r_0(x)u(x-\tau_0),\dots, r_n(x)u(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix} = 0 \quad \dots (7)$$

with initial functions:

$$\left.\begin{array}{l} u(x)=\phi(x) \\ u'(x)=\phi'(x) \\ \quad\vdots \\ u^{(n-1)}(x)=\phi^{(n-1)}(x) \end{array}\right\} \quad x\le x_0\ ,\quad i=0,1,\dots,n$$

where $I[Q(x,t)]$ is the nonlinear finite integral on $[a,x]$, $x\ge a$ and $Q(x,t)=k(x,t)\,[u(t-\tau)]^w$.

Hence, eq.(7) can be written as:

$$\frac{d^n u(x)}{dx^n} = f\begin{pmatrix} x, p_0(x)u(x), p_1(x)u'(x),\dots, p_{n-1}(x)u^{(n-1)}(x), q_1(x)u'(x-\tau_1),\dots, \\ q_n(x)u^{(n)}(x-\tau_n), r_0(x)u(x-\tau_0),\dots, r_n(x)u(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix} \quad \dots (8)$$

with initial functions:

$$\left.\begin{array}{l} u(x)=\phi(x) \\ u'(x)=\phi'(x) \\ \quad\vdots \\ u^{(n-1)}(x)=\phi^{(n-1)}(x) \end{array}\right\} \quad x\le x_0\ ,\quad i=0,1,\dots,n$$

Obviously, the n^th^ order equation (8) with difference argument may be replaced by a system of n^th^ equation of first order DVIDE's as follows:
Let

$$\begin{aligned}
v_1(x) &= u(x)\\
v_2(x) &= u'(x)\\
&\vdots\\
v_{n-1}(x) &= u^{(n-2)}(x)\\
v_n(x) &= u^{(n-1)}(x)
\end{aligned}$$

and the above initial functions become:

$$\left.\begin{aligned}
v_1(x) &= \phi(x)\\
v_2(x) &= \phi'(x)\\
&\vdots\\
v_n(x) &= \phi^{(n-1)}(x)
\end{aligned}\right\} \quad for \quad x \le x_0\ , \quad i = 0,1,\ldots,n$$

Then, one gets the following system of the first order equations:

$$\begin{aligned}
v_1'(x) &= v_2(x)\\
v_2'(x) &= v_3(x)\\
&\vdots\\
v_{n-1}'(x) &= v_n(x)\\
v_n'(x) &= f\begin{pmatrix} x, p_0(x)v_1(x), p_1(x)v_2(x),\ldots, p_{n-1}(x)v_n(x), q_1(x)\phi'(x-\tau_1),\ldots, q_n(x)\\ \phi^{(n)}(x-\tau_n), r_0(x)\phi(x-\tau_0),\ldots, r_n(x)\phi(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix}
\end{aligned} \qquad \ldots (9)$$

Hence, applying Bool method eq.(2) for computing $I[Q(x,t)]$ of equation (9) yields:

$$\begin{aligned}
I[Q(x,t)] &= \int_a^x Q(x,t)dt = Bool\big(Q(x,t),a,x,N\big)\\
&= \frac{2H}{45}\begin{bmatrix} 7Q(x,t_0)+32Q(x,t_1)+12Q(x,t_2)+32Q(x,t_3)+14Q(x,t_4)+32Q(x,t_5)\\ +\cdots+14Q(x,t_{N-4})+32Q(x,t_{N-3})+12Q(x,t_{N-2})+32Q(x,t_{N-1})+Q(x,t_N) \end{bmatrix}
\end{aligned} \qquad \ldots (10)$$

where $t_0 = a \ and \ t_N = x$ are the limits of the integral in nonlinear DVIDE, $H = \frac{(x-a)}{N}$, and $t_i = a + iH$ where $i = 0,1,\ldots,N$.

