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Performance Analysis of Classical and Fuzzy Series Components System of Reliability Estimation of New X-Lindley Distribution Simulation with Comparative

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ABSTRACT

This paper deal with analysis of classical and fuzzy series components systems of reliability shortly respectively R_s and R_Fs, based on estimation of these systems for New X –Lindley (NXL) distribution using eight different techniques. The techniques used include maximum likelihood estimation (MLE), weighted least squared (WLS), Approximate least squared (ALS), precise moments Method (PMM), percentile estimation (PE), and three shrinkage (Sh1, Sh2, and Sh3) techniques. Monte Carlo simulation is applied to test the performance of these techniques in estimating classical and fuzzy series distribution systems and analyze their accuracy according to criteria as mean absolute percentage error (MAPE), mean square error (MSE), and bias. The proofs of these distribution systems are presented, and then the extent of the impact of fuzziness in series components systems data on the estimation results is evaluated. This study founded that the performance of estimation techniques varied dependent on sample size, with MLE, Sh2, and Sh3 being the most accurate according to the evaluation criteria used. The results also showed that some traditional techniques, as WLS and ALS, suffer from greater bias when using fuzzy data, highlighting the impact of fuzzy estimation on the results of statistical models.

1. Introduction

Reliability systems plays important role in analyzing a performance for engineering systems and evaluating their operational efficiency. These systems (series systems) are popular model in which the reliability of the system is determined based on the performance of its individual components. With the development of analysis techniques, it has become necessary to study of effect of fuzzy in reliability data, which has implies to emergence of concept of fuzzy reliability systems that handle uncertain data more realistically.

Many studies have been presented in this field, examples of which as follows: the probability estimation of $R = P(Y < X)$ under a two-parameters exponential distribution focuses o the application of these estimate in areas such as engineering and reliability [1], a study also presented a model for improving the design of structures based on reliability in the presence of uncertain and ambiguous variables [2], another study discusses the concept of bounded reliability and is based on Alfred Chandler's ideas on the foundations of managements, it presents three main aspects of bounded reliability and its impact on decision making [3], a study on the estimation of stress-

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strength reliability under generalized exponential distribution using shrinkage estimation techniques with compares different estimation techniques [4], the relationships between bounded rationality and bounded reliability is also studied in the context of family businesses [5], another study proposed an interval estimation of the stress –strength reliability coefficient in the inverse exponential model, it uses frequentist and Bayesian estimation approaches to compare the performance of different techniques and evaluate their accuracy in estimation [6], a new class of Lindley distribution is also presented with applications in reliability analysis, simulation is used to study performance of the model in systems reliability assessment [7], finally, the Lomax-Lindley distribution is applied to the reliability assessment of industrial system, actual data is used to test the performance of the model in risk analysis and system reliability in industrial environments [8].

This study, the deals to estimating classical and fuzzy series components systems of reliability for NXL distribution. The systems of NXL distribution are estimated using eight different techniques, and their performance is evaluated through Monte Carlo simulation to test the accuracy of the estimates and their agreement with real data. The effect of using the fuzzy model on the accuracy of the estimation compared to the classical model is also analyzed.

Series components systems rely on the individual performance of their components, but in many real applications, reliability data are inaccurate or fuzzy. This raises questions about the impact of using fuzzy models compared to classical model in reliability analysis. Furthermore, the NXL distribution is an effective model for representing failure data, but its performance when estimating its systems using different techniques has not been tested. This paper to answer the following questions:

- How does fuzziness affect the estimation of the systems for NXL distribution in series components systems?
- Which of the eight estimation techniques provides the best accuracy in parameter estimation?
- This analysis includes comparing the performance of each technique in terms of the stability of its estimates, their convergence to the true values, and effect of sample size on the estimation results. Also, this simulation can be used to examine each technique's sensitivity to variation in the data and identify instances where each method achieves optimal performance. In this way, the most efficient technique can be identified based on precise statistical criteria, contributing to the selection of the most appropriate methodology for data analysis in different settings.
- Addressing this issue will provide the deeper insights into improving reliability analysis in engineering systems, helping to develop more accurate and effective strategies.

The research aims to identify the best estimation technique for series components systems of reliability for NXL distribution, which helps in developing more accurate reliability models in engineering and statistical applications.

2. New X-Lindley (NXL) distribution

This distribution was first presented by Khodja, et al in 2023, and the CDF and PDF for it are provided by forms, respectively, for the random variable X~NXL(β) [9][10]:

$$F_{NXL}(x) = 1 - \left(1 + \frac{\beta x}{2}\right) e^{-\beta x} \qquad (1)$$

$$f_{NXL}(x) = \beta \left(1 + \frac{\beta x}{2}\right) e^{-\beta x} \qquad (2)$$

where $x, \beta \geq 0$, and β is shape parameter for NXL distribution.

3. The classical and fuzzy series components systems of reliability

In reliability systems, a series system refers to a system consisting of several components arranged in a sequence such that the failure of any individual component causes the failure of the entire system. This type of system in common in engineering and industrial applications, where operations depend on a specific sequence of functions to ensure proper operation.

In a system containing n components, the system is in working condition only if all components are working. If any component fails, the entire system stops working. Mathematically, the reliability of a series system R_s and fuzzy series system R_Fs in calculated as follows respectively:

$$R_s = R_1 \times R_2 \times \dots \times R_n \quad (3)$$

$$R_{Fs} = R_{F1} \times R_{F2} \times \dots \times R_{Fn} \quad (4)$$

where R_i, R_{Fi} are the probability and fuzzy probability of component i.

Since reliability decreases when values are multiplied together, series systems are more susceptible to failure than other systems such as parallel systems. Because of their failure-sensitive nature, several strategies are used to improve the reliability of series systems, including:

- Redundancy: adding redundant components reduces the impact of the single component failure.
- Preventive maintenance: reducing the probability of failure through regular maintenance.
- Failure analysis: studying potential failure patterns to take corrective measures.

