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On Left σ-Centralizers of Jordan Ideals And Generalized Jordan Left (σ,τ)-Derivations of Prime Rings

Dr. Anwar Khaleel Faraj & Dr. Abdulrahman H. Majeed**

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Abstract

In this paper we generalize the result of S. Ali and C. Heatinger on left σ-centralizer of semiprime ring to Jordan ideal, we proved that if R is a 2-torsion free prime ring, U is a Jordan ideal of R and G is an additive mapping from R into itself satisfying the condition $G(ur+ru)=G(u)\sigma(r)+G(r)\sigma(u)$, for all $u\in U, r\in R$. Then $G(ur)=G(u)\sigma(r)$, for all $u\in U, r\in R$. Also, we extend the result of S. M. A. Zaidi, M. Ashraf and S. Ali on left (σ, σ)-derivation of prime ring to Jordan ideal by introducing the concept of generalized Jordan left (σ, τ)-derivation.

Keywords: centralizer, σ-centralizer, (σ, τ)-derivation, left (σ, τ)-derivation, generalized (σ, τ)- derivation, prime ring.

حول تمركز- σ الايسر على مثاليات جوردان و مشتقات - (σ,τ) جوردان اليسرى المعممه للحلقات الاوليه

الخلاصة

في هذا البحث عممنا نتيجة S. Ali و C. Heatinger على تمركز- σ الايسر للحلقه شبه الاوليه الى مثالي جوردان, برهنا اذا كانت R حلقه اوليه طليقة الالتواء من النمط 2, U مثالي جوردان في R و G داله تجميعيه من R الى R بحيث $G(ur+ru)=G(u)\sigma(r)+G(r)\sigma(u)$, لكل $u\ni U$, $r\ni R$. فأنً $G(ur)=G(u)\sigma(r)$, لكل $u\ni U$, $r\ni R$. وكذلك, عممنا نتيجة S. M. A. Zaidi, M. Ashraf و S. Ali على مشتقة -(σ, σ) اليسرى للحلقه الاوليه الى مثالي جوردان بتقديم مفهوم مشتقة- (σ,τ) جوردان اليسرى المعممه.

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$, not necessarily with an identity element. We will write for all $x,y\in R, [x,y]=xy-yx$ and $x\circ y=xy+yx$ for the Lie product and Jordan product, respectively. A ring R is said to be prime if $xRy=0$ implies that $x=0$ or $y=0$ and R is semiprime in case $xRx=0$ implies $x=0$, [1]. An additive subgroup U of R is said to be Jordan ideal (resp. Lie ideal) of R if $u\circ r\in U$ (resp. $[u,r]\in R$), for all $u\in U, r\in R$, [1]. A ring R is called n-torsion free,

* Applied Sciences Department, University of Technology /Baghdad
** College of Science, University of Baghdad /Baghdad

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2412-0758/University of Technology-Iraq, Baghdad, Iraq

where n is an integer in case $nx = 0$, for $x \in R$, implies $x = 0$, [1]. An additive mapping $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$, [2]. An additive mapping $d : R \to R$ is called Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$, [2]. It is clear that each derivation is a Jordan derivation. The converse is not true in general. Hersetine's result [2], states that each Jordan derivation of 2-torsion free prime ring is a derivation. Awtar [3] generalized this result on Lie ideals. M. Bresar [4], introduced the definition of generalized derivation to be an additive mapping $F : R \to R$ such there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. We call an additive mapping $F : R \to R$ is a generalized Jordan derivation if there exists a Jordan derivation $d : R \to R$ such that $F(x^2) = F(x)x + xd(x)$, for all $x \in R$, [5]. M. Ashraf and N. Rehman and S. Ali in [6], showed that in a 2-torsion free prime ring R , every generalized Jordan derivation on Lie ideal U of R such that $u^2 \in U$ for all $u \in U$ is a generalized derivation on U .

