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Shuker khalil
Department of Mathematics, College of Science, University of Basrah, Basrah 61004, Iraq, shuker.khalil@uobasrah.edu.iq

Ali Abbas Asmae
Department of Mathematics, College of Science, University of Basrah, Basrah 61004, Iraq

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ORIGINAL STUDY

The Permutation Annihilator Ideals in Commutative Permutation *BCK*–Algebras with their Applications

Shuker khalil *, Ali Abbas Asmael

Department of Mathematics, College of Science, University of Basrah, Basrah 61004, Iraq

ABSTRACT

This paper introduces new concepts such as permutation *BCK*–algebra, permutation involutory ideal, commutative permutation *BCK*–algebra, and prime permutation ideal. Additionally, their attributes are examined. This paper elucidates a method for determining a relationship between the chemical structure of atoms for the chemical element Cadmium, and some of our suggestions are given here. In this work, the structure of the sets $\mathcal{A}^{\ddagger}$ and $\lambda_n^{\beta**}A$ are defined. Next, we show that if A is a permutation ideal, then $\lambda_n^{\beta**}A$ is a permutation ideal that contains A. Also, in any commutative permutation *BCK*–algebra the inequality $(\lambda_i^{\beta} \# \lambda_j^{\beta}) \# \lambda_m^{\beta} = (\lambda_i^{\beta} \# \lambda_m^{\beta}) \# \lambda_j^{\beta}$, for any $\lambda_i^{\beta}, \lambda_j^{\beta}, \lambda_m^{\beta} \in X$ is provided. After that, we show that $(\lambda_n^{\beta} \circledast \lambda_m^{\beta}) \# (\lambda_n^{\beta} \circledast \lambda_k^{\beta}) \leq (\lambda_n^{\beta} \circledast \lambda_m^{\beta}) \leq \lambda_m^{\beta}$, for any $\lambda_i^{\beta}, \lambda_j^{\beta}, \lambda_m^{\beta} \in X$ in any commutative permutation *BCK*–algebra, where $\lambda_n^{\beta} \circledast \lambda_j^{\beta} = \lambda_n^{\beta} \# (\lambda_n^{\beta} \# \lambda_j^{\beta})$. The notion of triple merge permutation (TMP) in S_{n+k+m} for any permutations β, δ and γ in symmetric groups S_m, S_k and S_n respectively is shown and then the concepts of triple merge permutation *BCK*–algebra and triple merge permutation bounded are studied. Also, the atomic shells for three chemical elements they are Carbon, Beryllium and Hydrogen are studied and the structure of the triple merge permutation is given from their atomic shells. Some interesting inequalities in commutative permutation *BCK*–algebra and in triple merge permutation *BCK*–algebra are given to support this work.

Keywords: Symmetric groups, Permutation sets, *BCK*–algebra, Annihilator ideal, Involutory ideal

1. Introduction

In 1966, Y. Imai and K. Iseki [1] defined two types of abstract algebras, *BCK* and *BCI*–algebras, to widen the concept of set-theoretic difference and non-classical propositional calculi. Every *BCI*–algebra M with $0\#r = 0$ for every $r \in X$ is a *BCK*–algebra. Any abelian group is a *BCK*–algebra, where # denotes group subtraction and 0 represents group identity. As a result, some scholars have looked into generalizations of *BCK*/*BCI*/*RHO* algebras (see [2–9]). Furthermore, there are other classes that are examined by several scholars using permutation sets [10–23].

The ideals closed by this closure operator are sometimes referred to as annihilator ideals. This name refers to the fact that A is an annihilator ideal if and only if it equals $A = ann(B)$ for some ideal B. Recent research has explored the annihilator ideals of graph $C*$–algebras [24], as well as Leavitt route algebras [25].

The purpose of this activity is to investigate and discuss some new concepts such as permutation *BCK*–algebra, permutation involutory ideal, commutative permutation *BCK*–algebra, and prime permutation ideal. Furthermore their qualities are discussed. Some of our recommendations are provided here. This study describes a technique for establishing a relationship between the chemical structure of atoms for the element cadmium. In this work, the structure of the sets $\mathcal{A}^{\ddagger}$ and $\lambda_n^{\beta**}A$ are defined. Next, we show that if A is a permutation ideal, then $\lambda_n^{\beta**}A$ is a permutation ideal which contains A. Also, in any commutative permutation *BCK*–algebra the inequlity $(\lambda_i^{\beta}\#\lambda_j^{\beta})\#\lambda_m^{\beta} = (\lambda_i^{\beta}\# \lambda_m^{\beta}) \# \lambda_j^{\beta}$, for any

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* Corresponding author.
E-mail address: shuker.khalil@uobasrah.edu.iq (S. khalil).
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$\lambda_i^\beta,\ \lambda_j^\beta,\ \lambda_m^\beta \in X$ is provided. After that, we show that $(\lambda_n^\beta \circledast \lambda_m^\beta)\ \#\ (\lambda_n^\beta \circledast \lambda_k^\beta) \le (\lambda_n^\beta \circledast \lambda_m^\beta) \le \lambda_m^\beta$, for any $\lambda_i^\beta,\ \lambda_j^\beta,\ \lambda_m^\beta \in X$ in any commutative permutation BCK–algebra, where $\lambda_n^\beta \circledast \lambda_j^\beta = \lambda_n^\beta \# (\lambda_n^\beta\ \#\ \lambda_j^\beta)$. Some intriguing inequalities in commutative permutation BCK-algebra are presented and provided.

The concepts of triple merge permutation BCK–algebra and triple merge permutation bounded are next examined after the idea of triple merge permutation (TMP) in S_{n+k+m} for any permutations β, δ and γ in symmetric groups S_m, S_k and S_n, respectively, is demonstrated. Additionally, the structure of the triple merge permutation is obtained from the atomic shells of the three chemical elements such as carbon, beryllium, and hydrogen that are explored. This study is supported by some intriguing inequalities in triple merge permutation BCK–algebra and commutative permutation BCK–algebra.

2. Preliminary

In this section, we review some concepts and outcomes that will be useful throughout the study.

Definition 2.1: [26] Let $m = (m_1, m_2, \ldots, m_k)$ be a sequence of integers with $m_1 \ge m_2 \ge \cdots \ge m_k \ge 0$. The length $l(m)$ of m is recognized by $I(m) = Max\ \{t \in N; m_t \ne 0\}$ and the size $|m|$ of m is recognized by $|m| = \sum_{t=1}^{k} m_t$, we say m is a partition of v, if $|m| = v$.

Remark 2.2: [26] For any $B \in S_n$ we can write it as $B = \mu_1\mu_2 \ldots \mu_{c(\beta)}$, where μ_i disjoint cycles (DCs) with length m_i and $c(B)$ is the number of disjoint cycle factors (NDCF) overall the 1-cycle of B. Because of any two (DCs) are commute, we can put $m_1 \ge m_2 \ge \cdots \ge m_{c(B)}$. Therefore $m = (m_1, m_2, \ldots, m_k)$ is a partition of n and each m_i is called part of m.

Definition 2.3: [27] For any $B \in S_n$ over the set $\Omega = \{1, 2, \ldots, n\}$ with $m(B) = (m_1, m_2, \ldots, m_{c(\beta)})$, then B composite of pairwise (DCs) $\{\mu_i\}_{i=1}^{c(\beta)}$ where $\mu_i = (t_1^i, t_2^i, \ldots, t_{\alpha_i}^i)$, $1 \le i \le c(B)$. If $\mu = (t_1, t_2, \ldots, t_r)$ is r–cycle in S_n we recognize B–set by $\mu^B = \{t_1, t_2, \ldots, t_r\}$ and it is said to be B–set of cycle μ. The B–sets of $\{\mu_i\}_{i=1}^{c(B)}$ are recognized as $\{{\mu_i}^B = \{t_1^i, t_2^i, \ldots, t_{m_i}^i\} 1 \le i \le c(B)\}$.

Definition 2.4: [28] An algebra $(X, *, 0)$ of type (2,0) is called a BCK–algebra if for any $x, y, z \in X$ such that

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $0 * x = 0$
5. $x * y = 0$ and $y * x = 0$ imply $x = y$.

3. Some ideals in commutative permutation BCK–algebras

This section provides new ideas including permutation BCK–algebra, permutation involutory ideal, commutative permutation BCK–algebra, and prime permutation ideal. Also, investigates their characteristics.