Understanding series systems is essential in various fields, as it helps improve the performance of critical systems and increase their continuity

Theorem.1: let $X \sim NXL(\beta)$ and $Y \sim NXL(\beta)$ are independent and define The series components systems of reliability by $R_s = P(X < \min(y_1, y_2, \dots, y_n))$, then:

$$R_s = P(X < \min(y_1, y_2, \dots, y_n))$$

$$R_s = \frac{\beta_1}{2}\left[\frac{2}{\beta_1} - \frac{1}{\beta_1+\beta_2} - \frac{\beta_1}{(\beta_1+\beta_2)^2} - \frac{\beta_2}{2(\beta_1+\beta_2)^2} - \frac{\beta_1\beta_2}{2(\beta_1+\beta_2)^3}\right]$$

Proof. From definition of series components systems of reliability the equation of R_s has a form:

$$R_s = P(X < \min(y_1, y_2, \dots, y_n))$$

$$R_s = P(X < Z)$$

$$R_s = P(X < Z) = \int_0^\infty \int_0^z F(x, z)\,dx\,dz$$

Where $Z = \min(y_1, y_2, \dots, y_n)$

$$R_s = P(X < Z) = \int_0^\infty \int_0^z F(x)F(z)\,dx\,dz \quad (5)$$

$$R_s = P(X < Z) = \int_0^\infty F_x(z)F(z)\,dz$$

$$R_s = \int_0^\infty \left(1 - \left(1 + \frac{\beta_2 z}{2}\right)e^{-\beta_2 z}\right)\left(1 - \left(1 + \frac{\beta_1 z}{2}\right)e^{-\beta_1 z}\right)dz$$

$$R_s = \frac{\beta_1}{2}\int_0^\infty \left(e^{-\beta_1 z} + \beta_1 z e^{-\beta_2 z} - e^{-(\beta_1+\beta_2)z} - \beta_1 z e^{-(\beta_1+\beta_2)z} - \frac{\beta_2}{2} z e^{-(\beta_1+\beta_2)z} - \frac{\beta_1\beta_2}{2} z^2 e^{-(\beta_1+\beta_2)z}\right)dz$$

$$R_s = \frac{\beta_1}{2}\left[\int_0^\infty e^{-\beta_1 z}dz + \beta_1\int_0^\infty z e^{-\beta_2 z}dz - \int_0^\infty e^{-(\beta_1+\beta_2)z}dz - \beta_1\int_0^\infty z e^{-(\beta_1+\beta_2)z}dz - \frac{\beta_2}{2}\int_0^\infty z e^{-(\beta_1+\beta_2)z}dz - \frac{\beta_1\beta_2}{2}\int_0^\infty z^2 e^{-(\beta_1+\beta_2)z}dz\right]$$

$$R_s = \frac{\beta_1}{2}[I_1 + I_2 - I_3 - I_4 - I_5 - I_6]$$

Where

$$I_1 = \int_0^\infty e^{-\beta_1 z}dz = \frac{1}{\beta_1}$$

$$I_2 = \beta_1 \int_0^\infty z e^{-\beta_2 z} dz = \frac{1}{\beta_1}$$

$$I_3 = \int_0^\infty e^{-(\beta_1+\beta_2)z} dz = \frac{1}{\beta_1+\beta_2}$$

$$I_4 = \beta_1 \int_0^\infty z e^{-(\beta_1+\beta_2)z} dz = \frac{\beta_1}{(\beta_1+\beta_2)^2}$$

$$I_5 = \frac{\beta_2}{2} \int_0^\infty z e^{-(\beta_1+\beta_2)z} dz = \frac{\beta_2}{2(\beta_1+\beta_2)^2}$$

$$I_6 = \frac{\beta_1\beta_2}{2} \int_0^\infty z^2 e^{-(\beta_1+\beta_2)z} dz = \frac{\beta_1\beta_2}{2(\beta_1+\beta_2)^3}$$

Hence

$$R_s = \frac{\beta_1}{2}\left[\frac{2}{\beta_1} - \frac{1}{\beta_1+\beta_2} - \frac{\beta_1}{(\beta_1+\beta_2)^2} - \frac{\beta_2}{2(\beta_1+\beta_2)^2} - \frac{\beta_1\beta_2}{2(\beta_1+\beta_2)^3}\right] \quad (6)$$

To discuss R_{Fs} for $X \sim NXL(\beta)$ and $Y \sim NXL$ are independent, which defined the function as following:

$$R_{Fs} = P(X < \min(y_1, y_2, \dots, y_n))$$

$$R_{Fs} = P(X < Z)$$

Where $Z = \min(y_1, y_2, \dots, y_n)$

$$R_{Fs} = P(X < Z) = \int_0^\infty \int_0^z \mu_{(A)} F(x) F(z) dx dz \quad (7)$$

$\mu_A(x)$ is the membership degree of element for fuzzy set A, which defined as $\mu_x = (1 - e^{k(z-x)})$, $k \in \mathbb{R}^+$, and as $Z > X$ [12].

Theorem.2: let $X \sim NXL(\beta)$ and $Y \sim NXL(\beta)$ are independent and define fuzzy series components systems of reliability by $R_{Fs} = P(X < \min(y_1, y_2, \dots, y_n))$, then:

$$R_{Fs} = \frac{\beta_1\beta_2}{4}\left[\frac{4}{\beta_1\beta_2} + \frac{1}{(\beta_1+\beta_2)^2} + \frac{\beta_1}{(\beta_1+\beta_2)^3} - \frac{1}{(\beta_2+k)(\beta_1+\beta_2)} - \frac{1}{(\beta_2+k)(\beta_1-k)} - \frac{\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_2}{(\beta_1+\beta_2)(\beta_2+k)^2} - \frac{\beta_2}{(\beta_2+k)^2(\beta_1-k)} + \frac{\beta_1}{(\beta_2+k)(\beta_1+\beta_2)^2} - \frac{\beta_1}{(\beta_1-k)^2(\beta_2+k)} - \frac{\beta_1\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_1\beta_2}{(\beta_2+k)^2(\beta_1+\beta_2)^2} - \frac{\beta_1\beta_2}{(\beta_1-k)^2(\beta_2+k)^2}\right]$$