By using the result of the integral $I[Q(x,t)]$ in eq.(10), the numerical solution of the system of first order nonlinear DVIDE's in eq.(9) can be treated numerically by using fourth order block method as follows:

$$v_i(x_{j+1}) = v_i(x_j) + \frac{h}{12}(5B_{1i} + 8B_{3i} - B_{4i}) \qquad \ldots (11)$$

$$v_i(x_{j+2}) = v_i(x_j) + \frac{h}{3}(B_{1i} + 4B_{5i} + B_{6i}) \qquad \ldots (12)$$

where

$$B_{1i}=f_i\begin{pmatrix} x_j, p_0(x_j)v_1(x_j),\ldots,p_{n-1}(x_j)v_n(x_j),q_1(x_j)\phi'(x_j-\tau_1),\ldots,q_n(x_j)\phi^{(n)}(x_j-\tau_n),\\ r_0(x_j)\phi(x_j-\tau_0),\ldots,r_n(x_j)\phi(x_j-\tau_n),g(x_j),Bool\left(Q(x_j,t),a,x_j,N\right)\end{pmatrix}$$

$$B_{2i}=f_i\begin{pmatrix} x_j+h, p_0(x_j+h)v_1(x_j)+hB_{11},\ldots,p_{n-1}(x_j+h)v_n(x_j)+hB_{1n},q_1(x_j+h)\\ \phi'(x_j+h-\tau_1),\ldots,q_n(x_j+h)\phi^{(n)}(x_j+h-\tau_n),r_0(x_j+h)\phi(x_j+h-\tau_0)\\ ,\ldots,r_n(x_j+h)\phi(x_j+h-\tau_n),g(x_j+h),Bool\left(Q(x_j+h,t),a,x_j+h,N\right)\end{pmatrix}$$

$$B_{3i}=f_i\begin{pmatrix} x_j+h, p_0(x_j+h)v_1(x_j)+\frac{h}{2}B_{11}+\frac{h}{2}B_{21},\ldots,p_{n-1}(x_j+h)v_n(x_j)+\frac{h}{2}B_{1n}+\frac{h}{2}B_{2n},\\ q_1(x_j+h)\phi'(x_j+h-\tau_1),\ldots,q_n(x_j+h)\phi^{(n)}(x_j+h-\tau_n),r_0(x_j+h)\phi(x_j+h-\tau_0)\\ ,\ldots,r_n(x_j+h)\phi(x_j+h-\tau_n),g(x_j+h),Bool\left(Q(x_j+h,t),a,x_j+h,N\right)\end{pmatrix}$$

$$B_{4i}=f_i\begin{pmatrix} x_j+2h, p_0(x_j+2h)v_1(x_j)+2hB_{31},\ldots,p_{n-1}(x_j+2h)v_n(x_j)+2hB_{3n},q_1(x_j+2h)\\ \phi'(x_j+2h-\tau_1),\ldots,q_n(x_j+2h)\phi^{(n)}(x_j+2h-\tau_n),r_0(x_j+2h)\phi(x_j+2h-\tau_0)\\ ,\ldots,r_n(x_j+2h)\phi(x_j+2h-\tau_n),g(x_j+2h),\ Bool\left(Q(x_j+2h,t),a,x_j+2h,N\right)\end{pmatrix}$$

$$B_{5i}=f_i\begin{pmatrix} x_j+h, p_0(x_j+h)v_1(x_j)+\frac{h}{12}(5B_{11}+8B_{31}-B_{41}),\ldots,p_{n-1}(x_j+h)v_n(x_j)+\\ \frac{h}{12}(5B_{1n}+8B_{3n}-B_{4n}),q_1(x_j+h)\phi'(x_j+h-\tau_1),\ldots,q_n(x_j+h)\phi^{(n)}(x_j+h-\tau_n)\\ ,r_0(x_j+h)\phi(x_j+h-\tau_0),\ldots,r_n(x_j+h)\phi(x_j+h-\tau_n),g(x_j+h),\\ Bool\left(Q(x_j+h,t),a,x_j+h,N\right)\end{pmatrix}$$

$$B_{6i}=f_i\begin{pmatrix} x_j+2h, p_0(x_j+2h)v_1(x_j)+\frac{h}{3}(B_{11}+B_{41}+4B_{51}),\ldots,p_{n-1}(x_j+2h)v_n(x_j)+\\ \frac{h}{3}(B_{1n}+B_{4n}+4B_{5n}),q_1(x_j+2h)\phi'(x_j+2h-\tau_1),\ldots,q_n(x_j+2h)\\ \phi^{(n)}(x_j+2h-\tau_n),r_0(x_j+2h)\phi(x_j+2h-\tau_0),\ldots,r_n(x_j+2h)\phi(x_j+2h-\tau_n),\\ g(x_j+2h),Bool\left(Q(x_j+2h,t),a,x_j+2h,N\right)\end{pmatrix} \quad \ldots (13)$$

for each i=1,2,…,n. and j=0,1,…,m where (m + 1) is the number of points ($x_0,x_1,...,x_m$) and $Bool(Q(x,t),a,x,N)$ is Bool method in eq.(10).

