Eng. & Tech. Journal, Vol.29, No.4, 2011

Numerical Methods for Fractional Reaction-Dispersion Equation with Riesz Space Fractional Derivative

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Received on: 3/10/2010
Accepted on: 3/3/2011

Abstract

In this paper, a numerical solution of fractional reaction-dispersion equation with Riesz space fractional derivative has been presented. The algorithm for the numerical solution for this equation is based on two finite difference methods. The consistency, stability, and convergence of the fractional order numerical method are described.

The numerical methods have been applied to solve a practical numerical example and comparing results with the exact solution. The results were presented in tables using the MathCAD 12 software package when it is needed. The two finite difference methods appeared to be effective and reliable in solving fractional reaction-dispersion equation with Riesz space fractional derivative.

Keywords: Riesz fractional derivative, two finite difference methods, fractional reaction- dispersion equation, stability, convergence.

الطرق العدديةّ لمعادلة تشتت - رد الفعل الكسرية مع مشتقة فضاء ريزالكسري

الخلاصة

في هذا البحث قدمنا الحل العددي لمعادلة تشتت رد الفعل الكسرية مع مشتقة فضاء ريز الكسري ِ. وان خوارزمية الحل العددي لتلك المعادلات قائمة على اساس طريقتين لفروق المنتهية.حيث تم مناقشة: الاتساق والاستقرارية والتقارب للطرق العددية ذات الرتب الكسرية.

تم تطبيق الطرق العددية لحل مثال عددي تطبيقي ومقارنة النتائج مع الحل المضبوط . تم عرض النتائج على شكل جداول باستخدام برنامج ماث كاد12عند الحاجة. لوحظ إن الطريقتين لفروق المنتهية ذات كفاءة ودقة عالية في حل معادلة تشتت رد الفعل الكسرية مع مشتقة فضاء ريز الكسري.

Introduction

Various fields of science and engineering deals with the dynamical systems that can be described by fractional partial differential equations, for example, system of biology, chemistry and biochemistry, applications due to anomalous diffusion effects in constrained environments. However, effective numerical methods and numerical analysis for fractional partial differential equations are still in their infancy, [1, 2, 3, 4, 5, 6].

Liu F. et al. [7] considered the fractional Fokker-Planck equation and presented its numerical solution. Recently, Liu F. et al. [8] also treated the fractional advection-dispersion equations and derived the complete

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https://doi.org/10.30684/ etj.29.4.6
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solution of this equation with an initial condition. Chen and Liu [9] considered the space Riesz fractional reaction -dispersion equation and gave error analysis. Meerschaert M. et al. [10] considered the finite difference approximations for two-sided space-fractional partial differential equations and discussed their stability, consistency and convergence of the method.

In this paper implicit and explicit numerical methods for solving the fractional reaction-dispersion equation with Riesz space fractional derivatives are presented. Its stability and convergence are analyzed.

Two Finite Difference Methods for Solving the Fractional Reaction-Dispersion Equation

In this section, we propose two finite difference methods, i.e., an implicit finite difference method and explicit finite difference method for solving the fractional reaction-dispersion equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + D_x^{\alpha} u(x,t) \quad \ldots. (1)$$

In this problem initial and boundary conditions are considered which are:

$u(x,0)=f(x), L<x<R$ ....(2)

$u(L,t) = \psi_1(t),\ 0 \le t \le T$ ....(3)

$u(R,t) = \psi_2(t)(t),\ 0 \le t \le T$ ....(4)

where [L,R] is bounded space domain, f is a known function of x, ψ_1 and ψ_2 are known functions of t. the Riesz space-fractional derivative of order α .

D_x^{α} , is the Riesz operator, which is defined as:

$$D_x^{\alpha} = -C[D_{+x}^{\alpha} + D_{-x}^{\alpha}]$$

Where

$$C = \frac{1}{9999\cos(\alpha\pi/2)}$$

And $D_{\pm x}^{\alpha}$ are defined as the shifted Grunwald estimate to the α-the fractional derivative, [10]:

$$\frac{\partial^{\alpha} u(x,t)}{\partial_{+} x^{\alpha}} = \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{M_+} g_k u(x-(k-1)\Delta x, t) + O(\Delta x) \quad \ldots (5)$$

$$\frac{\partial^{\alpha} u(x,t)}{\partial_{-} x^{\alpha}} = \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{M_-} g_k u(x+(k-1)\Delta x, t) + O(\Delta x)$$

Where

$$g_k = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, k=0,1,2,\ldots$$

The finite difference method starts by dividing the x-interval [L, R] into n subintervals to get the grid points $x_i = L + i\Delta x$, where $\Delta x = (R-L)/n$ and i=0,1,…,n. Also, the t-interval [0,T] is divided into m subintervals to get the grid points $t_j = j\Delta t$, j = 0,1,…,m, where $\Delta t = T/m$.