Definition 3.1: Assume that $X = \{\lambda_i^\beta\}_{i=1}^{c(\beta)} = \{t_1^i, t_2^i, \ldots, t_{\alpha_i}^i\} | 1 \le i \le c(\beta)\}$ is a collection of β-sets, where $\beta \in S_n$. For some $\lambda_k^\beta \in X$ put $T = \lambda_k^\beta$, where λ_k^β satisfies $\sum_{s=1}^{\alpha_k} t_s^k \le \sum_{s=1}^{\alpha_i} t_s^i\ \&\ |\lambda_k| \le |\lambda_i|, \forall (1 \le i \le c(\beta)) \ldots (*)$. In other side, for any $\lambda_h^\beta,\ \lambda_g^\beta \in X$ two disjoint β-sets satisfy (*). Let $T = \lambda_h^\beta$ if $\exists t_r^h \in \lambda_h^\beta$ with $t_r^h < t_s^g,\ \forall\ (1 \le s \le \alpha_g)$ or $T = \lambda_g^\beta$ if $\exists t_r^g \in \lambda_g^\beta$ with $t_r^g < t_s^h,\ \forall\ (1 \le s \le \alpha_h)$.

Suppose that $\# : X \times X \to X$ is a map, then $(X,\ \#,\ T)$ is called be a *permutation BCK–algebra* $(PBCK - A)$, if # such that:

$(PBCK-1)\ ((\lambda_i^\beta\ \#\ \lambda_j^\beta)\ \#\ ((\lambda_i^\beta\ \#\ \lambda_m^\beta))\ \#\ (\lambda_m^\beta\ \#\ \lambda_j^\beta) = T,$

$(PBCK-2)\ (\lambda_i^\beta\ \#\ (\lambda_i^\beta\ \#\ \lambda_j^\beta))\ \#\ \lambda_j^\beta = T,$

$(PBCK-3)\ \lambda_i^\beta\ \#\ \lambda_i^\beta = T,$

$(PBCK-4)\ T \# \lambda_i^\beta = T,$

$(PBCK-5)$ If $\lambda_i^\beta\ \lambda_j^\beta = T$ and $\lambda_j^\beta \lambda_i^\beta = T$, then $\lambda_i^\beta = \lambda_j^\beta,\ \forall\ \lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in X$. Let $\emptyset \ne \mathcal{A} \subseteq X$, where $(X, \#, T)$ is $(PBCK - A)$. Then $\mathcal{A}$ is a permutation ideal if $T \in \mathcal{A}$ and such that for any $\lambda_n^\beta, \lambda_j^\beta\ \#\ \lambda_n^\beta \in \mathcal{A}$ imply $\lambda_j^\beta \in \mathcal{A}$. Also, $\mathcal{A}$ is a maximal, if there is no proper permutation ideal H contains $\mathcal{A}$ and $H \ne \mathcal{A}$. We define $\mathcal{A}^\ddagger = \{\lambda_n^\beta \in X : \lambda_n^\beta \circledast \lambda_j^\beta = T, \forall\ \lambda_j^\beta \in \mathcal{A}\}$, where $\lambda_n^\beta \circledast \lambda_j^\beta = \lambda_n^\beta\ \#\ (\lambda_n^\beta \# \lambda_j^\beta)$ and it is called a permutation annihilator of $\mathcal{A}$. $\mathcal{A}^\ddagger$ is an ideal of X. If $\mathcal{A}^\ddagger = \{\lambda_\mathcal{A}^\beta\}$ (singleton), then its can be written as $\{\lambda_\mathcal{A}^\beta\}^\ddagger = (\lambda_\mathcal{A}^\beta)^\ddagger$. In general, for any permutation ideal $\mathcal{A}$, $\mathcal{A} \cap \mathcal{A}^\ddagger = \{T\}$. If $\mathcal{A}$ *and* $\mathcal{B}$ are subsets of X such that $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{B}^\ddagger \subseteq \mathcal{A}^\ddagger$. If $\mathcal{A} = \mathcal{A}^\ddagger$ then $\mathcal{A}$ is called a permutation involutory ideal.

Definition 3.2: Let A be a permutation ideal of X and $\lambda_n^\beta \in X$. We define $\lambda_n^{\beta**}A = \{\lambda_j^\beta \in X : \lambda_n^\beta \circledast \lambda_j^\beta \in A$, where $\lambda_n^\beta \circledast \lambda_j^\beta = \lambda_n^\beta\ \#\ (\lambda_n^\beta\ \#\ \lambda_j^\beta)\}$.

Lemma 3.3: *Let $(X, \#, T)$ be a permutation BCK–algebra and A be a permutation ideal of X. Then $T \in \lambda_n^{\beta**}A$, for any $\lambda_n^\beta \in X$, (i.e, $\lambda_n^{\beta**}A$ is nonempty).*

Proof: Since A is a permutation ideal of X, then $T \in A$. Also, $\lambda_n^\beta \circledast T = \lambda_n^\beta \# (\lambda_n^\beta \# T) = \lambda_n^\beta \# \lambda_n^\beta = T \in A$. Then $T \in \lambda_n^{\beta**}A$. Thus $\lambda_n^{\beta**}A$ is nonempty set.

Definition 3.4: Let $(X, \#, T)$ be a permutation *BCK*–algebra, then its called a commutative permutation *BCK*–algebra, if $\lambda_n^\beta \circledast \lambda_m^\beta = \lambda_m^\beta \circledast \lambda_n^\beta$, $\forall\, \lambda_n^\beta,\ \lambda_m^\beta \in X$. Also, X is bounded if it contians an element λ_n^β such that $\lambda_m^\beta \le \lambda_n^\beta$, $\forall\, \lambda_m^\beta \in X$. Moreover, X is an implicative if $\lambda_m^\beta \# (\lambda_k^\beta \# \lambda_m^\beta) = \lambda_m^\beta$, $\forall\, \lambda_m^\beta,\ \lambda_k^\beta \in X$.

Example 3.5: *Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 5 & 1 & 9 & 6 & 2 & 11 & 10 & 4 & 8 & 7 \end{pmatrix}$ be a permutation in S_{11}. So, $\beta = (1\ 3)(2\ 5\ 6)(4\ 9)(7\ 11)(8\ 10)$. Therefore, we have $X = \{\lambda_i^\beta\}_{i=1}^{6} = \{\{1,3\}, \{2,5,6\}, \{4,9\}, \{7,11\}, \{8,10\}\}$ and $T = \{1,3\}$. Define $\# : X \times X \to X$ by Table 1*

Table 1. $(X, \#, T)$ is a $(P - BCK - A)$.

#	{1,3}	{2,5,6}	{4,9}	{7,11}	{8,10}
{1,3}	{1,3}	{1,3}	{1,3}	{1,3}	{1,3}
{2,5,6}	{2,5,6}	{1,3}	{2,5,6}	{2,5,6}	{2,5,6}
{4,9}	{4,9}	{4,9}	{1,3}	{4,9}	{4,9}
{7,11}	{7,11}	{7,11}	{7,11}	{1,3}	{7,11}
{8,10}	{8,10}	{8,10}	{8,10}	{8,10}	{1,3}

Therefore, we consider $\circledast : X \times X \to X$ by Table 2

Table 2. $(X, \#, T)$ is a commutative $(P - BCK - A)$.

⊛	{1,3}	{2,5,6}	{4,9}	{7,11}	{8,10}
{1,3}	{1,3}	{1,3}	{1,3}	{1,3}	{1,3}
{2,5,6}	{1,3}	{2,5,6}	{1,3}	{1,3}	{1,3}
{4,9}	{1,3}	{1,3}	{4,9}	{1,3}	{1,3}
{7,11}	{1,3}	{1,3}	{1,3}	{7,11}	{1,3}
{8,10}	{1,3}	{1,3}	{1,3}	{1,3}	{8,10}

Hence, $(X, \#, T\ )$ is a commutative $(P - BCK - A)$.

Example 3.6: *The chemical element Cadmium, denoted as (Cd), possesses 48 protons in its nucleus, and its electron count equals the proton count, resulting in 48 electrons revolving around it. Additionally, there are five atomic shells labelled as $Cd = \{K, L, M, N, O\}$ surrounding the nucleus, each containing a specific number of electrons, as illustrated in Fig. 1.*

Hence $K = \{e_1, e_2\}$, $L = \{e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$, $M = \{e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}\}$, $N = \{e_{29}, e_{30}, e_{31}, e_{32}, e_{33}, e_{34}, e_{35}, e_{36}, e_{37}, e_{38}, e_{39}, e_{40}, e_{41}, e_{42}, e_{43}, e_{44}, e_{45}, e_{46}\}$, $O = \{e_{47}, e_{48}\}$. Let $\beta = (1\ 2)(3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15\ 16\ 1718\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28)$

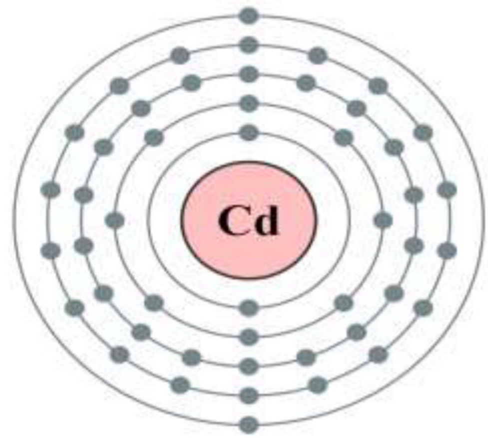

Fig. 1. Atomic shells for Cadmium.