The numerical solution using fourth order ***b****lock and* ***B****ool* ***m****ethods* of **nth** order ***n****onlinear* ***DVIDE*** can be summarized by the following algorithm:

BBM-nNDVIDE Algorithm :

Step 1: Input a, x_0 , m, N, n where x_0 is the initial value and n is the order of DVIDE.

Step 2: Define $Q(x,t)$ as in eq.(7) and the function $g(x)$ of nonlinear DVIDE.

Step 3: Set $h=\frac{(x_m-x_0)}{m}$ and $j = 0$.

Step 4: For each i=1,2,…,n compute B_{1i} in eq.(13).

Step 5: $\forall$ i=1,2,…,n compute B_{2i} in eq.(13).

Step 6: $\forall$ i=1,2,…,n compute B_{3i} in eq.(13).

Step 7: $\forall$ i=1,2,…,n compute B_{4i} in eq.(13).

Step 8: $\forall$ i=1,2,…,n compute B_{5i} in eq.(13).

Step 9: $\forall$ i=1,2,…,n compute B_{6i} in eq.(13).

Step 10: $\forall$ i=1,2,…,n compute:

$$v_i(x_{j+1})=v_i(x_j)+\frac{h}{12}(5B_{1i}+8B_{3i}-B_{4i})$$

$$v_i(x_{j+2})=v_i(x_j)+\frac{h}{3}(B_{1i}+4B_{5i}+B_{6i}) \quad \text{and} \quad x_{j+1}=x_j+h$$

Step 11: Put $j = j+1$

Step 12: If $j = m$ then stop.

Else go to (step 4)

5- Numerical Examples:

Example (1):

Consider the following first order nonlinear *Neutral* Volterra integro-differential equation:

$$\frac{du(x)}{dx}+\frac{du(x-1)}{dx}=\left(e^x+e^{(x-1)}-\frac{1}{4}e^{4x}+\frac{1}{4}\right)+\int_0^x u^4(t)dt \qquad x\geq 0 \qquad \dots (14)$$

with initial function : $u(x)=e^x \qquad -1\leq x\leq 0$.

The exact solution of the above nonlinear DVIDE is: $u(x)=e^x \qquad 0\leq x\leq 1$.

When the algorithm (BBM-nNDVIDE) is applied, table (1) presents the comparison between the exact and numerical solutions of eq.(14) using block and Bool methods for m=10, h=0.1, $x_j=jh$, $j=0,1,\dots,m$ and m=100, h=0.01 , with least square error (L.S.E.).

Table (1) The solution of DVIDE for Ex.(1).

x	***Exact***	***Block and Bool (BBM-nNDVIDE) u(x)***	
		h=0.1	***h=0.01***
0	1.0000	1.0000	1.0000
0.1	1.1052	1.1052	1.1052

0.2	1.2214	1.2214	1.2214
0.3	1.3499	1.3498	1.3499
0.4	1.4918	1.4918	1.4918
0.5	1.6487	1.6487	1.6487
0.6	1.8221	1.8221	1.8221
0.7	2.0138	2.0137	2.0138
0.8	2.2255	2.2255	2.2255
0.9	2.4596	2.4596	2.4596
1	2.7183	2.7183	2.7183
L.S.E.		**0.107e-8**	**0.484e-12**

Figure (2) shows the solution of nonlinear neutral *Volterra integro-differential* equation, which was given in example (1) by using block and Bool methods with the exact solution.