An additive mapping $d : R \to R$ is called left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$,(resp. $d(x^2) = 2xd(x)$, for all $x \in R$) for all $x, y \in R$, [7]. Clearly , every left derivation is a Jordan left derivation and the converse is not true in genral. In [7] M. Ashraf and N. Rehman proved that every Jordan left derivation of 2-torsion free prime ring on Lie ideal U of R is a left derivation on U . According to S. Ali and C. Heatinger [8], $F : R \to R$ is a generalized derivation iff F is of the form $F = d + G$, where d is a derivation and G is a left centralizer on R. Following B. Zalar [9], an additive mapping $G : R \to R$ is called left (resp. right) centralizer if $G(xy) = G(x)y$ (resp. $G(xy) = xG(y)$), for all $x, y \in R$. If $a \in R$, then $L_a(x) = ax$ is left centralizer and $R_a(x) = xa$ is a right centralizer. If G is a left and right centralizer, then G is centralizer, [9]. An additive mapping $G : R \to R$ is called Jordan left (right) centralizer in case $G(x^2) = G(x)x$ (resp. $G(x^2) = xG(x)$), for all $x \in R$, [8]. Obviously every left (right) centralizer is a Jordan left (right) centralizer. The converse is in general not true (see [10],Exapmple 1). In [9], B.Zalar proved that every Jordan left centralizer (resp. Jordan centralizer) on a 2-torsion free semiprime ring R is a left centralizer (resp. centralizer). Recently, E. Albas [10] introduced the following definitions which are generalizations of the definitions of centralizer and Jordan centralizer. Let σ be an endomorphism of R. A Jordan σ -centralizer of R is an additive mapping $G : R \to R$ satisfying $G(xy + yx) = G(x)\sigma(y) + \sigma(y)G(x) = G(y)\sigma(x) + \sigma(x)G(y)$, for all $x, y \in R$. An additive mapping

$G: R \to R$ is called a left (resp. right) σ-centralizer of R if $G(xy) = G(x)\sigma(y)$ (resp. $G(xy) = \sigma(x)G(y)$), for all $x, y \in R$. If G is a left and right σ-centralizer, it is natural to call G is an σ-centralizer. It is clear that for an additive mapping $G: R \to R$ associated with a homomorphism $\sigma: R \to R$, if $L_a(x) = a\sigma(x)$ and $R_a(x) = \sigma(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then $L_a(x)$ is a left σ-centralizer and $R_a(x)$ is a right σ-centralizer. Clearly every centralizer is special case of a 1-centralizer, where 1 is the identity mapping on R.

Let $G: R \to R$ be an additive mapping and is a σ be an endomorphism of R. We call G a Jordan left (resp. right) σ-centralizer if $G(x^2) = G(x)\sigma(x)$ (resp. $G(x^2) = \sigma(x)G(x)$), for all $x \in R$. Oboivously every left (resp. right) σ-centralizer is Jordan left (resp. right) σ-centralizer.

In [10], Albas proved, under some conditions, that in a 2-torsion free semiprime ring R, every Jordan left σ-centralizer of R is a left σ-centralizer of R.

If $G: R \to R$ is a centralizer, then an easy compution gives that $G(xyx) = xG(y)x$, for all $x, y \in R$. A natural question is to ask wether the converse is also true.

In [11], J. Vukman gave the affirmative answer in case R is a 2 - torsion free semiprime ring.

In [10], Albas proved, under some conditions, that in a 2-torsion free semiprime ring R, every Jordan σ-centralizer of R is a σ-centralizer of R. According to [8], M. N. Daif, M. S. Tammam El-Sayiad and C. Heatinger proved that in a 2-torsion free semiprime ring R, for an endomorphism σ of R and for an additive mapping $G: R \to R$ such that $G(xyx) = \sigma(x)G(y)\sigma(x)$, for all $x, y \in R$, then G is a σ-centralizer of R. In [12], L. Molnar proved that if R is a 2-torsion free semiprime ring and $G: R \to R$ is an additive mapping such that $G(xyx) = G(x)yx$, for all $x, y \in R$, then G is a left (right) centralizer. In 2008, S. Ali and C. Heatinger [8] generalized Molnar's result as follows: if R is a 2-torsion free semiprime ring, σ be an endomorphism of R and $G: R \to R$ is an additive mapping such that $G(xyx) = G(x)\sigma(y)\sigma(x)$ (resp. $G(xyx) = \sigma(x)\sigma(y)G(x)$), for all $x, y \in R$, then G is a left (right) σ - centralizer of R.