Proof. From definition of fuzzy series components systems of reliability the equation of R_{Fs} has a form:

$$R_{Fs} = P(X < Z) = \int_0^\infty \int_0^z \mu_{(A)} F(x) F(z) dx dz$$

$$R_{Fs} = P(X < Z) = \int_0^\infty \int_0^z \mu_{(A)} F(x) F(z) dx dz$$

$$R_{Fs} = \int_0^\infty \int_0^z (1 - e^{k(z-x)}) \left(1 - \left(1 + \frac{\beta_2 x}{2}\right) e^{-\beta_2 x}\right) \left(1 - \left(1 + \frac{\beta_1 z}{2}\right) e^{-\beta_1 z}\right) dx dz$$

$$R_{Fs} = \frac{\beta_1\beta_2}{4}\left[\int_0^\infty \int_0^z e^{-(\beta_1 z+\beta_2 x)} dx dz + \int_0^\infty \int_0^z \beta_2 x e^{-(\beta_1 z+\beta_2 x)} dx dz + \int_0^\infty \int_0^z \beta_1 z e^{-(\beta_1 z+\beta_2 x)} dx dz + \int_0^\infty \int_0^z (\beta_1+\beta_2) x z e^{-(\beta_1 z+\beta_2 x)} dx dz - \int_0^\infty \int_0^z e^{-(\beta_1-k)z-(\beta_2-k)x} dx dz - \int_0^\infty \int_0^z \beta_2 x e^{-(\beta_1-k)z-(\beta_2-k)x} dx dz - \int_0^\infty \int_0^z \beta_1 z e^{-(\beta_1-k)z-(\beta_2-k)x} dx dz - \int_0^\infty \int_0^z \beta_2\beta_1 z x e^{-(\beta_1-k)z-(\beta_2-k)x} dx dz\right]$$

$$I_1 = \int_0^\infty \int_0^z e^{-(\beta_1 z+\beta_2 x)} dx dz = \int_0^\infty e^{-\beta_1 z} \left(\int_0^z e^{-\beta_2 x} dx\right) dz$$

$$I_1 = \int_0^\infty \frac{e^{-\beta_1 z}}{\beta_2} (e^{-\beta_2 z} - 1) dz = \frac{1}{\beta_2} \int_0^\infty \left(e^{-(\beta_1+\beta_2)z} - e^{-\beta_1 z}\right) dz$$

$$I_1 = \int_0^\infty \frac{e^{-\beta_1 z}}{\beta_2}(e^{-\beta_2 z} - 1)dz = \frac{1}{\beta_2}\left[\frac{1}{\beta_1+\beta_2} + \frac{1}{\beta_1}\right] = \frac{1}{\beta_2(\beta_1+\beta_2)} + \frac{1}{\beta_1\beta_2}$$

By same way for all integrals

$$I_2 = \int_0^\infty \int_0^z \beta_2 x e^{-(\beta_1 z + \beta_2 x)} dxdz = \frac{1}{(\beta_1+\beta_2)^2} - \frac{1}{\beta_2(\beta_1+\beta_2)} + \frac{1}{\beta_1\beta_2}$$

$$I_3 = \int_0^\infty \int_0^z \beta_1 z e^{-(\beta_1 z + \beta_2 x)} dxdz = \frac{\beta_1}{\beta_2(\beta_1+\beta_2)^2} + \frac{1}{\beta_1\beta_2}$$

$$I_4 = \int_0^\infty \int_0^z (\beta_1+\beta_2) xz e^{-(\beta_1 z + \beta_2 x)} dxdz = \frac{\beta_1}{(\beta_1+\beta_2)^3} - \frac{\beta_1}{\beta_2(\beta_1+\beta_2)^2} + \frac{1}{\beta_1\beta_2}$$

$$I_5 = \int_0^\infty \int_0^z e^{-(\beta_1 - k)z - (\beta_2 - k)x} dxdz = \frac{1}{(\beta_2+k)(\beta_1+\beta_2)} + \frac{1}{(\beta_2+k)(\beta_1-k)}$$

$$I_6 = \int_0^\infty \int_0^z \beta_2 x e^{-(\beta_1 - k)z - (\beta_2 - k)x} dxdz = \frac{\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} - \frac{\beta_2}{(\beta_1+\beta_2)(\beta_2+k)^2} + \frac{\beta_2}{(\beta_2+k)^2(\beta_1-k)}$$

$$I_7 = \int_0^\infty \int_0^z \beta_1 z e^{-(\beta_1 - k)z - (\beta_2 - k)x} dxdz = -\frac{\beta_1}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_1}{(\beta_1-k)^2(\beta_2+k)}$$

$$I_8 = \int_0^\infty \int_0^z \beta_2\beta_1 zx e^{-(\beta_1 - k)z - (\beta_2 - k)x} dxdz = \frac{\beta_1\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} - \frac{\beta_1\beta_2}{(\beta_2+k)^2(\beta_1+\beta_2)^2} + \frac{\beta_1\beta_2}{(\beta_1-k)^2(\beta_2+k)^2}$$

Finally we get:

$$R_{Fs} = \frac{\beta_1\beta_2}{4}\left[\frac{1}{\beta_2(\beta_1+\beta_2)} + \frac{1}{\beta_1\beta_2} + \frac{1}{(\beta_1+\beta_2)^2} - \frac{1}{\beta_2(\beta_1+\beta_2)} + \frac{1}{\beta_1\beta_2} + \frac{\beta_1}{\beta_2(\beta_1+\beta_2)^2} + \frac{1}{\beta_1\beta_2} + \frac{\beta_1}{(\beta_1+\beta_2)^3} - \frac{\beta_1}{\beta_2(\beta_1+\beta_2)^2} + \frac{1}{\beta_1\beta_2} - \frac{1}{(\beta_2+k)(\beta_1+\beta_2)} - \frac{1}{(\beta_2+k)(\beta_1-k)} - \frac{\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_2}{(\beta_1+\beta_2)(\beta_2+k)^2} - \frac{\beta_2}{(\beta_2+k)^2(\beta_1-k)} + \frac{\beta_1}{(\beta_2+k)(\beta_1+\beta_2)^2} - \frac{\beta_1}{(\beta_1-k)^2(\beta_2+k)} - \frac{\beta_1\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_1\beta_2}{(\beta_2+k)^2(\beta_1+\beta_2)^2} - \frac{\beta_1\beta_2}{(\beta_1-k)^2(\beta_2+k)^2}\right]$$

And simplify above equation to get:

$$R_{Fs} = \frac{\beta_1\beta_2}{4}\left[\frac{4}{\beta_1\beta_2} + \frac{1}{(\beta_1+\beta_2)^2} + \frac{\beta_1}{(\beta_1+\beta_2)^3} - \frac{1}{(\beta_2+k)(\beta_1+\beta_2)} - \frac{1}{(\beta_2+k)(\beta_1-k)} - \frac{\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_2}{(\beta_1+\beta_2)(\beta_2+k)^2} - \right. \quad (8)$$

$$\frac{\beta_2}{(\beta_2+k)^2(\beta_1-k)} + \frac{\beta_1}{(\beta_2+k)(\beta_1+\beta_2)^2} - \frac{\beta_1}{(\beta_1-k)^2(\beta_2+k)} - \frac{\beta_1\beta_2}{(\beta_2+k)(\beta_1+\beta_2)^2} + \frac{\beta_1\beta_2}{(\beta_2+k)^2(\beta_1+\beta_2)^2} - \frac{\beta_1\beta_2}{(\beta_1-k)^2(\beta_2+k)^2}\Bigg]$$