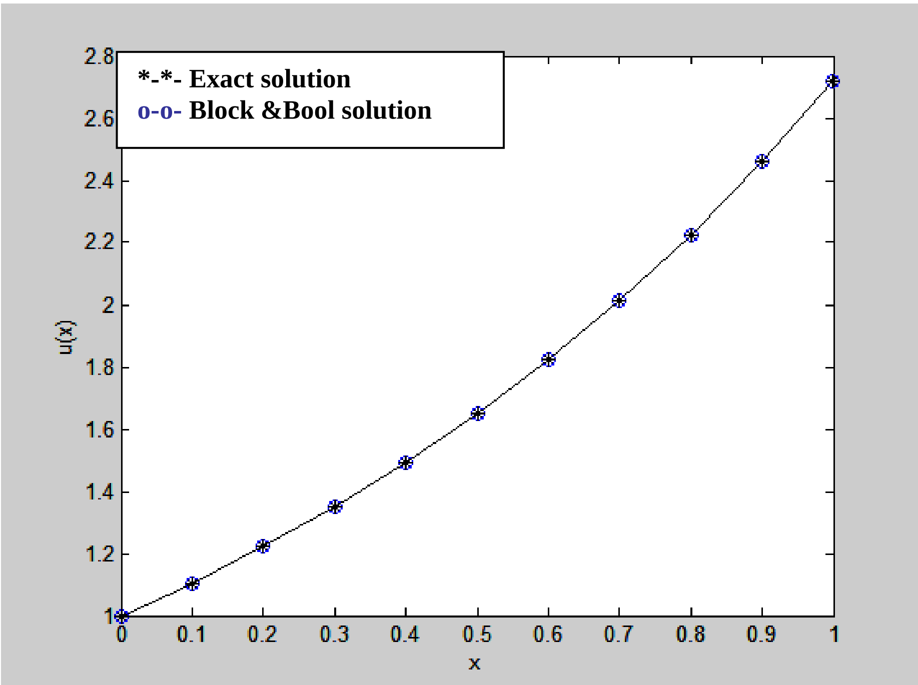

Fig.(2) The comparison between the exact and block & Bool solution for neutral Volterra integro-differential equation in Ex.(1).

Example (2):

Consider the following second order nonlinear *retarded* Volterra integro-differential equation:

$$\frac{d^2u(x)}{dx^2}-2xu(x-1)=\begin{pmatrix} e^{x+\frac{1}{2}}+\cos x-2x-\frac{1}{2}x^2e^{(x-0.5)^2}+ \\ \frac{1}{4}xe^{(x-0.5)^2}-2x^2e^{(x-0.5)}-\frac{1}{2}x^3- \\ \frac{1}{4}xe^{(-0.5)^2}-2xe^{-0.5} \end{pmatrix}+\int_0^x xt[u(t-1)]^2dt \qquad x\ge 0 \qquad \dots (15)$$

with initial functions : $\left.\begin{array}{l} u(x)=e^{x+\frac{1}{2}}+1 \\ u'(x)=e^{x+\frac{1}{2}} \end{array}\right\} \quad -1\le x\le 0$

The exact solution of the above nonlinear DVIDE is:

$$u(x)=2-\cos x+e^{x+\frac{1}{2}} \qquad 0\le x\le 1 \ .$$

The above DVIDE can be replaced by a system of two first order DVIDE's as:

$$\left.\begin{array}{ll} v_1'(x)=v_2(x), & x\ge 0 \\ v_2'(x)=2xv_1(x-1)+e^{x+\frac{1}{2}}+\cos x-2x-\frac{1}{2}x^2e^{(x-0.5)^2}+\frac{1}{4}xe^{(x-0.5)^2}-2x^2e^{(x-0.5)}- & \\ \qquad \frac{1}{2}x^3-\frac{1}{4}xe^{(-0.5)^2}-2xe^{-0.5}+\int_0^x xt[v_1(t-1)]^2dt & x\ge 0 \end{array}\right\} \qquad \dots (16)$$

with initial functions: $\left.\begin{array}{l} v_1(x)=e^{x+\frac{1}{2}}+1 \\ v_2(x)=e^{x+\frac{1}{2}} \end{array}\right\} \quad -1\le x\le 0$

and exact solutions: $\begin{array}{ll} exact_1=v_1(x)=2-\cos x+e^{x+\frac{1}{2}} & 0\le x\le 1 \\ exact_2=v_2(x)=\sin x+e^{x+\frac{1}{2}} & 0\le x\le 1 \end{array}$

When the algorithm (BBM-nNDVIDE) is applied, table (2) presents the comparison between the exact and numerical solutions of eq.(16) using block and Bool methods for m=10, h=0.1, $x_j=jh$, $j=0,1,\dots,m$ with least square error (L.S.E.).