In section 3, we generalize the above mentioned results for a Jordan ideal.

Given some endomorphisms σ and τ of R, an additive mapping $d: R \to R$ is called a (σ, τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$. Recall that a Jordan (σ, τ)-derivation, as defined in [13], is an additive mapping $d: R \to R$ satisfying $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$, for

all $x \in R$. An additive mapping $F: R \rightarrow R$ is called a generalized (σ, τ)-derivation on R if there exists an (σ, τ)- derivation $d: R \rightarrow R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$, [13]. An additive mapping $F: R \rightarrow R$ is called a generalized Jordan (σ, τ)-derivation if there exists a Jordan (σ, τ) -derivation such that
$F(x^2) = F(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$, [14].

An additive mapping $d: R \rightarrow R$ is called left (σ, τ)-derivation if $d(xy) = \sigma(x)d(y) + \tau(y)d(x)$, for all $x, y \in R$, [13]. Clearly, every left (1,1)-derivation is a left derivation on R. Shaheen [15], introduced the concept of generalized left derivation as an additive mapping $F: R \rightarrow R$, if there exist a left derivation $d: R \rightarrow R$ such that $F(xy) = xF(y) + yd(x)$, for all $x, y \in R$. The author in [16], introduced the concept of generalized left (σ, τ)-derivation to be an additive mapping $F: R \rightarrow R$ such that there exists a left (σ, τ) -derivation $d: R \rightarrow R$ such that
$F(xy) = \sigma(x)F(y) + \tau(y)d(x)$, for all $x, y \in R$. In the year 2003, S. M. A. Zaidi, M. Ashraf and S. Ali [13] proved that every Jordan left (σ, σ) -derivation on a Jordan ideal U of a 2-torsion free prime ring is a left (σ, σ)-derivation on U.

In section 3, we discuss the application of theory of σ -centralizers and extend the last result by introducing the concept of generalized Jordan left (σ, τ) -derivation. Throughout this paper consider σ is an automorphism of R.

2. Prelimineries

Now we will introduce the definition of generalized Jordan left (σ, τ)- derivation and some basic results which extensively to prove our theorems.

2.1 Definition:

Let S be a non empty set of R An additive mapping $F: R \rightarrow R$ is called generalized Jordan left (σ, τ) - derivation on S if there exist a Jordan left (σ, τ)-derivation $d: R \rightarrow R$ such that $F(xy) = \sigma(x)F(x) + \tau(x)d(x)$, for all $x \in S$.

2.2 Example:

Consider the ring
$R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in Z \right\}$, where Z denotes the set of integer numbers. Define $F: R \rightarrow R$ by
$F\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}$. Then it is easy to check F is a generalized Jordan left (σ, τ)- derivation on R with endomorphisms
$\sigma\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and
$\tau\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ since there exists a Jordan left (σ, τ)

-derivation $d:R\rightarrow R$ which is defined by $d\left(\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\right)=\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}$.

Lemma (2.3) and Lemma (2.4) can be found in [13].

2.3 Lemma:

Let R be a 2-torsion free prime ring and U a nonzero Jordan ideal of R. If $aU=0$ or ($Ua=0$), then $a=0$.

2.4 **Lemma**:

Let R be a 2-torsion free prime ring and U a nonzero Jordan ideal of R. If $aUb=0$, then $a=0$ or $b=0$.

2.5 Lemma:

Let R be a 2-torsion free ring, U a Jordan ideal of R and $G:R\rightarrow R$ an additive mapping defined by $G(ur+ru)=G(u)\sigma(r)+G(r)\sigma(u)$, for all $u\in U, r\in R$. Then for every $u\in U, r\in R$, the following statements are hold:

(i) $G(uru)=G(u)\sigma(r)\sigma(u)$.