4. Estimation Techniques

Eight estimating techniques are devised in order to confirm the effectiveness of R_s and R_{Fs} estimation for NXL distribution. These are as follows:

4.1 Maximum Likelihood estimation [11],[15]

NXL distribution parameters were employed in Maximum likelihood estimate approach. From equation (2), we obtain the following form for the random sample $x_1, x_2, .., x_n$:

$$L(\emptyset) = \prod_{i=1}^{n} f\,(x_i) = \prod_{i=1}^{n} \beta\left(1 + \frac{\beta x_i}{2}\right) e^{-\beta x_i} \qquad (9)$$

Where $\emptyset$ is the parameter for distribution, by take log function for above function and derivative by parameters to get a last forms for parameters as follows:

$$\hat{\beta}_{1MLE} = \frac{n}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{1 + \beta_{1_0} x_i}} \qquad (10)$$

$$\hat{\beta}_{2MLE} = \frac{n}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{1 + \beta_{2_0} x_i}} \qquad (11)$$

4.2 The Approximate Least Squares Estimator (ALS)

NXL distribution parameters were employed in Approximate Least Squares estimate approach. From equation (1), we obtain the following form for the random sample $x_1, x_2, .., x_n$: [12]

$$Q_1(\Theta, x) = \sum_{i=1}^{n}\left(F(x_i) - \frac{i}{n+1}\right)^2 = \sum_{i=1}^{n}\left(1 - \left(1 + \frac{\beta x_i}{2}\right)e^{-\beta x_i} - \frac{i}{n+1}\right)^2 \qquad (12)$$

By same way in MLE we get:

$$\hat{\beta}_{1_{ALST}} = -\frac{1}{x_i}\ln\left[\frac{n - \sum_{i=1}^{n}\frac{i}{n+1}}{\sum_{i=1}^{n}\left(1 + \frac{\beta_{1_0} x_i}{2}\right)}\right] \qquad (13)$$

$$\hat{\beta}_{2_{ALST}} = -\frac{1}{x_i}\ln\left[\frac{n - \sum_{i=1}^{n}\frac{i}{n+1}}{\sum_{i=1}^{n}\left(1 + \frac{\beta_{2_0} x_i}{2}\right)}\right] \qquad (14)$$

4.3 The weighted least squares estimator (WLS)

NXL distribution parameters were employed in weighted least squares estimate approach. From equation (1), we obtain the following form for the random sample $x_1, x_2, .., x_n$: [12], [16]

$$W_1\,(\Theta, x) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(x_i) - \frac{i}{n+1}\right)^2 \qquad (15)$$

$$W_1\,(\Theta, x) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1 - \left(1 + \frac{\beta x_i}{2}\right)e^{-\beta x_i} - \frac{i}{n+1}\right)^2 \qquad (16)$$

By same way in MLE, and ALS we get:

$$\hat{\beta}_{1_{WLSE}} = -\frac{n}{x_i}\ln\left[\frac{\sum_{i=1}^{n}\left(\frac{(n+1)^2(n+2)}{i(n-i+1)} - \frac{(n+1)^2(\cdot}{(n-i+1)}\right.}{\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1 + \frac{\beta_{1_0}\cdot}{2}\right.}\right. \qquad (17)$$

$$\hat{\beta}_{2_{WLSE}} = -\frac{n}{x_i}\ln\left[\frac{\sum_{i=1}^{n}\left(\frac{(n+1)^2(n+2)}{i(n-i+1)} - \frac{(n+1)^2(\cdot}{(n-i+1)}\right.}{\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1 + \frac{\beta_{2_0}\cdot}{2}\right.}\right. \qquad (18)$$

4.4 The Precise Moments Technique Estimator (PMM) [13]

This technique depended on moments function for NXL distribution, in this technique we get the estimation for parameters by form:

$$cv(r) = \frac{\sqrt{var}}{\mu_1} \quad (19)$$

$$\hat{\beta}_{1PMME} = \frac{\sqrt{var}}{\mu_1} = \frac{\sqrt{\frac{7}{4\beta_{1_0}^2}}}{\frac{3}{2\beta_{1_0}}} \quad (20)$$

$$\hat{\beta}_{2PMME} = \frac{\sqrt{var}}{\mu_1} = \frac{\sqrt{\frac{7}{4\beta_{2_0}^2}}}{\frac{3}{2\beta_{2_0}}} \quad (21)$$

The Shrinkage Estimator Technique (Sh) is a technique that estimates shrinkage based on historical data by form: [14]

$$\hat{\beta}_{sh} = \Omega(\beta)\hat{\beta}_{MLE} + (1 - \Omega(\beta))\hat{\beta}_0 \quad (22)$$

4.5 The Weight Function for Shrinkage (Sh1)

To find the weight function for shrinkage, substitute $\frac{\sin n}{n}$ for $\Omega(\beta)$ in equation (22) to obtain the estimators in the following forms:

$$\hat{\beta}_{1sh} = \left|\frac{\sin n}{n}\right| \hat{\beta}_{1_{MLE}} + \left(1 - \left|\frac{\sin n}{n}\right|\right) \hat{\beta}_{1_0} \quad (23)$$

$$\hat{\beta}_{2sh} = \left|\frac{\sin m}{m}\right| \hat{\beta}_{2_{MLE}} + \left(1 - \left|\frac{\sin m}{m}\right|\right) \hat{\beta}_{2_0} \quad (24)$$

4.6 Constant Shrinkage (sh2)

To find the weight function for shrinkage, substitute for $\Omega(\beta) = 0.001$ in equation (22) to obtain the estimators in the following forms:

$$\hat{\beta}_{1sh} = 0.001\hat{\beta}_{1_{MLE}} + 0.999\hat{\beta}_{1_0} \quad (25)$$

$$\hat{\beta}_{2sh} = 0.001\hat{\beta}_{2_{MLE}} + 0.999\hat{\beta}_{2_0} \quad (26)$$

4.7 Function Shrinkage (sh3)

To find the weight function for shrinkage, substitute for $\Omega(\beta_1) = e^{-g}$, and $\Omega(\beta_2) = e^{-h}$ in equation (22) to obtain the estimators in the following forms:

$$\hat{\beta}_{1sh} = e^{-g}\hat{\beta}_{1_{MLE}} + (1 - e^{-g})\hat{\beta}_{1_0} \quad (27)$$

$$\hat{\beta}_{2sh} = e^{-h}\hat{\beta}_{2_{MLE}} + (1 - e^{-h})\hat{\beta}_{2_0} \quad (28)$$

4.8 Percentile estimator

Finding the inverse of CDF function, which is described by the following equation, yields the NXL distribution estimates:

$$q = 1 - \left(1 + \frac{\beta x}{2}\right) e^{-\beta x} \quad (29)$$

Then simplify this equation to obtain estimates for β parameter.