Table (2) The solution of nonlinear DVIDE for Ex.(2).

x	***Exact₁***	***Block and Bool Methods (BBM-nNDVIDE)*** $v_1(x)$	***Exact₂***	***Block and Bool Methods (BBM-nNDVIDE)*** $v_2(x)$
0	**2.6487**	**2.6487**	**1.6487**	**1.6487**
0.1	**2.8271**	**2.8271**	**1.9220**	**1.9219**
0.2	**3.0337**	**3.0337**	**2.2124**	**2.2124**
0.3	**3.2702**	**3.2702**	**2.5211**	**2.5210**
0.4	**3.5385**	**3.5386**	**2.8490**	**2.8490**
0.5	**3.8407**	**3.8407**	**3.1977**	**3.1977**
0.6	**4.1788**	**4.1788**	**3.5688**	**3.5687**

0.7	**4.5553**	**4.5553**	**3.9643**	**3.9643**
0.8	**4.9726**	**4.9726**	**4.3867**	**4.3866**
0.9	**5.4336**	**5.4336**	**4.8385**	**4.8384**
1	**5.9414**	**5.9414**	**5.3232**	**5.3231**
L.S.E		**0.2709e-9**	**L.S.E**	**0.5142e-8**

Figure (3) shows the solution of nonlinear retarded *Volterra integro-differential* equation, which was given in example (2) by using block and Bool methods (BBM-nNDVIDE algorithm) with the exact solutions.

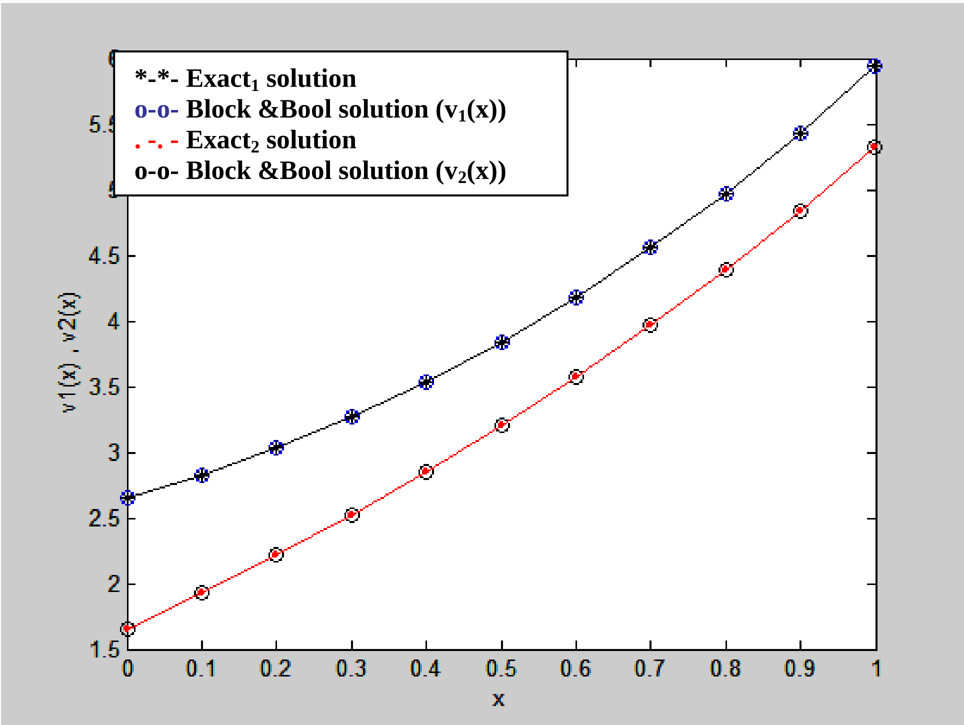

Fig.(3) The comparison between the exact and block & Bool solutions for retarded Volterra integro-differential equation in Ex.(2)

Example (3):

