On MP–rings and DS–rings

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Abstract:

The research aims to study two kinds of rings,MP-rings and DS-rings . The researcher gave some binding relation with other modules and rings . The researcher put the hypothesis that condition.

على الحلقات من النمط-MP والحلقات من النمط-DS

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ملخص البحث :

يهدف البحث إلى دراسة نوعين من الحلقات هي الحلقات اليسرى من النمط–MP والحلقات اليسرى من النمط–DS .وأعطينا بعض العلاقات التي تربط بين هذه الحلقات وحلقات أخرى ,وأعطينا شرط(*) في أثبات علاقة الحلقات اليسرى من النمط–DS مع حلقات أخرى .

1. Introduction :

To study a left MP-rings and a left DS-rings[3] requires our knowledge of other definitions as :

1. A right R-module M is said to be P-injective if and only if ,for each principal right ideal I of R ,and every right R- homomorphism $f:I\rightarrow M$, there exists y in M such that $f(x)=yx$ for all x in I,[2].
2. An R-module M is called simple when its only sub modules are 0 and M, [5].

3. A right annihilator of a non-zero element a in a ring R is defined by r(a)={b∈R: ab=0},a left annihilator l(a) is similarly defined,[5].
4. An R-module M is called faithful if and only if Ann(M)=0,[5].

Many scientists studied rings such Nicholson,Yousif and Watter, Nicholson proved that [3](Every MP-ring is DS-ring) ,if R is P-injective or R is commutative. So (Every MP-ring is A left faith R-module).

2- MP–Rings

Following [3]

A ring R is called a left MP-ring if every minimal left ideal of R is a P-injective module. And a left R–module M is called an MP-module if every simple sub module is a P-injective module .

Theorem (2-1)[3]

The following conditions are equivalent for a ring R :-

1. R is a left MP-ring.
2. R is has a faithful left MP–module.
3. If L is a maximal left ideal of R then either r(L) =0 or R/L is P–injective module.
4. Every simple left R-module K either is P-injective or satisfies hom(K,R)=0.

Lemma (2-2)

Let R be a left MP-ring , if for each minimal left ideal L of R ,and every $0 \neq a \in R$ Then $r(l(a)) = aL$.

Proof :

Let L be a minimal left ideal of R and $0 \neq a \in R$ since $_RL$ is P-injective, then $r(l(a)) = aL$.

Theorem (2-3)

Let R be a left MP-ring, such that for each minimal left ideal L of R, and every $0 \neq a \in R$. If $f: Ra \to L$ is any R-linear map, then $f(a) \in aL$.

Proof :

Let L be a minimal left ideal of R and $0 \neq a \in R$

Then $rl(a) = aL$

If $f: Ra \to L$ is any R-linear map, Then:

$l(a)\, f(a) = f(l(a)a) = f(0) = 0$

So $f(a) \in rl(a) = aL$

Then $f(a) \in aL$

Theorm (2-4)

Let R be a left MP-ring, such that for each minimal left ideal L of R and every $0 \neq a \in R$, if $l(a) \subseteq l(k)$, where $0 \neq k \in L$, then $0 \neq kL \subseteq aL$

Proof :

Let L be a minimal left ideal of R and $0 \neq a \in R$, if $l(a) \subseteq l(k)$, where $k \in L$, then $kL = rl(k) \subseteq rl(a) = aL$, $L=Re$, $e^2=e \in R$,since $k = ke \in kL$, $kL \neq 0$ then $0 \neq kL \subseteq aL$

Theorem (2-5)

The following conditions are equivalent

1. R is a left MP-ring.

2. For each minimal ideal L of R and every $0 \neq a \in R$, $r(Rb \cap l(a)) = r(b) + aL$

Proof :

$1 \longrightarrow 2$

Let L be a minimal left ideal of R and $0 \neq a , b \in R$.

We can suppose $a \in r(b) + aL$, we know $a \in r(b) \subset r(Rb)$, so $a \in aL = r(l(a))$

Therefore $a \in r(Rb) \cap r(l(a)) \subseteq r(Rb \cap l(a)$

$\therefore$ $r(Rb \cap l(a)) \supseteq r(b) + aL$ ……..1

Now suppose $x \in r(Rb \cap l(a))$ ………2

Then $l(ba) \subseteq l(bx)$. If bx=0 then $x \in r(b) + aL$

If $bx \neq 0$ then by theorem (2-4) $0 \neq bxL \subseteq baL$

So L=Re,from the same theorem, where $e^2=e$.

Hence $bx = bxe \in bxL \subseteq baL$, bx=bay, where $y \in L$.

Then b(x-ay)=0 and $x - ay \in r(b)$.

Hence $x \in r(b) + aL$ ………3

From(2),(3) we get $r(Rb \cap l(a)) \subseteq r(b) + aL$ ……..4,

from (1),(4)

Then $r(Rb \cap l(a)) = r(b) + aL$.

$2 \longrightarrow 1$

If for every minimal L left ideal L of R and $0 \neq a, b \in R$

We have $r(Rb \cap l(a)) = r(b) + aL$.

Then we let b=1 and then rl(a) = aL.

Hence R is a left MP-ring by theorem (2-2).