(ii) $G(urv+vru)=G(u)\sigma(r)\sigma(v)$

$$+G(v)\sigma(r)\sigma(u)$$

(iii)

$(G(u^2r)-G(u^2)\sigma(r))\sigma[u^2,r]=0$.

Proof:

(i) Replace r by $2ur+r2u$. Then

$$G(u(2ur+r2u)+(2ur+r2u)u)$$
$$=G(u)\sigma(2ur+r2u)$$
$$+G(2ur+r2u)\sigma(u)$$
$$=2(G(u)\sigma(u)\sigma(r)+G(u)\sigma(r)\sigma(u))$$
$$+G(u)\sigma(r)\sigma(u)+G(r)\sigma(u)\sigma(u)) \qquad (1)$$

On the other hand,

$$G(u(2ur+r2u)+(2ur+r2u)u)$$
$$=G(2u^2r+r2u^2)+4G(uru)$$
$$=(G(u^2)\sigma(r)+G(r)\sigma(u^2))+4G(uru)$$
$$=2(G(u)\sigma(u)\sigma(r)+G(r)\sigma(u)\sigma(u))$$
$$+4G(uru) \qquad (2)$$

By comparing equation(1) and equation(2) and since R is 2-torsion free, we get $G(uru)=G(u)\sigma(r)\sigma(u)$.

(ii) If we replace u by $u+v$ in (i), we get the required result.

(iii) Let $u,v\in U$, such that $uv\in U$. Let $W=G(uvuv+uvvu)$. Then by (ii), we get

$$W=G(u)\sigma(v)\sigma(uv)+G(uv)\sigma(v)\sigma(u) \qquad (3)$$

On the other hand,

$$W=G(uv)^2+G(uv^2u)$$
$$=G(uv)\sigma(uv)+G(u)\sigma(v^2)\sigma(u) \qquad (4)$$

By comparing equation (3) and equation (4), we get

$$0=(G(uv)-G(u)\sigma(v))\sigma(uv)-(G(uv)$$
$$-G(u)\sigma(v))\sigma(vu)$$
$$=(G(uv)-G(u)\sigma(v))\sigma[u,v]$$

For any $u\in U$ and $r\in R$, the element $v=ur+ru$ satisfies the criterion $uv\in U$, hence by above, we get

$$0=(G(u(ur+ru))$$
$$-G(u)\sigma(ur+ru))\sigma[u,ur+ru]$$
$$=(G(u^2r)+G(uru)-G(u)\sigma(u)\sigma(r)$$
$$-G(u)\sigma(r)\sigma(u))\sigma[u^2,r]$$

$= (G(u^2r) - G(u)\sigma(u)\sigma(r))\sigma[u^2, r]$
$= (G(u^2r) - G(u^2)\sigma(r))\sigma[u^2, r]$

Note that Lemma (2.5) holds in case σ is just endomorphism of the ring R.

2.6 Lemma:

Let R be a 2-torsion free prime ring and $G(ur + ru) = G(u)\sigma(r) + G(r)\sigma(u)$, for all $u \in U, r \in R$. If $u \in U$ such that $u \in Z(R)$, then $G(ur) - G(u)\sigma(r) = 0$

Proof:

Let $W = G(vur + urv)$
$= G(v)\sigma(ur) + G(ur)\sigma(v)$ (5)

On the other hand since $u \in Z(R)$, then
$W = G(vur + urv)$

$= G(u)\sigma(r)\sigma(v) + G(v)\sigma(r)\sigma(u)$ (6)

Compare equation (5) and equation (6) to get
$(G(ur) - G(u)\sigma(r))\sigma(v) = 0$, for all $u, v \in U, r \in R$, i.e.
$\sigma^{-1}(G(ur) - G(u)\sigma(r))U = 0$.