Consider the following *nonlinear mixed* Volterra integro-differential equation of third order:

$$\frac{d^3u(x)}{dx^3}+\frac{d^2u(x-2)}{dx^3}+x\frac{du(x-2)}{dx}+2\frac{du(x)}{dx}-u(x)+u(x-\tfrac{1}{2})= \left(x+\frac{3}{2}-\frac{3}{80}x^2\left(12x^3+70x^2+150x+135\right)\right)+\int_0^x (x+t)\left[u(t-\tfrac{1}{2})\right]^3 dt \quad x\ge 0 \qquad \ldots (17)$$

with initial functions :
$$u(x)=x+2 \qquad x\le 0$$
$$u'(x)=1 \qquad x\le 0$$
$$u''(x)=0 \qquad x\le 0$$

The exact solution of the above nonlinear DVIDE is: $u(x) = x + 2 \qquad 0 \le x \le 0.5$

When the above DVIDE is replaced by a system of three first order DVIDE's, the algorithm (BBM-nNDVIDE) is applied to solve this equation. Table (3) presents the comparison between the exact and numerical solutions of eq.(17) using block and Bool methods for m=10, h=0.05, $x_j = jh$, $j = 0,1,\dots,m$ and m=100, h=0.005, depending on least square error (L.S.E.).

Table (3) The solution of DVIDE for Ex.(3).

x	*Exact*$_1$	*Block and Bool (BBM-nNDVIDE)* $v_1(x)$		*Exact*$_2$	*Block and Bool (BBM-nNDVIDE)* $v_2(x)$		*Exact*$_3$	*Block and Bool (BBM-nNDVIDE)* $v_3(x)$	
		h=0.05	*h=0.005*		*H=0.05*	*h=0.005*		*h=0.05*	*h=0.005*
0	2.0000	2.0000	2.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.05	2.0500	2.0500	2.0500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.10	2.1000	2.1000	2.1000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.15	2.1500	2.1500	2.1500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.20	2.2000	2.2000	2.2000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.25	2.2500	2.2500	2.2500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.30	2.3000	2.3000	2.3000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.35	2.3500	2.3500	2.3500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.40	2.4000	2.4000	2.4000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.45	2.4500	2.4500	2.4500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.50	2.5000	2.5000	2.5000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
L.S.E.		0.11e-29	0.0000	**L.S.E.**	0.0000	0.0000	**L.S.E.**	0.15e-39	0.0000

6- Conclusions

Block and Bool methods have been presented to find the numerical solutions for three types (retarded, neutral and mixed) of n^{th} order nonlinear delay Volterra integro-differential equations. The results show a marked improvement in the least square errors (L.S.E.). Some numerical examples are concludes the following points:

1. Block and Bool methods proved their effectiveness in solving n^{th} order nonlinear DVIDE's where they give a better accuracy to the solution of three types DVIDE's.
2. BBM-nNDVIDE algorithm gives qualified way for solving first order nonlinear DVIDE as well as n^{th} order nonlinear DVIDE
3. The good advantage in numerical computation depends on the size of h, if h is decreased then the number of nodes increases and the L.S.E. approaches to zero.
4. Block and Bool methods solved nonlinear DVIDE of any order by reducing the equation to a system of first order nonlinear equations.

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طريقتي بلوك و بول لحل نوع خاص من معادلات فولتيرا التكاملية- التفاضلية التباطؤية اللاخطية من الرتبة n

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الخلاصــة :

يقدم البحث طريقة مقترحة لحل نوع خاص من معادلات فولتيرا التكاملية التفاضلية التباطؤية اللاخطية من الرتبة n عدديا باستخدام طريقة بلوك من الرتبة الرابعة و طريقة بول. حيث تم اشتقاق خوارزمية تمت برمجتها بلغة (Matlab) لمعالجة ثلاثة أنواع (التراجعية ، المتعادلة والمختلطة) من معادلات فولتيرا التكاملية– التفاضلية التباطؤية اللاخطية من الرتبة n عددياً باستخدام طريقتي بلوك و بول. كما تمت مقارنة النتائج العددية و الحقيقية من خلال ثلاثة أمثلة مع الرسوم التوضيحية وقد تم الحصول على نتائج دقيقة للطريقة المقترحة.