3- DS-ring:

Following [3]

A ring R is called a left DS-ring if every minimal left ideal of R is a direct summand.

Definition (3-1) [4]

A ring R is called a right (left) minijective ring if and only if for any minimal right (left) ideal E of R. every R-homomorphism of E into R extends to one of right (left) R into R.

Following [3] A left R-module M is called a DS-module if every simple sub module is a minijective.

Definition (3-2)

A right R has condition (*) if K≅Re are simple, $e^2=e$, then K=Rg for some $g^2=g$.

Obviously a left DS-ring and a left minijective ring have condition (*).

Lemma (3-3)[3]

The following condition are equivalent:

1. R is a left Ds-ring.
2. Soc(R) is a minijective module.
3. R has a faithful left DS-module.

Theorem (3-4)

If L is a maximal left ideal of R and either r(L)=0 or R/L is a minijective module then R is a left DS-ring.

Proof :

Let Rk be a minimal left of R.

If $k^2 \neq 0$, then Rk = Re, e being an idempotent otherwise k∈l(k), and then R/l(k) is a minijective let f: Rk→ R/l(k), by f(rk) = r+l(k), then there exists ad∈R such that 1-kd∈l(k)

Henece k=kdk.

Let g=dk, then g is an idempotent ,and Rk=Rkdk = Rkg⊆Rg = Rdk⊆Rk.

Hence Rk = Rg.

Therefore R is a left DS-ring, Definition (3-2)

Theorem (3-5)

A ring R is a left DS-ring if and only if J(R)∩soc(R)=0 .

Proof :

Let R be a left DS-ring, If J(R)∩Soc (R)≠0 then there exists a minimal left ideal M of R with M⊆J(R).

But M=re for some $0\neq e^2 = e \in R$

So e∈J(R), a contradiction

Therefore J(R)∩soc(R)=0

Conversely:

If M is a minimal left ideal of r, then J(R)∩Soc(R) =0, implies $M^2\neq 0$

So M=Re, where $e^2=e\in R$. Thus R is a left DS-ring.

Definition (3-6) [2]

Let R be a ring and x be an element in R, then x is said to be left singular if and only if L(x) is essential ideal in R. The set of all left singular elements in R is denotef by Z(R).

Z(R) is an ideal in R which is the left singular ideal of R.

Definition (3-7)

A ring R is said to be SSM-ring if and only if every singular simple left R-module is minijective.

Theorem (3-8)

Let R is a SSM-ring and has condition (*), then R is a left DS-ring.

Proof :

Let Rk be a minimal left ideal of R

l(k) be a maximal left ideal of R.

If l(k) is not essential. Then l(k) is a direct summand of R.

Hence Rk≅R/l(k) is projective, Rk ≅Re, where $e^2=e$.

Then Rk=Rg ,since R has condition (*) where $g=g^2$.

If l(k) is essential, then Rk is singular simple so is minjective and we easily show that k=kdk [by proof Lemma (3-3)] let e=dk

Then Rk=Re and $e^2=e$

Then R is a left DS-ring.

Theorem (3-9)

A sub direct product of a left DS-ring is a gain a left DS-ring.

Proof :

Let R /A_i be a left DS-ring for each $i\in I$ where $\cap_{i\in I}$ $A_i=0$ If M is a minimal left ideal of R, Then $M\not\subset Ai$ for some I , So $(M+A_i)/A_i$ is a minimal left ideal of R/A_i. It follows from Theorem (3-3) that $M^2\not\subset A_i$ So $M^2=M$.

Hence M=Re where $e^2=e$ and then R is a left DS-ring.

Theorem (3-10)

A ring R is a left DS-ring if and only if for each minimal left ideal K is $K \not\subset r(K)$.

Proof :

Suppose that R is a left DS-ring ,

Let K be a minimal left ideal of R. Then K=Re, where e is an idempotent. Hence $K^2 \neq 0$, and $K \not\subset r(K)$.

Conversely,

If for each minimal left ideal K of R, $K \not\subset r(K)$,

Then $K^2 \neq 0$ and $K^2=K$, so K=Re, $e^2=e$, R is a left DS-ring**.**

Definition (3-11) [1]

A left R-module M is said to be flat if for any monomorphism N $\to$Q of right R-module N, Q, the induced homomorphism N $\otimes$M$\to$Q$\otimes$M is also homomorphism.

Theorem (3-12)

If J(R) $\cap$ Soc(R) is a flat left R-module, then R is a left DS-ring.

Proof :

Let $\{M_i \backslash\ i \in \Omega\}$ be a set of representative of non-isomorphic class of simple right R-module and $U=\Sigma_{i\in\Omega} \otimes M_i$.

Then we have an exact sequence:

L: $0 \to U \to E(U) \to E(U)/U \to 0$.

Where E(U) is the injective hull of U. Since $_R(J(R) \cap Soc(R))$ is flat, $0=U(J(R)) \cap Soc(_RR)= E(U)(J(R) \cap Soc(_RR)) \cap U$.

Hence $E(U)(J(R) \cap Soc(_RR))=0$

As U is essential in E(U) , Since E(U) is an injective [see(2)] co generator it is faithful so $J(R) \cap Soc(_RR)=0$

By theorem (3-5), R is a left DS-ring.

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