$(29\ 30\ 31\ 32\ 33\ 34\ 35\ 36\ 37\ 38\ 39\ 40\ 41\ 42\ 43\ 44\ 45\ 46)(47\ 48)$ be a permutation in S_{48}. Therefore, we have $X = \{\lambda_i^\beta\}_{i=1}^{5} = \{\{1, 2\}, \{3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}, \{11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20,\ 21, 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28\},\ \{29,\ 30,\ 31,\ 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46\}\ \{47, 48\}\}$. Define a map $f :\ Cd \to X$ by $f(A) = \{i, j, \ldots, r\}$, $\forall A = \{e_i, e_j, \ldots, e_r\} \in Cd$. Then

$$f(A) = \begin{cases} \lambda_1^\beta, & if\ A = K, \\ \lambda_2^\beta, & if\ A = L \\ \lambda_3^\beta, & if\ A = M \\ \lambda_4^\beta, & if\ A = N \\ \lambda_5^\beta, & if\ A = O \end{cases} \text{. for any } A \in Cd.$$

Moreover, for any $f(A), f(B) \in f(Cd) = \{f(K), f(L), f(M), f(N), f(O)\}$ define $\#\!:\ f(Cd) \times f(Cd) \to f(Cd)$ by $(A)\ \#\ f(B) = \begin{cases} f(K), & if\ f(A) = f(B) \\ f(A), & if\ f(A) \ne f(B) \end{cases}$. Therefore we can consider Table 3.

Table 3. $(f(Cd), \#, f(K)\ )$ is a $(P - BCK - A)$.

#	$f(K)$	$f(L)$	$f(M)$	$f(N)$	$f(O)$
$f(K)$	$f(K)$	$f(K)$	$f(K)$	$f(K)$	$f(K)$
$f(L)$	$f(L)$	$f(K)$	$f(L)$	$f(L)$	$f(L)$
$f(M)$	$f(M)$	$f(M)$	$f(K)$	$f(M)$	$f(M)$
$f(N)$	$f(N)$	$f(N)$	$f(N)$	$f(K)$	$f(N)$
$f(O)$	$f(O)$	$f(O)$	$f(O)$	$f(O)$	$f(K)$

Hence, $(f(Cd), \#, f(K)\ )$ is a commutative $(P - BCK - A)$.

Remark 3.7: In any commutative permutation *BCK*–algebra $(X, \#, T)$, the inequalities are held:

(1) $\lambda_i^\beta \le \lambda_j^\beta$ iff $\lambda_i^\beta \# \lambda_j^\beta = T$,

(2) $If\ \lambda_i^\beta \le \lambda_j^\beta$, then $\lambda_i^\beta \# \lambda_m^\beta \le \lambda_j^\beta \# \lambda_m^\beta$ and $\lambda_m^\beta \# \lambda_j^\beta \le \lambda_m^\beta \# \lambda_i^\beta$,

(3) $\lambda_m^\beta \circledast (\lambda_n^\beta \circledast \lambda_i^\beta) = (\lambda_n^\beta \circledast \lambda_m^\beta) \circledast \lambda_i^\beta$,

(4) $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_i^\beta) \leq \lambda_n^\beta \circledast (\lambda_m^\beta \# \lambda_i^\beta)$, $\forall\ \lambda_m^\beta, \lambda_j^\beta, \lambda_n^\beta, \lambda_i^\beta \in X$.

Proposition 3.8: If $(X, \#, T)$ is a $(PBCK-A)$, then $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta = (\lambda_i^\beta \# \lambda_m^\beta) \# \lambda_j^\beta$.

Proof: By [(PBCK – A) – (2)], and from [Remark 3.7-(1)], we get $\lambda_i^\beta \#(\lambda_i^\beta \# \lambda_m^\beta) \leq \lambda_m^\beta$. Also, from [$(PBCK-A)-(2)$] and [Remark 3.7-(2)], we obtain that $(\lambda_i^\beta \# \lambda_j^\beta)\#\lambda_m^\beta \leq (\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_i^\beta \#(\lambda_i^\beta \# \lambda_m^\beta))$. This yields, $((\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta) \# ((\lambda_i^\beta \# \lambda_m^\beta) \# \lambda_j^\beta) \leq (\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_i^\beta \# (\lambda_i^\beta \# \lambda_m^\beta)) \# ((\lambda_i^\beta \# \lambda_m^\beta) \# \lambda_j^\beta) = T$ [from [$(PBCK-A)-(1)$]. Hence $((\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta) \# ((\lambda_i^\beta \# \lambda_m^\beta) \# \lambda_j^\beta) \leq T$, and from [(PBCK – A) – (4)] imply $T \leq ((\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta)\#((\lambda_i^\beta \# \lambda_m^\beta) \# \lambda_j^\beta)$. Hence $((\lambda_i^\beta \# \lambda_j^\beta) \#\lambda_m^\beta) \# ((\lambda_i^\beta \# \lambda_m^\beta) \#\lambda_j^\beta) = T \ldots (i)$. Similarly, $\lambda_i^\beta \#(\lambda_i^\beta \# \lambda_j^\beta) \leq \lambda_j^\beta$ by [(PBCK–A) – (2)] and [Remark 3.7-(2)]. Also, by [$(PBCK-A)-(1)$] and [$(PBCK-A)-(4)$], we have $((\lambda_i^\beta \# \lambda_m^\beta)\#\lambda_j^\beta) \# ((\lambda_i^\beta \# \lambda_j^\beta)\#\lambda_m^\beta) = T \ldots (ii)$. Which by [$(PBCK-A)-(5)$], gives $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta = (\lambda_i^\beta \# \lambda_m^\beta) \# \lambda_j^\beta$.

Proposition 3.9: The relation $\leq$ define on a $(PBCK-A)$ $(X, \#, T)$ by [Remark 3.7] then the following condition are satisfies.

i. $(\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_m^\beta \# \lambda_j^\beta) \leq \lambda_i^\beta \# \lambda_m^\beta$,
ii. $\lambda_i^\beta \#(\lambda_j^\beta \# \lambda_m^\beta) \leq \lambda_i^\beta \# ((\lambda_j^\beta \# \lambda_n^\beta) \# (\lambda_m^\beta \# \lambda_n^\beta))$,
iii. $((\lambda_i^\beta \# \lambda_n^\beta) \# (\lambda_j^\beta \# \lambda_n^\beta)) \# \lambda_m^\beta \leq (\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta$,
iv. $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq \lambda_i^\beta \# \lambda_m^\beta \leq \lambda_i^\beta \# (\lambda_m^\beta \# \lambda_n^\beta)$,
v. $\lambda_i^\beta \# \lambda_j^\beta \leq \lambda_i^\beta$,
vi. $(\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_i^\beta \# \lambda_m^\beta) \leq \lambda_i^\beta \# (\lambda_i^\beta \# \lambda_m^\beta)$,
vii. $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq (\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_m^\beta \# \lambda_j^\beta)$,
viii. $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq \lambda_i^\beta \# (\lambda_j^\beta \# \lambda_m^\beta)$, for any $\lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta, \lambda_n^\beta \in X$.

Proof (i) Since $(\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_i^\beta \# \lambda_m^\beta) \leq (\lambda_m^\beta \# \lambda_j^\beta)$ [By $(PBCK-A-1)$ & Remark (3.5)]. $(\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_m^\beta \# \lambda_j^\beta) \leq (\lambda_i^\beta \# \lambda_j^\beta) \# ((\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_i^\beta \# \lambda_m^\beta)) \leq \lambda_i^\beta \# \lambda_m^\beta$ [From $(PBCK-A-2)$ and Remark 3.7]. Thus $(\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_m^\beta \# \lambda_j^\beta) \leq \lambda_i^\beta \# \lambda_m^\beta$. Hence (i) is held. Condition (ii) and (iii) follows from (i) and [Remark 3.7]. Using (ii) and (iii), we get $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta = ((\lambda_i^\beta \# \lambda_j^\beta) \# (T \# \lambda_j^\beta)) \# \lambda_m^\beta \leq (\lambda_i^\beta \# T) \# \lambda_m^\beta = \lambda_i^\beta \# \lambda_m^\beta$. Now $\lambda_i^\beta \# \lambda_m^\beta = \lambda_i^\beta \# (\lambda_m^\beta \# T) \leq \lambda_i^\beta \# ((\lambda_m^\beta \# \lambda_n^\beta) \# (T \# \lambda_n^\beta)) = \lambda_i^\beta \# (\lambda_m^\beta \# \lambda_n^\beta)$. Thus $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq \lambda_i^\beta \# \lambda_m^\beta \leq \lambda_i^\beta \# (\lambda_m^\beta \# \lambda_n^\beta)$. That means (iv) is held. As a simple consequence of (iv), we obtain (v). Also, since $\lambda_i^\beta \# \lambda_j^\beta \leq \lambda_i^\beta$ [by (v)], then from [Remark 3.7] we have $(\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_i^\beta \# \lambda_m^\beta) \leq \lambda_i^\beta \# (\lambda_i^\beta \# \lambda_m^\beta)$. Hence (vi) is held. Moreover, since $\lambda_m^\beta \# \lambda_j^\beta \leq \lambda_m^\beta$ [by (v)], then from [Remark 3.7] we have $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq (\lambda_i^\beta \# \lambda_j^\beta) \# (\lambda_m^\beta \# \lambda_j^\beta)$, thus (vii) is held Furthermore, since $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq (\lambda_i^\beta \# \lambda_j^\beta)$ [by (v)] and $(\lambda_i^\beta \# \lambda_j^\beta) \leq \lambda_i^\beta \# (\lambda_j^\beta \# \lambda_m^\beta)$ [by (iv)]. Then $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta \leq (\lambda_i^\beta \# \lambda_j^\beta) \leq \lambda_i^\beta \# (\lambda_j^\beta \# \lambda_m^\beta)$, which gives (viii).