First, we present the following implicit finite difference method for the initial-boundary value problem of the fractional reaction-dispersion equation. By Reisz fractional derivative of the shifted Grunwald estimate to the α-the fractional derivative eq.(5) where

$M_{-} = n-i+1$ and $M_{+} = i+1$ [10], one can get:

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = -u_{i,j+1} - \frac{C}{(\Delta x)^{\alpha}}\left[\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j+1} + \sum_{k=0}^{i+1} g_k u_{i-k+1,j+1}\right],$$

$i = 1,2,...,n-1,\ j = 0,1,...,m-1$ …. (6)

Where $u_{i,j} = u(x_i, t_j)$.

The resulting equation can be implicitly solved for $u_{i,j+1}$ to give

$$u_{i,j+1} + \beta C\left[\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j+1} + \sum_{k=0}^{i+1} g_k u_{i-k+1,j+1}\right] = \eta u_{i,j},$$

$i = 1,2,...,n-1,\ j = 0,1,...,m-1$ …. (7)

Where $\beta = \frac{\Delta t}{(1+\Delta t)(\Delta x)^{\alpha}}$,

$\eta = \frac{1}{(1+\Delta t)}$

Secondly, we present the following explicit finite difference method for solving the fractional reaction-dispersion equation eq.(1) with the boundary conditions (3), (4), and the initial condition (2), also use is made of Reisz fractional derivative of the shifted Grunwald estimate to the α-th fractional derivative given by eq.(5) to reduce it as in the following form:

$$u_{i,j+1} = \eta u_{i,j} - \beta C\left[\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j} + \sum_{k=0}^{i+1} g_k u_{i-k+1,j}\right],$$

$i = 1,2,...,n-1,\ j = 0,1,...,m-1$ …. (8)

Where $\eta = (1-\Delta t)$, $\beta = \frac{\Delta t}{(\Delta x)^{\alpha}}$, $u_{i,j} = u(x_i, t_j)$.

After evaluating eq.(7) and eq.(8) at i=1,…,n-1, j=1,…,m-1 and s=0,…,M one can get a system of algebraic equations which can be solved.

Also form the initial condition and boundary conditions one can get

$u_{i,0} = f(x_i)$, i=0,1,…, n

$u_{L,j} = \psi_1(t_j)$, j=0,1,…, m

$u_{R,j} = \psi_2(t_j)$, j=0,1,…, m

Consistency, Stability and Convergent

The methods implicit Euler and explicit defined by eq.(7) and eq.(8) are consistent with order $O(\Delta t) + O(\Delta x^{[\alpha]})$, where $[\alpha]$ denotes the largest integer that is less than or equal to α. That consistency of two finite difference methods together with the results theories (3.1) and (3.2) located at the bottom on unconditionally stability of implicit and conditionally stability of explicit implies that the two finite difference methods are convergent.

Theorem 3.1: The implicit system defined by the linear difference eq.(7) for eq.(1) is unconditionally stable for all $1 < \alpha < 2$.

Proof:

The system of equations defined by (7), together with the initial and boundary condition can be written in the implicit matrix form $\underline{A}\underline{U_{j+1}} = \beta \underline{U_j}$

where

$\underline{U_j} = [u_{0,j}, u_{1,j}, \ldots, u_{n,j}]^T$, and

A is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix. Therefore the resulting matrix entries $A_{i,j}$ for $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,n-1$ are defined by:

$$A_{i,j} = \begin{cases} 1 + \beta_i C g_1 + \beta_i C g_1 & for \quad j = i \\ \beta_i C g_0 + \beta_i C g_2 & for \quad j = i-1 \\ \beta_1 C g_2 + \beta_1 C g_0 & for \quad j = i+1 \\ \beta_{i-1} C g_{i-j+1} & for \quad j < i-1 \end{cases}$$