By Lemma (2.1), we get $G(ur) - G(u)\sigma(r) = 0$.

2.7 Lemma:

Let R be a 2-torsion free prime ring, U be a Jordan ideal of R and $G(ur + ru) = G(u)\sigma(r) + G(r)\sigma(u)$, for all $u \in U, r \in R$. Then $G(u^2r) - G(u^2)\sigma(r) = 0$.

Proof:

Let $u, v \in U$ such that $2uv \in U$ and $2vu \in U$.

Let $W = G(uvsvu + vusuv)$.

By Lemma ((2.5),(ii)), we get
$W = G(uv)\sigma(s)\sigma(vu)$
$+ G(vu)\sigma(s)\sigma(uv)$ (7)

On the other hand by Lemma (2.3,(ii)), we get
$W = G(u(vsv)u) + G(v(usu)v)$
$= G(u)\sigma(vsv)\sigma(u) + G(v)\sigma(usu)\sigma(v)$
$= G(u)\sigma(v)\sigma(s)\sigma(v)\sigma(u)$
$+ G(v)\sigma(u)\sigma(s)\sigma(u)\sigma(v)$ (8)

By comparing equation (7) and equation (8), we get
$0 = (G(uv) - G(u)\sigma(v))\sigma(s)\sigma(vu)$
$+ (G(vu) - G(v)\sigma(u))\sigma(s)\sigma(uv)$
$= (G(uv) - G(u)\sigma(v))\sigma(s)\sigma(vu)$
$- (G(vu) - G(u)\sigma(v))\sigma(s)\sigma(uv)$
$= (G(uv) - G(u)\sigma(v))\sigma(s)\sigma[u, v]$

For any $u \in U, s \in R$, the element $v = ur + ru$ satisfies the criterion $uv \in U$ and $vu \in U$, hence by above we get
$0 = (G(u(ur + ru))$
$- G(u)\sigma(ur + ru))\sigma(s)\sigma[u, ur + ru]$
$= (G(u^2r) - G(u)\sigma(u)\sigma(r))R\sigma[u^2, r]$

$= (G(u^2r) - G(u^2)\sigma(r))R\sigma[u^2, r]$

Since R is prime, either $G(u^2r) - G(u^2)\sigma(r) = 0$ or $\sigma[u^2, r] = 0$.

If $\sigma[u^2, r] = 0$, then $u^2 \in Z(R)$. By Lemma (2.6), we get $G(u^2r) - G(u^2)\sigma(r) = 0$.

3. Main Results

In this section we introduce our main results.

3.1 Theorem:

Let R be a 2-torsion free prime ring, be a U Jordan ideal of R and G be an additive mapping from R into itself satisfying the condition $G(ur+ru)=G(u)\sigma(r)+G(r)\sigma(u)$, for all $u\in U, r\in R$. Then $G(ur)=G(u)\sigma(r)$, for all $u\in U$, $r\in R$.

Proof:

Let $W=G(uur+uru)$

$$=G(u)\sigma(ur)+G(ur)\sigma(u) \quad (9)$$

On the other hand,

$$W=G(u^2r+uru)$$
$$=G(u^2)\sigma(r)+G(u)\sigma(r)\sigma(u)$$
$$=G(u)\sigma(u)\sigma(r)+G(u)\sigma(r)\sigma(u) \quad (10)$$

By comparing equation (9) and equation (10), we get $(G(ur)-G(u)\sigma(r))\sigma(u)=0$, for all $u\in U, r\in R$. (11)

Replace u by $u+v$ in equation (11), we get

$$(G(ur)-G(u)\sigma(r))\sigma(v)$$
$$+(G(vr)-G(v)\sigma(r))\sigma(u)=0$$

Replace v by v^2 in the last equation and by using Lemma (3), we get $(G(ur)-G(u)\sigma(r))\sigma(v^2)=0$, for all $u,v\in U, r\in R$. (12)

Now linearize equation (12) on v and use equation (12) and equation (11) to get $(G(ur)-G(u)\sigma(r))\sigma(v)\sigma(u)=0$, for all $u,v\in U, r\in R$ and this implies that $\sigma^{-1}(G(ur)-G(u)\sigma(r))Uu=0$.