Proposition 3.10: Let $(X, \#, T)$ be a commutative permutation BCK–algebra and $A \subseteq X$ is a permutation ideal. Then $\lambda_n^{\beta **}A$ is a permutation ideal which contains A.

Proof: It is obvious that $T \in \lambda_n^{\beta **}A$ [From Lemma 3.3]. Now, assume that $\lambda_j^\beta, \lambda_m^\beta \# \lambda_j^\beta \in \lambda_n^{\beta **}A$. We need to prove $\lambda_m^\beta \in \lambda_n^{\beta **}A$, (i.e, $\lambda_n^\beta \circledast \lambda_m^\beta \in A$). Then $\lambda_n^\beta \circledast \lambda_j^\beta, \lambda_n^\beta \circledast (\lambda_m^\beta \# \lambda_j^\beta) \in A$, by [Definition 3.2]. Now, $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_j^\beta)) \leq \lambda_n^\beta \circledast (\lambda_m^\beta \# \lambda_j^\beta) \in A$ [From Remark 3.7-(4)], and A is a permutation ideal, therefore $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_j^\beta) \in A$. Again using the fact that A is permutation ideal and $(\lambda_n^\beta \circledast \lambda_j^\beta) \in A$, we get that $(\lambda_n^\beta \circledast \lambda_m^\beta) \in A$. This means $\lambda_m^\beta \in \lambda_n^{\beta **}A$, which proves that $\lambda_n^{\beta **}A$ is a permutation ideal. To prove that $A \subseteq \lambda_n^{\beta **}A$, let $\lambda_m^\beta \in A$. Then $\lambda_n^\beta \circledast \lambda_m^\beta \leq \lambda_m^\beta \in A$ implies that $\lambda_n^\beta \circledast \lambda_m^\beta \in A$, thus $\lambda_m^\beta \in \lambda_n^{\beta **}A$. This ends the proof.

Lemma 3.11: *If $(X, \#, T)$ is a commutative permutation BCK–algebra, then*

(i). $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_k^\beta) \leq \lambda_m^\beta$,
(ii). $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_i^\beta) \leq (\lambda_m^\beta \# \lambda_i^\beta)$, $\forall\ \lambda_n^\beta, \lambda_m^\beta, \lambda_k^\beta \in X$.

Proof: (i) from [Proposition 3.9-(v)] such that $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_k^\beta) \leq (\lambda_n^\beta \circledast \lambda_m^\beta)$. However, $(\lambda_n^\beta \circledast \lambda_m^\beta) = \lambda_n^\beta \# (\lambda_n^\beta \# \lambda_m^\beta) = \lambda_m^\beta \# (\lambda_m^\beta \# \lambda_n^\beta)$ [since X is a commutative]. Then, from [Proposition 3.9-(v)], we have $\lambda_m^\beta \# (\lambda_m^\beta \# \lambda_n^\beta) \leq \lambda_m^\beta$. Hence $(\lambda_n^\beta \circledast \lambda_m^\beta) \#(\lambda_n^\beta \circledast \lambda_k^\beta) \leq (\lambda_n^\beta \circledast \lambda_m^\beta) = \lambda_n^\beta \# (\lambda_n^\beta \# \lambda_m^\beta) = \lambda_m^\beta \# (\lambda_m^\beta \# \lambda_n^\beta) \leq \lambda_m^\beta$, $\forall\ \lambda_n^\beta, \lambda_m^\beta, \lambda_k^\beta \in X$.

Proof: (ii) since X is a commutative and [BCK–A–(1)], we have $(\lambda_n^\beta \circledast \lambda_m^\beta) \# (\lambda_n^\beta \circledast \lambda_k^\beta) = (\lambda_n^\beta \#(\lambda_n^\beta \#$

$\lambda_m^\beta))\#\ (\lambda_n^\beta\ \#\ (\lambda_n^\beta\ \#\ \lambda_k^\beta)) \leq (\lambda_n^\beta\ \#\ \lambda_k^\beta)\ \#\ (\lambda_n^\beta\ \#\ \lambda_m^\beta) \leq (\lambda_m^\beta\ \#\ \lambda_k^\beta)$.

Definition 3.12: Let $(X, \#, T)$ be a commutative permutation BCK–algebra and A be aproper permutation ideal. We say A is prime permutation ideal if $(\lambda_n^\beta \circledast \lambda_m^\beta) \in A$, then $\lambda_n^\beta \in A$ or $\lambda_m^\beta \in A$.

Proposition 3.13: Let $(X, \#, T)$ be acommutative permutation BCK–algebra and A be a permutation ideal of X. Then the following statements hold:

(1) $\lambda_n^{\beta**}A = X$ if and only if $\lambda_n^\beta \in A$.
(2) If $\lambda_n^\beta \leq \lambda_i^\beta$ then $\lambda_i^{\beta**}A \subseteq \lambda_n^{\beta**}A$.
(3) If A and B are permutation ideals of X such that $A \subseteq B$, then $\lambda_n^{\beta**}A \subseteq \lambda_n^{\beta**}B, \forall\ \lambda_n^\beta \in X$.
(4) $(\lambda_n^\beta)^\ddagger \subseteq \lambda_n^{\beta**}A$, for all $\lambda_n^\beta \in X$.
(5) For any ideals A, B *of* X and any $\lambda_n^\beta \in X$, $\lambda_n^{\beta**}(A \cap B) = \lambda_n^{\beta**}A \cap \lambda_n^{\beta**}B$.
(6) Let A be a permutation ideal and P be a prime permutation ideal such that $A \subseteq P$. Then $\lambda_n^{\beta**}A \subseteq P$, for all $\lambda_n^\beta \in X - P$.
(7) $(\lambda_n^\beta \circledast \lambda_m^\beta)^{**}A = \lambda_n^{\beta**}(\lambda_m^{\beta**}A)$, for all $\lambda_n^\beta \in X$.

Proof (1): If $\lambda_n^{\beta**}A = X$. Let A be an ideal of X and $\lambda_n^\beta \in X$. So, $\lambda_n^{\beta**}A = \{\lambda_j^\beta \in X : \lambda_n^\beta \circledast \lambda_j^\beta \in A\}$, imply $\lambda_n^\beta \in X = \lambda_n^{\beta**}A$, we have $\lambda_n^\beta \circledast \lambda_n^\beta \in A$, but $\lambda_n^\beta \circledast \lambda_n^\beta = \lambda_n^\beta\ \#\ (\lambda_n^\beta\ \#\ \lambda_n^\beta) = \lambda_n^\beta\ \#\ T \in A$ and A is permutation ideal. Hence $\lambda_n^\beta \in A$.

Conversely, if $\lambda_n^\beta \in A$, clearly we conclude easily $\lambda_n^{\beta**}A \subseteq X$, because $\lambda_n^{\beta**}A$ is ideal of X. So, we need only to show that $X \subseteq \lambda_n^{\beta**}A$, let $\lambda_m^\beta \in X$, then $\lambda_m^\beta \circledast \lambda_n^\beta \leq \lambda_n^\beta \in A$, imply $\lambda_m^\beta \circledast \lambda_n^\beta \in A$, but $\lambda_n^\beta \circledast \lambda_m^\beta = \lambda_m^\beta \circledast \lambda_n^\beta \in A$ [From Definition 3.4] . Hence [From Definition 3.2], we have $\lambda_m^\beta \in \lambda_n^{\beta**}A$. Thus $X \subseteq \lambda_n^{\beta**}A$. This means $\lambda_n^{\beta**}A = X$.