5. Simulation experiments of R_s and R_{Fs}

To measure the dependability of functions R_s and R_{Fs} for the estimation techniques described in section 4, A Monte Carlo simulations experiment was created. The following is an algorithm for implementing a Monte Carlo simulation to determine the most efficient estimation technique based on statistical accuracy criteria:

1. Define the statistical model distribution (NXL), then determine the actual parameters to be estimated, represented by three cases below:
   - Case 1: as $\beta_1 = 0.90$; $\beta_2 = 0.60$, i.e. $\beta_1 > \beta_2$.
   - Case 2: as $\beta_1 = 1$; $\beta_2 = 1$, i.e. $\beta_1 = \beta_2$.
   - Case 3: as $\beta_1 = 2$; $\beta_2 = 3$, i.e. $\beta_1 < \beta_2$.

   While for R_{Fs} adding $k = 1$ for above three cases.
2. List the different estimation technique.
3. Select multiple sample sizes (n=25, 50, 75, 100) to test the effect of sample size. Then, generate N iteration (e.g., 1000 samples) using a random number generator based on chosen distribution.
4. Apply the estimation technique to each sample. Parameter estimates are calculated using all the selected estimation techniques.
5. Compute the statistical accuracy criteria for each technique across all iterations, and then the values are compared between techniques to determine the best performer. The equation for accuracy criteria (MAPE, MSE [15], and Bias [16]), respectively:

$$MAPE = \frac{1}{N}\sum_{i=0}^{N}\left|\frac{\hat{\theta}_i - \theta_{true}}{\theta_{true}}\right| \times 100$$

$$MSE = \frac{1}{N}\sum_{i=0}^{N}(\hat{\theta}_i - \theta_{true})^2$$

$$Bias = \frac{1}{N}\sum_{i=0}^{N}(\hat{\theta}_i - \theta_{true})$$

6. Rank techniques based on their performance according to three statistical criteria. Then, identify the technique with the lowest MSE, MAPE,

and Bias values as a measure of performance advantage.

7. Present the results and make recommendations.

Rank

Table 1 : Estimation value of R_s= 0.612000000000000, $\beta_1 = 0.90;\ \beta_2 = 0.60$

n ,m	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Sh3}$	$\widehat{R}_{PMM}$
25,25	0.609327129584777	0.593107569482301	0.612149983379952	0.592876852419012	0.612164897515930	0.612151412975722	0.612148194923270	0.728299323947406
25,50	0.608668504688173	0.638610527088063	0.616542288231711	0.638375895664924	0.612187480584842	0.612154957375367	0.612148194923089	0.730281310892855
25,75	0.617345531447808	0.663203142502334	0.618951389655788	0.662974607597198	0.612282987170776	0.612171947491626	0.612148194923258	0.738770658490463
25,100	0.615191306556481	0.670052370123134	0.619557163425943	0.669824131332148	0.612276444509780	0.612169127334073	0.612148194923207	0.738828908211181
50,25	0.608062246808529	0.565240406666743	0.603264461750313	0.564995970815766	0.612105075166973	0.612140780739083	0.612148194923006	0.724809716083574
50,50	0.609994425919757	0.607031668619580	0.610689975762318	0.606781086838719	0.612152814883385	0.612149101146004	0.612148194923053	0.731357927851568
50,75	0.609591460645521	0.619440539774059	0.613652590605129	0.619189942230433	0.612163523644064	0.612149973946024	0.612148194922952	0.734742122223649
50,100	0.610846321680893	0.630060573024550	0.614909331884369	0.629811552037996	0.612187153945051	0.612152865980916	0.612148194922991	0.735512279193785
75,25	0.599253116733228	0.536567840573848	0.604607257837996	0.536314616966741	0.612012175505345	0.612124265479868	0.612148194922914	0.717816844888059
75,50	0.608661074453599	0.589347404783514	0.609858621181402	0.589091533711573	0.612118189050526	0.612143644686095	0.612148194922926	0.732160265970215
75,75	0.610508317230064	0.608640117962870	0.611254347906592	0.608384629198562	0.612148477359536	0.612148262398277	0.612148194922900	0.735702769097502
75,100	0.616968847332386	0.623856852945623	0.611936469781964	0.623603466339915	0.612206885245984	0.612158014419619	0.612148194923005	0.740873743942741
100,25	0.598035792205751	0.530475168484126	0.603388696010582	0.530220312086358	0.611989521741222	0.612121552024440	0.612148194922908	0.718456395880071
100,50	0.607453032931018	0.579026505081166	0.608615566059408	0.578767415890796	0.612096730670233	0.612141033120737	0.612148194922871	0.730856517964316
100,75	0.609021952909782	0.597267233207181	0.610103321373774	0.597008053089654	0.612122578980232	0.612144834248515	0.612148194922889	0.734472670866098
100,100	0.609581336104525	0.606508074106290	0.613127174201270	0.606248603835509	0.612139660788087	0.612146516548662	0.612148194922866	0.735228926892284