By Lemma (2.4), either $G(ur)-G(u)\sigma(r)=0$ or $u=0$.

If $u=0$, for all $u\in U$ then $U=0$ and this a contradiction. Therefore, $G(ur)-G(u)\sigma(r)=0$, for all $u\in U, r\in R$.

As a consequence of Theorem (3.1) we get the following Corollaries:

3.2 Corollary:

Let R be a 2-torsion free prime ring, be a U Jordan ideal of R and G be a left Jordan σ-centralizer on U. Then G is a left σ-centralizer on U.

In Corollary (3.2), if $\sigma=1$, 1 where is the identity mapping, we get the following:

3.3 Corollary:

Let R be a 2-torsion free prime ring, be a U Jordan ideal of R and G be a left Jordan centralizer on U. Then G is a left centralizer on U.

3.4 Corollary:

Let R be a 2-torsion free prime ring. Then every left Jordan centralizer on R is a left centralizer on R.

Zalar in [9], proved that Corollary (3.4) in case is R semiprime ring.

If U is a Jordan ideal and a subring of R and $G:R\to R$ is a left σ-centralizer on U into R, then an easy computation ginen that $G(xyz)=G(x)\sigma(y)\sigma(z)$, for all $x,y,z\in U$. A natural question is to ask wether the converse is also true. We prove the following theorem which contains the answer on this question

3.5 Theorem:
Let R be a 2-torsion free semiprime ring, be a U Jordan ideal of R. If $G:R\to R$ is an additive mapping such that $G(uru)=G(u)\sigma(r)\sigma(u)$, for all $u\in U, r\in R$, then G is a left σ-centralizer on U.

Proof:
By the hypothesis, we get
$G(uru)=G(u)\sigma(r)\sigma(u)$, for all $u\in U, r\in R$. (13)

Replacing u by $u+v$ in equation (13), we get
$$G((u+v)r(u+v)) = G(u)\sigma(r)\sigma(u)+G(u)\sigma(r)\sigma(v) + G(v)\sigma(r)\sigma(u)+G(v)\sigma(r)\sigma(v) \quad (14)$$
On the other hand,
$$G((u+v)r(u+v)) = G(uru)+G(urv+vru)+G(vrv) \quad (15)$$
Combining equation (14) and equation (15), we get
$G(urv+vru)=G(u)\sigma(r)\sigma(v) + G(v)\sigma(r)\sigma(u)$, for all $u,v\in U$, $r\in R$. (16)

Replace v by $2u^2$ in equation (16), we get
$$G(uru^2+u^2ru) = 2(G(u)\sigma(r)\sigma(u^2)+G(u^2)\sigma(r)\sigma(u))$$
, for all $u\in U, r\in R$. (17)

Put $r=ur+ru$ in equation (13) and using equation (13), we get
$$2G(uru^2+u^2ru) = 2(G(u(ru)u+u(ur)u) = 2(G(u)\sigma(ru)\sigma(u) + G(u)\sigma(ur)\sigma(u)) \quad (18)$$
By comparing equation (17) and equation (18), and since R is a 2 -torsion free we get
$(G(u^2)-G(u)\sigma(u))\sigma(r)\sigma(u)=0$, for all $u\in U, r\in R$. (19)

Now we set $G(u^2)-G(u)\sigma(u) = A(u)$, for all $u\in U$.
Then equation (19) reduces to $A(u)\sigma(r)\sigma(u)=0$, for all $u\in U$, $r\in R$. (20)

Since is σ onto, equation (20) implies that
$A(u)s\sigma(u)=0$, for all $u\in U$, $s\in R$. (21)

Replacing s by $\sigma(u)zA(u)$ in equation (21), equation (21) gives that $A(u)\sigma(u)zA(u)\sigma(u)=0$, for all $u\in U, z\in R$. (22)