Proof (2): If $\lambda_n^\beta \leq \lambda_i^\beta$, then $\lambda_n^\beta\ \#\ \lambda_i^\beta = T$. Let $\lambda_m^\beta \in \lambda_i^{\beta**}A$, thus $\lambda_i^\beta \circledast \lambda_m^\beta \in A$. So $(\lambda_n^\beta \circledast \lambda_m^\beta)\ \#\ (\lambda_i^\beta \circledast \lambda_m^\beta) = (\lambda_m^\beta \circledast \lambda_n^\beta)\ \#\ (\lambda_m^\beta \circledast \lambda_i^\beta) \leq \lambda_n^\beta\ \#\ \lambda_i^\beta$ [by Remark 3.7-(3)]. But $\lambda_n^\beta\ \#\ \lambda_i^\beta = T \in A$ [since A is permutation ideal]. Then $(\lambda_n^\beta \circledast \lambda_m^\beta)\ \#\ (\lambda_i^\beta \circledast \lambda_m^\beta) \leq T \in A$, thus $(\lambda_n^\beta \circledast \lambda_m^\beta)\ \#\ (\lambda_i^\beta \circledast \lambda_m^\beta) \in A$, but $(\lambda_i^\beta \circledast \lambda_m^\beta) \in A$ and A is permutation ideal. Then $(\lambda_n^\beta \circledast \lambda_m^\beta) \in A$. Hence $\lambda_m^\beta \in \lambda_n^{\beta**}A$. Therefore, $\lambda_i^{\beta**}A \subseteq \lambda_n^{\beta**}A$.

Proof (3): Let $\lambda_m^\beta \in \lambda_n^{\beta**}A$, then we have $\lambda_n^\beta \circledast \lambda_m^\beta \in A \subseteq B$, imply that $\lambda_n^\beta \circledast \lambda_m^\beta \in B$. Hence $\lambda_m^\beta \in \lambda_n^{\beta**}B$. then $\lambda_n^{\beta**}A \subseteq \lambda_n^{\beta**}B$ for all $\lambda_n^\beta \in X$.

Proof (4): From [Definition 3.1] we can prove (4) easily.

Proof (5): Let $\lambda_i^\beta \in \lambda_n^{\beta**}(A \cap B)$, by [Definition 3.2]. We have $\lambda_n^\beta \circledast \lambda_i^\beta \in (A \cap B)$, then $\lambda_n^\beta \circledast \lambda_i^\beta \in A\ \wedge\ \lambda_n^\beta \circledast \lambda_i^\beta \in B$, imply $\lambda_i^\beta \in \lambda_n^{\beta**}A\ \wedge\ \lambda_i^\beta \in \lambda_n^{\beta**}B$. Then $\lambda_i^\beta \in (\lambda_n^{\beta**}A\ \cap\ \lambda_n^{\beta**}B)$ so that $\lambda_n^{\beta**}(A \cap B) \subseteq \lambda_n^{\beta**}A\ \cap\ \lambda_n^{\beta**}B$. In the same way we get $\lambda_n^{\beta**}A\ \cap\ \lambda_n^{\beta**}B \subseteq \lambda_n^{\beta**}(A \cap B)$, thus $\lambda_n^{\beta**}(A \cap B) = \lambda_n^{\beta**}A \cap \lambda_n^{\beta**}B$ for any $\lambda_n^\beta \in X$.

Proof (6): Let $\lambda_i^\beta \in \lambda_n^{\beta**}A$, by [Definition 3.2] we have $\lambda_n^\beta \circledast \lambda_i^\beta \in A$. this means $\lambda_n^\beta \circledast \lambda_i^\beta \in P$, because $A \subseteq P$. So $\lambda_i^\beta \in P$, since P is a prime and $\lambda_n^\beta \in X - P$. Thus $\lambda_n^{\beta**}A \subseteq P$.

Proof (7): Let $\lambda_i^\beta \in \lambda_n^{\beta**}(\lambda_m^{\beta**}A)$, imply $\lambda_m^\beta \circledast (\lambda_n^\beta \circledast \lambda_i^\beta) \in A$ from [Definition 3.2]. To prove $\lambda_i^\beta \in (\lambda_n^\beta \circledast \lambda_m^\beta)^{**}A$, we need to show that $(\lambda_n^\beta \circledast \lambda_m^\beta) \circledast \lambda_i^\beta \in A$. Now, $(\lambda_n^\beta \circledast \lambda_m^\beta) \circledast \lambda_i^\beta = (\lambda_m^\beta \circledast \lambda_n^\beta) \circledast \lambda_i^\beta$ [since X is a commutative]. Also, $(\lambda_m^\beta \circledast \lambda_n^\beta) \circledast \lambda_i^\beta = \lambda_m^\beta \circledast (\lambda_n^\beta \circledast \lambda_i^\beta)$ [since $\circledast$ is associative]. Hence $(\lambda_n^\beta \circledast \lambda_m^\beta) \circledast \lambda_i^\beta = (\lambda_m^\beta \circledast \lambda_n^\beta) \circledast \lambda_i^\beta = \lambda_m^\beta \circledast (\lambda_n^\beta \circledast \lambda_i^\beta) \in A$. Then $\lambda_i^\beta \in (\lambda_n^\beta \circledast \lambda_m^\beta)^{**}A$. This means $\lambda_n^{\beta**}(\lambda_m^{\beta**}A) \subseteq (\lambda_n^\beta \circledast \lambda_m^\beta)^{**}A$. In the same way we get $\lambda_n^{\beta**} \circledast (\lambda_m^{\beta**}A) \subseteq (\lambda_n^\beta \circledast \lambda_m^\beta)^{**}A$. Thus $(\lambda_n^\beta \circledast \lambda_m^\beta)^{**}A = \lambda_n^{\beta**}(\lambda_m^{\beta**}A)$.

Proposition 3.14: Let $(X, \#, T)$ be acommutative permutation BCK–algebra and A be a permutation ideal of X. Then A is prime permutation ideal of X if and only if $\lambda_n^{\beta**}A = A$, for all $\lambda_n^\beta \in X - A$.

Proof. Suppose that A is a prime permutation ideal of X and $\lambda_n^\beta \in X - A$. Let $\lambda_i^\beta \in A$, thus $\lambda_n^\beta \circledast \lambda_i^\beta = \lambda_n^\beta\ \#\ (\lambda_n^\beta\ \#\ \lambda_i^\beta) \leq \lambda_i^\beta \in A$, then $\lambda_n^\beta \circledast \lambda_i^\beta \in A$. Hence $\lambda_i^\beta \subseteq \lambda_n^{\beta**}A$. To prove the reverse inclusion, let $\lambda_i^\beta \in \lambda_n^{\beta**}A$. This implies that $\lambda_n^\beta \circledast \lambda_i^\beta \in A$ and A being a prime permutation ideal implies that $\lambda_i^\beta \in A$, (because $\lambda_n^\beta \notin A$ by assumption). This proves that $\lambda_n^{\beta**}A = A$. Conversely,assume that $\lambda_n^{\beta**}A = A$ for all $\lambda_n^\beta \in X - A$. Let $\lambda_i^\beta \circledast \lambda_m^\beta \in A$ and $\lambda_i^\beta \notin A$. By hypothesis $\lambda_m^\beta A = A$ and consequently $\lambda_m^\beta \in \lambda_m^\beta A = A$. This proves that A is a prime permutation ideal.

Proposition 3.15: Every maximal permutation ideal in a commutative permutation BCK–algebra is prime permutation ideal.

Proof. Let A be a maximal permutation ideal in a commutative permutation $BCK-$ algebra X. To show that A is prime, it is sufficient to prove that $\lambda_n^{\beta**}A = A$ for all $\lambda_n^\beta \in X - A$ (by Proposition 3.14). As proved earlier $A \subseteq \lambda_n^{\beta**}A$. If $A \neq \lambda_n^{\beta**}A$ then the maximality of A implies that $\lambda_n^{\beta**}A = X$. This happens only when $\lambda_n^\beta \in A$ [by Proposition 3.13-(1)] which is a contradiction because $\lambda_n^\beta \notin A$. This shows that $\lambda_n^{\beta**}A = A$ and consequently A is a prime ideal.

Proposition 3.16: Let $(X, \#, T)$ be acommutative permutation BCK–algebra and P be a bounded permutation involutory ideal of X. Then P is maximal permutation ideal if and only if it is prime permutation ideal.