Table 2: Bias, MSE and MAPE of R_s= 0.612000000000000, $\beta_1 = 0.90;\ \beta_2 = 0.60$

n ,m	Criteria	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Shr1}$	$\widehat{R}_{Shr2}$	$\widehat{R}_{Shr3}$	$\widehat{R}_{PMM}$
25,25	Bias	0.002672870415223	0.018892430517699	0.000149983379952	0.019123147580988	0.000164897515930 e-03	0.151412975721660 e-03	0.148194923269457 e-03	0.116299323947406
	MSE	0.011197420826898	0.035438623800845	0.002552085262739	0.035452100753440	0.000000817302486 e-03	0.000051290882255 e-03	0.000021961735283 e-03	0.022384354581141
	MAPE	0.137571877620534	0.233221151505841	0.064215779747774	0.233258237244048	0.001091698769188 e-03	0.301432947697126 e-03	0.242148567433753 e-03	0.212853539779835
25,50	Bias	0.00333149531182	0.026610527088064	0.004542288231711	0.026375895664924	0.187480584841882 e-03	0.154957375367106 e-03	0.148194923088870 e-03	0.118281310892856
	MSE	0.007561317775900	0.021514069449764	0.001956842571374	0.021514584275233	0.000478799945198 e-03	0.000040012005510	0.000021961735229 e-03	0.020057293722086
	MAPE	0.115438626176935	0.188963703042979	0.057698779940695	0.188961361225002	0.857203346550278 e-03	0.269741726197060 e-03	0.242148567138677 e-03	0.204386076446075
25,75	Bias	0.005345531447808	0.051203142502334	0.006951389655788	0.050974607597198	0.282987170775592 e-03	0.171947491626380 e-03	0.148194923258621 e-03	0.126770658490463
	MSE	0.007180996643182	0.021378298005198	0.001772047380915	0.021368285260450	0.000558855362956 e-03	0.000046908111362 e-03	0.000021961735280 e-03	0.021351283280942
	MAPE	0.109905241354899	0.190481192601721	0.053870474930767	0.190424804969271	0.882354133465245 e-03	0.290437716793409 e-03	0.242148567416048 e-03	0.213470556867234
25,100	Bias	0.003191306556481	0.058052370123134	0.007557163425943	0.057824131332148	0.276444509780654 e-03	0.169127334073075 e-03	0.148194923206740 e-03	0.126828908211182
	MSE	0.006583400693992	0.019896593296754	0.001734546277390	0.019883751027808	0.000510927921850 e-03	0.000044421613486 e-03	0.000021961735264 e-03	0.021177931593845
	MAPE	0.106812742015830	0.181933056107695	0.053246641210468	0.181855141740757	0.836630414151849 e-03	0.282196025178468 e-03	0.242148567331275 e-03	0.213416055750256
50,25	Bias	0.003937753191471	0.046759593333257	0.008735538249687	0.047004029184234	0.105075166972990 e-03	0.105075166972990 e-03	0.148194923005877 e-03	0.112809716083574
	MSE	0.008006857953761	0.029591820259539	0.002041987297553	0.029614791448772	0.000485235957380 e-03	0.000485235957380 e-03	0.000021961735205 e-03	0.019698025481825
	MAPE	0.117199485292094	0.210946197383332	0.057145748999158	0.211023859997601	0.884907873469992 e-03	0.884907873469992 e-03	0.242148567003067 e-03	0.201184044006793
50,50	Bias	0.002005574080242	0.004968331380420	0.001310024237682	0.005218913161281	0.152814883384883 e-03	0.149101146004063 e-03	0.148194923053251 e-03	0.119357927851568
	MSE	0.005123588031718	0.015540839663986	0.001121759172740	0.015549695404997	0.000305912803863 e-03	0.000032534249180 e-03	0.000021961735219 e-03	0.018774962706447
	MAPE	0.094153839465341	0.158841959200042	0.043980694971141	0.158871706845571	0.702407460899597 e-03	0.254391816141711 e-03	0.242148567080476 e-03	0.200316205155062
50,75	Bias	0.119357927851568	0.002408539354479	0.007440539774059	0.001652590605129	0.007189942230433 e-03	0.163523644063678 e-03	0.149973946024583 e-03	0.122742122223650
	MSE	0.018774962706447	0.004344919059055	0.011852422996125	0.000986596136093	0.011855008921110 e-03	0.000246566479815 e-03	0.000030596158821 e-03	0.018694476655873
	MAPE	0.2003162051	0.0867787315	0.1422217535	0.0412413886	0.1422298931	0.6265298848	0.2489856761	0.2035847485

		55062	64115	68433	58092	08905 e-03	18273 e-03	50601 e-03	82355
50,100	Bias	0.001153678319106	0.018060573024550	0.002909331884369	0.017811552037996	0.187153945051064 e-03	0.152865980916105 e-03	0.148194922991054 e-03	0.123512279193785
	MSE	0.003995282263455	0.010816243048123	0.000854235765844	0.010813479290670	0.000245180659821 e-03	0.000031196316777 e-03	0.000021961735200 e-03	0.018421692279620
	MAPE	0.082814786083334	0.134745408452027	0.038060286228416	0.134724174858665	0.611305071174549 e-03	0.252957661640689 e-03	0.242148566978846 e-03	0.203631988040865
75,25	Bias	0.012746883266772	0.075432159426152	0.007392742162004	0.075685383033259	0.012175505345482 e-03	0.124265479868199 e-03	0.148194922913759 e-03	0.105816844888060
	MSE	0.007530568636106	0.031150742477751	0.001870917757530	0.031186884951762	0.000457764127572 e-03	0.000031992962005 e-03	0.000021961735177 e-03	0.017276477395867
	MAPE	0.111444365748820	0.211234184968977	0.054924858822069	0.211370210273946	0.828382295799277 e-03	0.250769111239924 e-03	0.242148566852547 e-03	0.186430169724104
75,50	Bias	0.003338925546401	0.022652595216486	0.002141378818598	0.022908466288426	0.118189050525982 e-03	0.143644686095487 e-03	0.148194922926496 e-03	0.120160265970215
	MSE	0.004452651052632	0.014086909642607	0.001098405344739	0.014102385594691	0.000242525760337 e-03	0.000029052762740 e-03	0.000021961735181 e-03	0.018218644910211
	MAPE	0.086326722208505	0.149672853810855	0.042188193786742	0.149751617826050	0.633191021482495 e-03	0.247124540332156 e-03	0.242148566873360 e-03	0.199489420533243
75,75	Bias	0.001491682769936	0.003359882037130	0.000745652093408	0.003615370801438	0.148477359536115 e-03	0.148262398276760 e-03	0.148194922900404 e-03	0.123702769097502
	MSE	0.003345315032227	0.010083135481854	0.000814618062986	0.010089347240923	0.000181249400414 e-03	0.000027955223239 e-03	0.000021961735173 e-03	0.017958279013014
	MAPE	0.076006450439695	0.131119837949189	0.037185308636508	0.131149596221853	0.550752947181043 e-03	0.245146356526713 e-03	0.242148566830726 e-03	0.203045824063490
75,100	Bias	0.004968847332387	0.011856852945623	-0.000063530218036	0.011603466339915	0.206885245984499 e-03	0.158014419618948 e-03	0.148194923005203 e-03	0.128873743942741
	MSE	0.003012083631214	0.008457889972044	0.000687725860942	0.008456329813756	0.000187312737032 e-03	0.000030479177706 e-03	0.000021961735205 e-03	0.019093384942940
	MAPE	0.072039163622067	0.121187462853876	0.034160293241324	0.121156247792978	0.556469781244914 e-03	0.259693926391235 e-03	0.242148567001965 e-03	0.211535598461447
100,25	Bias	-0.013964207794249	-0.081524831515874	-0.008611303989418	-0.081779687913642	-0.010478258777487 e-03	0.121552024440556 e-03	0.148194922908336 e-03	0.106456395880071
	MSE	0.006898287171872	0.028330881737920	0.001663386641682	0.028369634935454	0.000424929012514 e-03	0.000030198785375 e-03	0.000021961735176 e-03	0.016919445666963
	MAPE	0.107302479815693	0.205039263940361	0.051238929260346	0.205217150477568	0.792495588068673 e-03	0.241898530279845 e-03	0.242148566843686 e-03	0.185093252585896
100,50	Bias	-0.004546967068981	-0.032973494918834	-0.003384433940592	-0.033232584109204	0.096730670232942 e-03	0.141033120737661 e-03	0.148194922871275 e-03	0.118856517964316
	MSE	0.004168105974088	0.013650822961540	0.000904620530745	0.013670710570725	0.000220491144469 e-03	0.000027752918487 e-03	0.000021961735165 e-03	0.017676974895903
	MAPE	0.082929880031080	0.145357220203084	0.038613072665556	0.145462721760247	0.596834488247163 e-03	0.242096665460468 e-03	0.242148566783130 e-03	0.197409187148407
100,75	Bias	-0.002978047090217	-0.014732766792819	-0.001896678626226	-0.014991946910346	0.122578980232693 e-03	0.144834248515427 e-03	0.148194922888395 e-03	0.122472670866098
	MSE	0.003121440805813	0.009480474734597	0.000685722703398	0.009491340530829	0.000158536407922 e-03	0.000026452363671 e-03	0.000021961735170 e-03	0.017669938426249
	MAPE	0.072714365520256	0.123686948271936	0.034119365447074	0.123741601678239	0.519945565628396 e-03	0.240675664901145 e-03	0.242148566811103 e-03	0.201187170209109
100,100	Bias	-0.002418663895475	-0.005491925893710	0.001127174201270	-0.005751396164491	0.139660788086692 e-03	0.146516548662560 e-03	0.148194922865679 e-03	0.123228926892284
	MSE	0.002540016113945	0.007160497102091	0.000591086753641	0.007166488766239	0.000131102581995 e-03	0.000025831227626 e-03	0.000021961735163 e-03	0.017310437598066
	MAPE	0.066853208117494	0.110586597751157	0.031626431790963	0.110629077885791	0.473138448658628 e-03	0.240281423766373 e-03	0.242148566773985 e-03	0.201584140770651