To illustrate this matrix pattern, we list the corresponding equations for the rows i =1, 2 and n-1:

$$(\beta_1 C g_0 + \beta_1 C g_2) u_{0,j+1} + (1 + \beta_1 C g_1 + \beta_1 C g_1) u_{1,j+1} + (\beta_1 C g_2 + \beta_1 C g_0) u_{2,j+1} = \eta u_{1,j}$$

$$\beta_2 C g_3 u_{0,j+1} + (\beta_2 C g_0 + \beta_2 C g_2) u_{1,j+1} + (1 + \beta_2 C g_1 + \beta_2 C g_1) u_{2,j+1} + (\beta_2 C g_2 + \beta_2 C g_0) u_{3,j+1} = \eta u_{2,j}$$

$$\beta_{n-1} C g_n u_{0,j+1} + \ldots + (\beta_{n-1} C g_0 + \beta_2 C g_2) u_{n-2,j+1} + (1 + \beta_{n-1} C g_1 + \beta_{n-1} C g_1) u_{n-1,j+1} + (\beta_{n-1} C g_2 + \beta_{n-1} C g_0) u_{n,j+1} = \eta u_{n-1,j}$$

According to the Greshgorin theorem [11], the eigenvalues of the matrix $\underline{A}$ lie in the union of the circles centered at $A_{i,i}$ with radius $r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} A_{i,l}$.

Here we have

$$A_{i,i} = 1 + \beta_i C g_1 + \beta_i C g_1 = 1 - 2\beta_i C \alpha$$

and

$$r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} A_{i,l} = \beta_i C \left[\sum_{\substack{l=0 \\ l \neq i}}^{n-i+1} g_{\alpha,i+l-1} + \sum_{\substack{l=0 \\ l \neq i}}^{i+1} g_{\alpha,i-l+1} \right] \leq 2\beta_i C \alpha$$

With strict inequality holding true when α is not an integer. This implies that the eigenvalue of the matrix $\underline{A}$ are all no less than 1 in magnitudes. Hence the spectral radius of the matrix $\underline{A^{-1}}$ is less than 1. Thus any error in $\underline{U^j}$ is not magnified, and therefore the implicit Euler method defined above is unconditionally stable. □

Theorem 3.2: The explicit finite difference method (8) is stable if $\frac{\Delta t}{\Delta x^\alpha} \leq \frac{1+\eta}{4\alpha}$, for all $1 < \alpha < 2$.

Proof:

The system of equations defined by eq.(8), together with the initial and boundary condition can be written in the explicit matrix form

$\underline{U}_{j+1} = B\underline{U}_j$ where

$\underline{U}_j = [u_{0,j}, u_{1,j}, \ldots, u_{n,j}]^T$, and

$\underline{B}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix.

To illustrate the matrix B pattern, we list the corresponding equations for i = 1, 2 and n-1:

$$u_{1,j+1} = -(\beta_1 Cg_0 + \beta_1 Cg_2)u_{0,j} + (\eta - \beta_1 Cg_1 - \beta_1 Cg_1)u_{1,j} - (\beta_1 Cg_2 + \beta_1 Cg_0)u_{2,j}$$

$$u_{2,j+1} = -\beta_2 Cg_3 u_{0,j} - (\beta_2 Cg_0 + \beta_2 Cg_2)u_{1,j} + (\eta - \beta_2 Cg_1 - \beta_2 Cg_1)u_{2,j} - (\beta_2 Cg_2 + \beta_2 Cg_0)u_{3,j}$$

$$u_{n-1,j+1} = -\beta_{n-1} Cg_n u_{0,j} \cdots - (\beta_{n-1} Cg_0 + \beta_{n-1} Cg_2)u_{n-2,j} + (\eta - \beta_{n-1} Cg_1 - \beta_{n-1} Cg_1)u_{n-1,j} - \beta_{n-1} Cg_0 u_{n,j}$$

Therefore the resulting matrix entries $B_{i,j}$ for $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,n-1$ are defined by:

$$B_{i,j} = \begin{cases} \eta - \beta_i Cg_1 - \beta_i Cg_1 & for \quad j = i \\ -(\beta_i Cg_0 + \beta_i Cg_2) & for \quad j = i-1 \\ -(\beta_1 Cg_2 + \beta_1 Cg_0) & for \quad j = i+1 \\ -\beta_{i-1} Cg_{i-j+1} & for \quad j < i-1 \end{cases}$$

According to the Greshgorin theorem [11], the eigenvalues of the matrix $\underline{B}$ lie in the union of the circles centered at $B_{i,i}$ with radius $r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} B_{i,l}$