Proof. Let P be bounded permutation involutory ideal of X. Suppose that P is maximal permutation ideal. Then P is prime permutation ideal by [Proposition 3.14]. Conversely, assume that P is prime permutation ideal. Let M be a proper maximal permutation ideal that contains P. We now show that $M = P$. Assume that $M \subsetneq P$. Now $M \cap M^{\ddagger} = \{T\} \subseteq P$. P being a prime ideal implies that $M \subseteq P$ or $M^{\ddagger} \subseteq P$. As M $\subsetneq P$, therefore $M^{\ddagger} \subseteq P$ Since $P \subseteq M$, therefore $M^{\ddagger} \subseteq P^{\ddagger}$. We get that $M^{\ddagger} \subseteq P \cap P^{\ddagger} = \{T\}$. That is $M^{\ddagger} = \{T\}$ and hence $M^{\ddagger\ddagger} = X$. As X is involutory we have $M^{\ddagger\ddagger} = M = X$, a contradiction. Therefore, $M \subseteq P$ and consequently $M = P$. This verifies the result.

Definition 3.17: An element λ_m^{β} in a permutation $BCK-$ algebra X is said to be a portion if $\lambda_i^{\beta} < \lambda_m^{\beta}$ for some $\lambda_i^{\beta} \in X$ implies $\lambda_i^{\beta} = T$ or $\lambda_i^{\beta} = \lambda_m^{\beta}$.

It follows that $\lambda_i^{\beta} \in A^{\ddagger\ddagger}$ and consequently $\lambda_i^{\beta} \in \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A \subseteq A^{\ddagger\ddagger}$. Hence $A^{\ddagger\ddagger} = \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A$.

Corollary 3.20: *Let $(X, \#, T)$ be acommutative permutation BCK–algebra and A be a prime permutation ideal in X and A be a prime ideal of X with $A^{\ddagger} = \{T\}$. Then A is an involutory ideal.*

Proof. To prove that we need show that $A^{\ddagger\ddagger} = A$. So from [Proposition 3.19], then $A^{\ddagger\ddagger} = \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A$. From [Proposition 3.13-(1)] and [Proposition 3.14], imply $A = \lambda_m^{\beta**} A$. Thus $\cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A = \cap_{\lambda_m^{\beta} \in A^{\ddagger}} A = A = A^{\ddagger\ddagger}$. That is the requirement wanted.

Definition 3.21: Assume $\beta = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & m \\ \beta(1) & \beta(2) & \beta(3) & \cdots & \beta(m) \end{smallmatrix}\right) = \lambda_1 \lambda_2 \ldots \lambda_{c(\beta)}$, $\delta = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & k \\ \delta(1) & \delta(2) & \delta(3) & \cdots & \delta(k) \end{smallmatrix}\right) = g_1 g_2 \ldots g_{c(\delta)}$, and $\gamma = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ \gamma(1) & \gamma(2) & \gamma(3) & \cdots & \gamma(n) \end{smallmatrix}\right) = f_1 f_2 \ldots f_{c(\gamma)}$ are three permutations in S_m, S_k and S_n respectively, where $n \leq k \leq m$. Let $\{\lambda_i^{\beta} = \{t_1^i, t_2^i, \ldots, t_{\alpha_i}^i\} | 1 \leq i \leq c(\beta)\}$, $\{\lambda_i^{\delta} = \{q_1^i, q_2^i, \ldots, q_{\alpha_i}^i\} | 1 \leq i \leq c(\delta)\}$ and $\{f_i^{\gamma} = \{d_1^i, d_2^i, \ldots, d_{\alpha_i}^i\} | 1 \leq i \leq c(\gamma)\}$ be three collection sets of permutation sets for β, δ and γ, respectively. Define

$$\omega_{\beta,\delta,\gamma} = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m+1 & \cdots & m+k & m+k+1 & m+k+2 & \cdots & m+k+n \\ \omega_{\beta,\delta,\gamma}(1) & \omega_{\beta,\delta,\gamma}(2) & \omega_{\beta,\delta,\gamma}(3) & \cdots & \omega_{\beta,\delta,\gamma}(m) & \omega_{\beta,\delta,\gamma}(m+1) & \cdots & \omega_{\beta,\delta,\gamma}(m+k) & \omega_{\beta,\delta,\gamma}(m+k+1) & \omega_{\beta,\delta,\gamma}(m+k+2) & \cdots & \omega_{\beta,\delta,\gamma}(m+k+n) \end{pmatrix}$$

Proposition 3.18: Let $(X, \#, T)$ be acommutative permutation BCK–algebra. If λ_m^{β} is a portion in X, then $(\lambda_m^{\beta})^{\ddagger} = \lambda_m^{\beta**} A$ for every permutation ideal A with $\lambda_m^{\beta} \notin A$, and $(\lambda_m^{\beta})^{\ddagger}$ is a prime and maximal permutation ideal.

Proof. $(\lambda_m^{\beta})^{\ddagger} \subseteq \lambda_m^{\beta**} A$ by [Proposition 3.13–(4)]. If $\lambda_j^{\beta} \in \lambda_m^{\beta**} A$, then $\lambda_m^{\beta} \circledast \lambda_j^{\beta} = \lambda_j^{\beta} \circledast \lambda_m^{\beta} \in$ A [since X is a commutative]. Thus $\lambda_j^{\beta} \circledast \lambda_m^{\beta} = T$ [Since λ_m^{β} is a portion and $\lambda_m^{\beta} \notin A$. Hence $\lambda_j^{\beta} \in (\lambda_m^{\beta})^{\ddagger}$. Then $(\lambda_m^{\beta})^{\ddagger} = \lambda_m^{\beta**} A$. If the permutation ideal $(\lambda_m^{\beta})^{\ddagger}$ was not maximal, then there would exist a proper permutation ideal A and $\lambda_j^{\beta} \in A$ such that $(\lambda_m^{\beta})^{\ddagger} \subseteq A$ and $\lambda_j^{\beta} \notin (\lambda_m^{\beta})^{\ddagger}$. Then $\lambda_j^{\beta} \circledast \lambda_m^{\beta} \neq T$. Since λ_m^{β} is a portion, $\lambda_j^{\beta} \circledast \lambda_m^{\beta} = \lambda_m^{\beta} \in A$, a contradiction. So $(\lambda_m^{\beta})^{\ddagger}$ is maximal. By [Proposition 3.15] it is prime also.

Proposition 3.19: Let $(X, \#, T)$ be acommutative permutation BCK–algebra and A be a permutation ideal in X. Then $A^{\ddagger\ddagger} = \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A$.

Proof. Let $\lambda_i^{\beta} \in A^{\ddagger\ddagger}$. Then $\lambda_i^{\beta} \circledast \lambda_m^{\beta} = T$, $\forall\ \lambda_m^{\beta} \in A^{\ddagger}$. Therefore $\lambda_i^{\beta} \in \lambda_m^{\beta**} A$, $\forall\ \lambda_m^{\beta} \in A^{\ddagger}$ and consequently $\lambda_i^{\beta} \in \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A$. That is, $A^{\ddagger\ddagger} \subseteq \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A$. Conversely, let $\lambda_i^{\beta} \in \cap_{\lambda_m^{\beta} \in A^{\ddagger}} \lambda_m^{\beta**} A$. Then $\lambda_i^{\beta} \in \lambda_m^{\beta**} A$, $\forall\ \lambda_m^{\beta} \in A^{\ddagger}$. This implies that $\lambda_m^{\beta} \circledast \lambda_i^{\beta} \in A$, $\forall\ \lambda_m^{\beta} \in A^{\ddagger}$ and hence $\lambda_m^{\beta} \circledast \lambda_i^{\beta} = (\lambda_m^{\beta} \circledast \lambda_i^{\beta}) \circledast \lambda_m^{\beta} =$ T, $\forall\ \lambda_m^{\beta} \in A^{\ddagger}$.

in S_{m+k+n} by $\omega_{\beta,\delta,\gamma}(j)$

$$= \begin{cases} \beta(j), & if\ 1 \leq j \leq m \\ \delta(m+k+1-j)+k, & if\ m < j \leq m+k \\ \gamma(m+k+n+1-j)+n, & if\ m+k < j \leq m+k+n \end{cases}.$$

Then, $\omega_{\beta,\delta,\gamma} = \prod_{i=1}^{c(\omega_{\beta,\delta,\gamma})} \sigma_i$ where $\prod_{i=1}^{c(\omega_{\beta,\delta,\gamma})} \sigma_i$ is a composite of pairwise disjoint cycles $\{\sigma_i\}_{i=1}^{c(\omega_{\beta,\delta,\gamma})}$. Moreover, $\omega_{\beta,\delta,\gamma}$ is called triple merge permutation (TMP) in S_{n+k+m} for β, δ and γ.