Table 3 : Estimation value of R_s= 0.500000000000000, $\beta_1 = 1;\ \beta_2 = 1$

n ,m	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Sh3}$	$\widehat{R}_{PMM}$
25,25	0.501000195883730	0.493767248904310	0.502421881829601	0.493523998513614	0.500011172082000	0.500002125425173	0.500000000001537	0.501824172417037
25,50	0.502448661529254	0.540250966357505	0.508605794479585	0.540002007539123	0.500071051689433	0.500012731674524	0.500000000001400	0.502375774850223
25,75	0.506189796633342	0.558948748759857	0.508096993603655	0.558698093819801	0.500117016083494	0.500020289289970	0.500000000001442	0.505759380543410
25,100	0.510368852106812	0.578389184987768	0.506990415701624	0.578137502778960	0.500156866868509	0.500026134396897	0.500000000001521	0.509942908012291
50,25	0.496943672713217	0.455453130094049	0.494476044936132	0.455203959482883	0.499939028223375	0.499989196462304	0.500000000001228	0.497763946271758
50,50	0.502123475726253	0.505711260057989	0.497657589559418	0.505451174456369	0.500014854966551	0.500002834243155	0.500000000001280	0.501845761712227
50,75	0.499793886847182	0.512707881990952	0.503486934052207	0.512443826140299	0.500017356787058	0.500002058655600	0.500000000001228	0.500561561371971
50,100	0.500277233570134	0.524783301999600	0.503371973824334	0.524515810313314	0.500039300238972	0.500004690629312	0.500000000001190	0.500993305033133

75,25	0.496169241427656	0.447739755544429	0.490727534494371	0.447485734729538	0.499913543977957	0.499985605700706	0.500000000001204	0.496166332259096
75,50	0.499241365448169	0.481026147522702	0.498726345080306	0.480761376851286	0.499974067676806	0.499996285445804	0.500000000001182	0.500085230561843
75,75	0.499015627168448	0.497633288940350	0.500556055887831	0.497364483695129	0.499994955119754	0.499999024940980	0.500000000001179	0.499321105225692
75,100	0.503416065343578	0.511620343858871	0.500817887036075	0.511351603004074	0.500039045304091	0.500005893729360	0.500000000001210	0.504264364381309
100,25	0.496085257358277	0.434553729210461	0.490160173419121	0.434303214444280	0.499874585010447	0.499979720862132	0.500000000001134	0.496306224521645
100,50	0.498024250364125	0.473722450730134	0.496128984914029	0.473457155133575	0.499949954634336	0.499993353345281	0.500000000001162	0.496598648576336
100,75	0.499665602471562	0.490753417034624	0.498816871629959	0.490483106704887	0.499982741954018	0.499998334789561	0.500000000001134	0.499622943001187
100,100	0.499631055026221	0.499325189360869	0.500957804099883	0.499053743473149	0.499996755223727	0.499999357564909	0.500000000001141	0.500101312692805