Since R is semiprime, then $A(u)\sigma(u)=0$, for all $u\in U$. (23)

Replace u by $u+v$ in equation (23), we get
$A(u+v)\sigma(u)+A(u+v)\sigma(v)=0$, for all $u\in U$. (24)

Since
$A(u+v)=B(u,v)+A(u)+A(v)$, for all $u,v\in U$. (25)

Where
$$B(u,v)=G(uv+vu)-G(u)\sigma(v) - G(v)\sigma(u)$$
In view of equation (25), expression (24) implies that
$A(u)\sigma(v)+B(u,v)\sigma(u)+A(v)\sigma(u) + B(u,v)\sigma(v)=0$, for all $u,v\in U$. (26)

Replace u by $-u$ in the last equation, to get

$A(u)\sigma(v)+B(u,v)\sigma(u)-A(v)\sigma(u)$
$-B(u,v)\sigma(v)=0$, for all $u,v\in U$. ….(27)

Adding equation (26) with equation (27) and using the fact that R is a 2 -torsion free semiprime ring, we find that $A(u)\sigma(v)+B(u,v)\sigma(u)=0$, for all $u,v\in U$. …..(28)

On right multiplication of equation (28) by $A(u)$, we get

$A(u)\sigma(v)A(u)+B(u,v)\sigma(u)A(u)=0$, for all $u,v\in U$. (29)

From equation (21), we get $\sigma(u)A(u)R\sigma(u)A(u)=0$, for all $u\in U$.

Since R is semiprime, then $\sigma(u)A(u)=0$, for all $u\in U$. (30)

On combining equation (29) and equation (30) and since R is semiprime, $A(u)=0$, for all $u\in U$, i.e., G is a Jordan left σ- centralizer and hence G is a left σ- centralizer on by Corollary (3.2) .

Now we present some application of the theory of σ-centralizer in rings. The following theorem is a generalization of main theorem of [7].

3.6 Theorem:

Let R be a 2-torsion free prime ring, U be a Jordan ideal and a subring of R. If F is a generalized Jordan left (σ,σ)-derivation on U, then F is a generalized left (σ,σ)-derivation on U.

Proof:

Since F is a generalized Jordan left (σ,σ)-derivation on U, then there exists a Jordan left (σ,σ)-derivation d on U such that $F(u^2)=\sigma(u)F(u)+\sigma(u)d(u)$, for all $u\in U$.

Now we write $G=F-d$. Then, we find that

$G(u^2)=(F-d)(u^2)=F(u^2)-d(u^2)$
$=\sigma(u)F(u)+\sigma(u)d(u)-2\sigma(u)d(u)$
$=\sigma(u)(F(u)-d(u))$
$=\sigma(u)G(u)$

That is, G is a Jordan left σ-centralizer on U. Thus by Corollary (3.2), G is a left σ-centralizer on U.

By [13], d is a left (σ,σ)-derivation on U. Therefore, $F=G+d$ and

$F(uv)=G(uv)+d(uv)$
$=\sigma(u)G(v)+\sigma(u)d(v)+\sigma(v)d(u)$
$=\sigma(u)(F(y)-d(y))+\sigma(u)d(v)$
$+\sigma(v)d(u)$
$=\sigma(u)F(v)+\sigma(v)d(u)$

Hence F is a generalized left (σ,σ)-derivation on U.

In Theorem (3.6), if $F=d$ where d is a Jordan left (σ,σ)-derivation associated with F, we get the main theorem of [13].

3.7 Corollary:

Let R be a 2-torsion free prime ring, U be a Jordan ideal and a subring of R. If d is a Jordan left

(σ , σ)-derivation on U , then d is a left (σ , σ)-derivation on U .

In Corollary (3.7), if $\sigma = 1$, where 1 is the identity mapping of R , we get

3.8 Corollary:

Let R be a 2-torsion free prime ring, U be a Jordan ideal and a subring of R. If d is a Jordan left derivation on U , then d is a left derivation on U .

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