Here we have

$$B_{i,i} = \eta + \beta_i Cg_1 + \beta_i Cg_1 = \eta - 2\beta_i C\alpha$$

and

$$r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} B_{i,l} = \beta_i C\left[\sum_{\substack{l=0 \\ l \neq i}}^{n-i+1} g_{\alpha,i+l-1} + \sum_{\substack{l=0 \\ l \neq i}}^{i+1} g_{\alpha,i-l+1}\right]$$

$$\leq 2\beta_i C\alpha$$

and therefore $B_{i,i} + r_i \leq 1$. We also have

$$B_{i,i} - r_i \geq \eta - 2\beta_i\alpha - 2\beta_i\alpha = \eta - 4\beta_i\alpha$$

$$= \eta - 4\left[\frac{\Delta t}{\Delta x^\alpha}\right]\alpha$$

Therefore for the spectral radius of the matrix $\underline{A}$ to be at most one, it suffices to have

$$\eta - 4\left[\frac{\Delta t}{\Delta x^\alpha}\right]\alpha \geq -1 \rightarrow \left[\frac{\Delta t}{\Delta x^\alpha}\right]\alpha \leq \frac{1+\eta}{4} \rightarrow$$

$$\left[\frac{\Delta t}{\Delta x^\alpha}\right]\alpha \leq \frac{1+\eta}{4} \rightarrow \quad [\alpha]\frac{\Delta t}{\Delta x^\alpha} \leq \frac{1+\eta}{4} \rightarrow$$

$$\frac{\Delta t}{\Delta x^\alpha} \leq \frac{1+\eta}{4\alpha}$$

where $\eta = (1 - \Delta t)$.

Therefore the explicit Euler method defined above is conditionally stable. □

Numerical Examples

In this section, two numerical examples are presented, showing the fractional reaction-dispersion equation with Riesz space fractional derivative behaviors of the solution with the two finite difference methods.

Example 1: Consider the fractional reaction-dispersion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} - x^{0.5} e^{-t}$$

subject to the initial condition

$u(x,0) = x^{0.5}$, $0 < x < 0.2$

and the boundary conditions

$u(0,t) = 0$, $0 \le t \le 0.025$

$u(0.2,t) = 0.44721\ e^{-t}$, $0 \le t \le 0.025$

This reaction-dispersion equation together with the above initial and boundary condition is constructed such that the exact solution is $u(x,t)= x^{0.5} e^{-t}$.

Table1 show the numerical solution using the implicit finite difference approximation. From table 1, it can be seen that that good agreement between the numerical solution and exact solution.

Example 2: Consider the fractional reaction-dispersion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} - x^{0.5} e^{-t}$$

subject to the initial condition

$u(x,0) = x^{0.5}$, $0 < x < 0.5$

and the boundary conditions

$u(0,t) = 0$, $0 \le t \le 0.02$

$u(0.5,t) = 0.70711\ e^{-t}$, $0 \le t \le 0.02$

This reaction-dispersion equation together with the above initial and boundary condition is constructed such that the exact solution is $u(x,t)= x^{0.5} e^{-t}$.

Table 2 show the numerical solution using the explicit finite difference approximation. From table 2, it can be seen that that good agreement between the numerical solution and exact solution.

Conclusions

In this paper

1-Numerical methods for solving the fractional reaction-dispersion equation with Riesz space fractional derivative has been described and demonstrated.

2-The two finite difference methods are proved to be stable and converge.

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Table 1: The numerical solution of example by using the implicit finite difference method for $\Delta x = 0.05$ and $\Delta t = 0.0125$

x	t	Numerical Solution	Exact Solution	Error
0.05	0.0125	0.22100	0.22083	-0.17089 E-3
0.10	0.0125	0.31200	0.31230	0.29952 E-3
0.15	0.0125	0.38300	0.38249	-0.51276 E-3
0.05	0.0250	0.21800	0.21809	0.85926 E-4
0.10	0.0250	0.30900	0.30842	0.57993 E-3
0.15	0.0250	0.37800	0.37774	-0.26410 E-3

Table 2: The numerical solution of example by using the explicit Finite difference method for $\Delta x = 0.125$ and $\Delta t = 0.01$

x	t	Numerical Solution	Exact Solution	Error
0.125	0.01	0.35000	0.35004	0.40000E-4
0.250	0.01	0.49500	0.49502	0.20000E-4
0.375	0.01	0.60600	0.60628	0.28000E-3
0.125	0.02	0.34700	0.34655	-0.45000 E-3
0.250	0.02	0.49000	0.49009	0.10000E-3
0.375	0.02	0.60000	0.60025	0.25000E-3