Example 3.22: *Let $\beta = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 5 & 1 & 9 & 6 & 2 & 11 & 10 & 4 & 8 & 7 \end{smallmatrix}\right)$, $\delta = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 9 & 1 & 2 & 7 & 3 & 6 & 8 \end{smallmatrix}\right)$ & $\gamma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{smallmatrix}\right)$ be three permutations in $S_m = S_{11}$, $S_k = S_9$ and $S_n = S_5$, respectively. Since $\beta = (1\ 3)(2\ 5\ 6)(4\ 9)(7\ 11)(8\ 10)$, $\delta = (1\ 5\ 2\ 4)(3\ 9\ 8\ 6\ 7)$ and $\gamma = (1\ 4)(2)(3\ 5)$. Then, we obtain that $X = \{\lambda_i^{\beta}\}_{i=1}^{5} = \{\{1,3\}, \{2,5,6\}, \{4,9\}, \{7,11\}, \{8,10\}\}$, $Y = \{\lambda_i^{\delta}\}_{i=1}^{2} = \{\{1,2,4,5\}, \{3,6,7,8,9\}\}$, and $W = \{\lambda_i^{\gamma}\}_{i=1}^{3} = \{\{1,4\}, \{2\}, \{3,5\}\}$. Here, we note that $n < k < m$ and hence we can find (TMP) by [Definition (9)] as following:*

$\omega_{\beta,\delta,\gamma}(j)$

$$= \begin{cases} \beta(j), & if\ 1 \leq j \leq m \\ \delta(m+k+1-j)+m, & if\ m < j \leq m+k \\ \gamma(m+k+n+1-j)+m+k, & if\ m+k < j \leq m+k+n \end{cases}.$$

That means

$$\omega_{\beta,\delta,\gamma}(j) = \begin{cases} \beta(j), & if\ 1 \leq j \leq 11 \\ \delta(21-j)+11, & if\ 11 < j \leq 20 \\ \gamma(26-j)+20, & if\ 20 < j \leq 25 \end{cases}$$

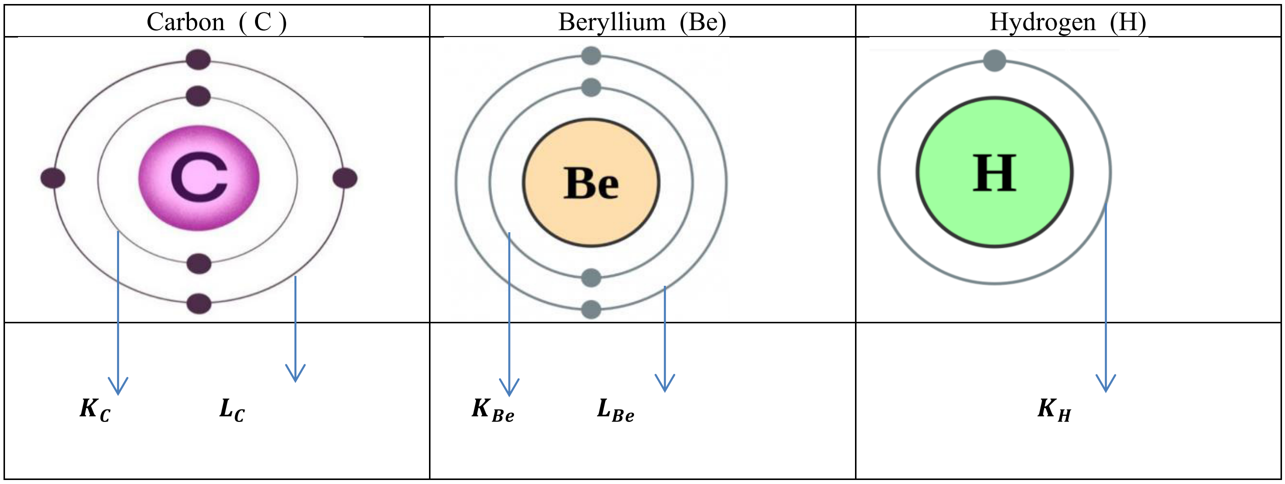

Fig. 2. Atomic shells for Carbon, Beryllium and Hydrogen.

Hence,

$$\omega_{\beta,\delta,\gamma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 3 & 5 & 1 & 9 & 6 & 2 & 11 & 10 & 4 & 8 & 7 & 19 & 17 & 14 & 18 & 13 & 12 & 20 & 15 & 16 & 23 & 21 & 25 & 22 & 24 \end{pmatrix}$$

is (TMP) in S_{25} for β, δ and γ.

Example 3.23: *The Carbon atom (C), Beryllium atom (Be) and Hydrogen atom (H), have 6, 4 and 1 electrons, respectively. Therefore, there are two atomic shells $C = \{K_C, L_C\}$ and $Be = \{K_{Be}, L_{Be}\}$ exist around nucleus (C) and (Be), respectively. Also, there is only one atomic shell $H = \{K_H\}$ around nucleus (H). See Fig. 2.*

Hence $K_C = \{e_1, e_2\}$, $L_C = \{e_3, e_4, e_5, e_6\}$, $K_{Be} = \{\acute{e}_1, \acute{e}_2\}$, $L_{Be} = \{\acute{e}_3, \acute{e}_4\}$, and $M_H = \{\acute{e}_1\}$.

Let $\beta = (1\ 2)(3\ 4\ 5\ 6)$, $\delta = (1\ 2)(3\ 4)$ and $\gamma = (1)$ be permutations in $S_m = S_6$, $S_k = S_4$ and $S_n = S_1$, respectively. Therefore, we have $X = \{\lambda_i^\beta\}_{i=1}^2 = \{\lambda_1^\beta = \{1,2\}, \lambda_2^\beta = \{3,4,5,6\}\}$, $Y = \{\lambda_i^\delta\}_{i=1}^2 = \{\lambda_1^\delta = \{1,2\}, \lambda_2^\delta = \{3,4,\}\}$ and $Y = \{\lambda_1^\gamma = \{1\}\}$. Define a map f_1: $C \to X$ by $f_1 A) = \{i, j, \ldots, r\}$, $\forall A = \{e_i, e_j, \ldots, e_r\} \in C$, f_2: $Be \to Y$ by $f_2(A) = \{i, j, \ldots, r\}$, $\forall A = \{\acute{e}_i, \acute{e}_j, \ldots, \acute{e}_r\} \in Be$, and f_3: $H \to W$ by $f_2(\{\acute{e}_1\}) = \{\{1\}\}$. Then $f_1(K_C) = \lambda_1^\beta = \lambda_1^\delta = f_2(K_{Be})$, $f_1(L_C) = \lambda_2^\beta$, $f_2(L_{Be}) = \lambda_2^\delta$, $f_3(K_H) = \lambda_1^\gamma$. Define f: $\{C, Be, H\} \to \{X, Y, W\}$ by $f(A) = \begin{cases} f_1(A) \text{ if } A \in C, \\ f_2(A) \text{ if } A \in Be, \\ f_3(A) \text{ if } A \in H. \end{cases}$ Since $n = 1 < k = 4 < m = 6$, then we will consider that

$$\omega_{\beta,\delta,\gamma}(j) = \begin{cases} \beta(j), \ if\ 1 \le j \le m \\ \delta(m+k+1-j)+m, \ if\ m < j \le m+k \\ \gamma(m+k+n+1-j)+m+k, \ if\ m+k < j \le m+k+n \end{cases}.$$

That means we will consider that

$$\omega_{\beta,\delta,\gamma}(j) = \begin{cases} \beta(j), \ if\ 1 \le j \le 6 \\ \delta(11-j)+6, \ if\ 6 < j \le 10 \\ \gamma(12-j)+10, \ if\ j = 11 \end{cases}.$$

Hence, $\omega_{\beta,\delta,\gamma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 1 & 4 & 5 & 6 & 3 & 9 & 10 & 7 & 8 & 11 \end{pmatrix}$ is (TMP) in S_{11} for β, δ and γ.

Definition 3.24: Assume that $\omega_{\beta,\delta,\gamma}$ is (TMP) in S_n for β, δ and γ. For some $T = \lambda_v^{\omega_{\beta,\delta,\gamma}} \in X = \{\lambda_i^{\omega_{\beta,\delta,\gamma}} = \{\lambda_1^i, \lambda_2^i, \ldots, \lambda_{\alpha_i}^i\} 1 \le i \le c(\omega_{\beta,\delta,\gamma})\}$ and the binary operation Δ on X, we say $(X, \Delta, \lambda_v^{\omega_{\beta,\delta,\gamma}})$ is a triple merge permutation $BCK-$algebra $(TMP-BCK-A)$ for β, δ and γ if ;

$(TMPBCK-1)$ $((\lambda_t^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_j^{\omega_{\beta,\delta,\gamma}}) \Delta ((\lambda_t^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_m^{\omega_{\beta,\delta,\gamma}})) \Delta (\lambda_m^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_j^{\omega_{\beta,\delta,\gamma}} \lambda_j^\beta) = T$,
$(TMPBCK-2)$ $(\lambda_i^\beta \Delta (\lambda_t^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_j^{\omega_{\beta,\delta,\gamma}})) \Delta \lambda_j^{\omega_{\beta,\delta,\gamma}} = T$,
$(TMPBCK-3)$ $\lambda_t^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_t^{\omega_{\beta,\delta,\gamma}} = T$,
$(TMPBCK-4)$ $T \Delta \lambda_i^{\omega_{\beta,\delta,\gamma}} = T$,
$(TMPBCK-5)$ If $\lambda_t^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_j^{\omega_{\beta,\delta,\gamma}} = T$ and $\lambda_j^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_t^{\omega_{\beta,\delta,\gamma}} = T$, then $\lambda_t^{\omega_{\beta,\delta,\gamma}} = \lambda_j^{\omega_{\beta,\delta,\gamma}}$, $\forall \lambda_t^{\omega_{\beta,\delta,\gamma}}, \lambda_j^{\omega_{\beta,\delta,\gamma}}, \lambda_m^{\omega_{\beta,\delta,\gamma}} \in X$

Example 3.35: *Let β, δ, and γ be three permutations in Example 3.22. Hence*

$$\omega_{\beta,\delta,\gamma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 3 & 5 & 1 & 9 & 6 & 2 & 11 & 10 & 4 & 8 & 7 & 19 & 17 & 14 & 18 & 13 & 12 & 20 & 15 & 16 & 23 & 21 & 25 & 22 & 24 \end{pmatrix}$$

Table 4. $(X, \Delta, \lambda_1^{\omega_{\beta,\delta,\gamma}})$ is a $(TMP - BCK - A)$.

Δ	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_3^{\omega_{\beta,\delta,\gamma}}$	$\lambda_4^{\omega_{\beta,\delta,\gamma}}$	$\lambda_5^{\omega_{\beta,\delta,\gamma}}$	$\lambda_6^{\omega_{\beta,\delta,\gamma}}$	$\lambda_7^{\omega_{\beta,\delta,\gamma}}$	$\lambda_8^{\omega_{\beta,\delta,\gamma}}$
$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_3^{\omega_{\beta,\delta,\gamma}}$	$\lambda_3^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_4^{\omega_{\beta,\delta,\gamma}}$	$\lambda_4^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_5^{\omega_{\beta,\delta,\gamma}}$	$\lambda_5^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_6^{\omega_{\beta,\delta,\gamma}}$	$\lambda_6^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_7^{\omega_{\beta,\delta,\gamma}}$	$\lambda_7^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$
$\lambda_8^{\omega_{\beta,\delta,\gamma}}$	$\lambda_8^{\omega_{\beta,\delta,\gamma}}$	$\lambda_7^{\omega_{\beta,\delta,\gamma}}$	$\lambda_6^{\omega_{\beta,\delta,\gamma}}$	$\lambda_5^{\omega_{\beta,\delta,\gamma}}$	$\lambda_4^{\omega_{\beta,\delta,\gamma}}$	$\lambda_3^{\omega_{\beta,\delta,\gamma}}$	$\lambda_2^{\omega_{\beta,\delta,\gamma}}$	$\lambda_1^{\omega_{\beta,\delta,\gamma}}$

is (TMP) in S_{25} for β, δ and γ. Also, $\omega_{\beta,\delta,\gamma} = (1\ 3)(2\ 5\ 6)(7\ 11)(8\ 10)(4\ 9)(12\ 19\ 15\ 18\ 20\ 16\ 13\ 17)(14)(21\ 23\ 25\ 24\ 22)$. Therefore, we have $X = \{\lambda_i^{\omega_{\beta,\delta,\gamma}}\}_{i=1}^{8} = \{\lambda_1^{\omega_{\beta,\delta,\gamma}} = \{1,3\}, \lambda_2^{\omega_{\beta,\delta,\gamma}} = \{2,5,6\}, \lambda_3^{\omega_{\beta,\delta,\gamma}} = \{7,11\}, \lambda_4^{\omega_{\beta,\delta,\gamma}} = \{8,10\}, \lambda_5^{\omega_{\beta,\delta,\gamma}} = \{4,9\}, \lambda_6^{\omega_{\beta,\delta,\gamma}} = \{12,19,15,18,20,16,13,17\}, \lambda_7^{\omega_{\beta,\delta,\gamma}} = \{14\}, \lambda_8^{\omega_{\beta,\delta,\gamma}} = \{21,23,25,24,22\}\}$ and $T = \lambda_1^{\omega_{\beta,\delta,\gamma}}$. Define $\Delta: X \times X \to X$ by Table 4.

Hence, $(X, \Delta, \lambda_1^{\omega_{\beta,\delta,\gamma}})\pm$ is a $(TMP - BCK - A)$.

Definition 3.36: Assume that $(X, \Delta, \lambda_k^{\omega_{\beta,\delta,\gamma}})$ is a $(TMP - BCK - A)$. We say that X is triple merge permutation bounded (TMPB) with unit $\lambda_r^{\omega_{\beta,\delta,\gamma}}$, if $\lambda_r^{\omega_{\beta,\delta,\gamma}} \in X$ satisfies $\lambda_j^{\omega_{\beta,\delta,\gamma}} \Delta \lambda_r^{\omega_{\beta,\delta,\gamma}} = \lambda_k^{\omega_{\beta,\delta,\gamma}}, \forall\, \lambda_j^{\omega_{\beta,\delta,\gamma}} \in X$.

Example 3.37: *From Example 3.35, we obtain that $(X, \Delta, \lambda_1^{\omega_{\beta,\delta,\gamma}})$ is a $(TMP - BCK - A)$ and (TMPB) with unit $\lambda_8^{\omega_{\beta,\delta,\gamma}}$.*

Proposition 3.38: Suppose that $(X, \Delta, \lambda_t^{\omega_{\beta,\delta,\gamma}})$ is a $(TMP - BCK - A)$. If X is (TMPB) with unit $\lambda_r^{\omega_{\beta,\delta,\gamma}}$, then $\lambda_r^{\omega_{\beta,\delta,\gamma}} \neq \lambda_t^{\omega_{\beta,\delta,\gamma}}$.

Proof. Let $(X, \Delta, \lambda_t^{\omega_{\beta,\delta,\gamma}})$ be $(TMP - BCK - A)$ and X be (TMPB) with unit $\lambda_r^{\omega_{\beta,\delta,\gamma}}$. Then $\lambda_j^{\omega_{\beta,\delta,\gamma}} \Delta\ \lambda_r^{\omega_{\beta,\delta,\gamma}} = \lambda_t^{\omega_{\beta,\delta,\gamma}}, \forall\, \lambda_j^{\omega_{\beta,\delta,\gamma}} \in X$. If $\lambda_r^{\omega_{\beta,\delta,\gamma}} = \lambda_t^{\omega_{\beta,\delta,\gamma}}$, so for any $\lambda_j^{\omega_{\beta,\delta,\gamma}} \neq \lambda_t^{\omega_{\beta,\delta,\gamma}}$ in X, we obtain $\lambda_r^{\omega_{\beta,\delta,\gamma}} \Delta\ \lambda_j^{\omega_{\beta,\delta,\gamma}} = \lambda_t^{\omega_{\beta,\delta,\gamma}}$ [By $(TMPBCK - 4)$]. But $\lambda_j^{\omega_{\beta,\delta,\gamma}} \Delta\ \lambda_r^{\omega_{\beta,\delta,\gamma}} = \lambda_t^{\omega_{\beta,\delta,\gamma}}$ [Since X is (TMPB) with unit $\lambda_r^{\omega_{\beta,\delta,\gamma}}$] and hence $\lambda_j^{\omega_{\beta,\delta,\gamma}} = \lambda_r^{\omega_{\beta,\delta,\gamma}} = \lambda_t^{\omega_{\beta,\delta,\gamma}}$ [By $(TMPBCK - 5)$], but $\lambda_j^{\omega_{\beta,\delta,\gamma}} \neq \lambda_t^{\omega_{\beta,\delta,\gamma}}$ and this contradiction. Then $\lambda_r^{\omega_{\beta,\delta,\gamma}} \neq \lambda_t^{\omega_{\beta,\delta,\gamma}}$.

4. Conclusion

Several new extensions to BCK–algebras are presented in this study, and their properties are examined using non–classical sets, especially permutation sets. There is a full explanation of how to correlate the ideas presented here with the chemical structure of the cadmium atom. The atomic shells of the three chemical elements under investigation (carbon, beryllium, and hydrogen) provide the structure of the triple merge permutation. The report suggests using non–classical sets, such as intuitionistic sets and nano sets, to improve the accuracy of ideas and results rather than depending solely on permutation sets in future studies. Additionally, the study intends to present a novel method for investigating the connections among other elements' chemical structures while also investigating novel mathematical ideas. The research specifically aims to use this information to determine how many electrons are in each electron shell, which is an essential component for understanding and talking about these elements' characteristics.

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The platforms' data redistribution policies were complied with.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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