Table 4: Bias, MSE and MAPE of R_s= 0.500000000000000, $\beta_1 = 1;\ \beta_2 = 1$

n ,m	Criteria	$\hat{R}_{MLE}$	$\hat{R}_{PE}$	$\hat{R}_{ALS}$	$\hat{R}_{WLS}$	$\hat{R}_{Shr1}$	$\hat{R}_{Shr2}$	$\hat{R}_{Shr3}$	$\hat{R}_{PMM}$
25,25	Bias	0.001000195883730	0.006232751095690	0.002421881829601	0.006476001486386	0.000011172082000	0.002125425173456 e-03	0.153723589413346 e-011	0.001824172417037
	MSE	0.011872770643485	0.034770353986803	0.002784525554137	0.034774135628817	0.000000864370381	0.000031058434726 e-03	0.000000000000543 e-011	0.012406406513802
	MAPE	0.177871025894064	0.292715129720519	0.082111266926024	0.292753557844755	0.001375626898522	0.260448790027662 e-03	0.332024541194187 e-011	0.181952888188488
25,50	Bias	0.002448661529254	0.040250966357505	0.008605794479585	0.040002007539123	0.000071051689433	0.012731674523744 e-03	0.140062828002385 e-011	0.002375774850223
	MSE	0.008613976221069	0.028261547762225	0.002252438552220	0.028249218724763	0.000000534314720	0.000019231539580 e-03	0.000000000000467 e-011	0.009293296624830
	MAPE	0.150391759152561	0.260151341738212	0.074096151765999	0.260104613875541	0.001107957822438	0.210227618910995 e-03	0.313156567344208 e-011	0.156731803920590
25,75	Bias	0.006189796633342	0.058948748759857	0.008096993603655	0.058698093819800	0.000117016083494	0.020289289970242 e-03	0.144172618288252 e-011	0.005759380543410
	MSE	0.008121909856788	0.026338297634951	0.002075320173754	0.026317747019605	0.000000602328955	0.000021807245957 e-03	0.000000000000537 e-011	0.008467547191177
	MAPE	0.143831964260950	0.251910427718508	0.071614636921543	0.251812196766862	0.001065852020243	0.202824309519240 e-03	0.315150772145500 e-011	0.147106681623422
25,100	Bias	0.010368852106812	0.078389184987768	0.006990415701624	0.078137502778960	0.000156866868509	0.026134396897035 e-03	0.152077450632504 e-011	0.009942908012291
	MSE	0.007440959421833	0.026927640109617	0.001765959558015	0.026898032848518	0.000000497729256	0.000017917427164 e-03	0.000000000000498 e-011	0.007929210041995
	MAPE	0.136650739950134	0.251652820989676	0.065493722297125	0.251492457393992	0.001016670816134	0.193577555140830 e-03	0.328271210214837 e-011	0.141245953714925
50,25	Bias	0.003056327286783	0.044546869905951	0.005523955063868	0.044796040517117	0.000060971776625	0.010803537696166 e-03	0.122837895411010 e-011	0.002236053728242
	MSE	0.008294930174923	0.027661842603636	0.002244902260242	0.027676674199366	0.000000491957732	0.000017717090748 e-03	0.000000000000260 e-011	0.008691193043596
	MAPE	0.147187840544141	0.259136084746929	0.073951814014656	0.259222654942543	0.001073541538741	0.203796963773027 e-03	0.257237120493414 e-011	0.150008690521467
50,50	Bias	0.002123475726253	0.005711260057989	0.002342410440582	0.005451174456369	0.014854966550992	0.002834243155298 e-03	0.128008748045971 e-011	0.001845761712227
	MSE	0.006056656187422	0.018143730733721	0.001422457941145	0.018140846583847	0.000330711085128	0.000012056638014 e-03	0.000000000000279 e-011	0.006224977714797
	MAPE	0.125465645758688	0.208720168465807	0.059872266220163	0.208705964354620	0.892229713113657	0.170306190888067 e-03	0.266482191868533 e-011	0.127876197401425
50,75	Bias	0.000206113152818	0.012707881990953	0.003486934052207	0.012443826140299	0.017356787058100	0.002058655599908 e-03	0.122842569449944 e-011	0.000561561371971
	MSE	0.005163298767016	0.014582574829086	0.001158218252301	0.014577020860947	0.000269175527738	0.000009919902093 e-03	0.000000000000260 e-011	0.005551187350694
	MAPE	0.114168719949837	0.190099836910651	0.054442029317100	0.190070810586017	0.799861520822204	0.153633484599165 e-03	0.255131049620161 e-011	0.119713288367719
50,100	Bias	0.000277233570134	0.024783301999600	0.003371973824334	0.024515810313314	0.039300238972289	0.004690629311575 e-03	0.119036425161312 e-011	0.000993305033133
	MSE	0.004425077201091	0.011674032121321	0.000915358743569	0.011662816620737	0.000234368705135	0.000008681101696 e-03	0.000000000000262 e-011	0.005036978443428
	MAPE	0.106324422337872	0.170076346365148	0.047620198780005	0.169999835320244	0.734368017489505	0.141831232111516 e-03	0.249143905506344 e-011	0.113999077145002
75,25	Bias	0.003830758572344	0.052260244455571	0.009272465505629	0.052514265270462	0.086456022043251 e-03	0.014394299293929 e-03	0.120464649366880 e-011	0.003833667740904
	MSE	0.007178582842432	0.023913703978419	0.001990083792735	0.023933962606938	0.000443634796216 e-03	0.000016017803424 e-03	0.000000000000223 e-011	0.008006751053911
	MAPE	0.135427791769332	0.242033846688845	0.069220245044945	0.242155247306049	0.989819464672764 e-03	0.188343253363874 e-03	0.244918907377212 e-011	0.144770483705555
75,50	Bias	0.000758634551831	0.018973852477298	0.001273654919694	0.019238623148714	0.025932323193740 e-03	0.003714554196167 e-03	0.118236842538977 e-011	0.000085230561843
	MSE	0.004874731812693	0.013968737225112	0.001134877969971	0.013976956997673	0.000244225981840 e-03	0.000008988231595 e-03	0.000000000000210 e-011	0.005097042293044
	MAPE	0.112568946910123	0.187056998925797	0.052847499520987	0.187104040981003	0.780475745568069 e-03	0.149793579393024 e-03	0.240474085089204 e-011	0.114873888118424

75,75	Bias	0.000984372831552	0.002366711059650	0.000556055887831	0.002635516304871	0.005044880245641 e-03	0.000975059019971 e-03	0.117895615492358 e-011	0.000678894774308
	MSE	0.003840272648952	0.010538405957605	0.000839375416233	0.010539603588733	0.000194345790537 e-03	0.000007294299225 e-03	0.000000000000213 e-011	0.004078515675117
	MAPE	0.096914454294395	0.162212739054209	0.045409657944628	0.162235824888403	0.668436160893855 e-03	0.129470490157889 e-03	0.239603648033437 e-011	0.101225830138157
75,100	Bias	0.003416065343578	0.011620343858871	0.000817887036075	0.011351603004074	0.039045304091085 e-03	0.005893729360064 e-03	0.121001075825689 e-011	0.004264364381309
	MSE	0.003454913515161	0.010604088782266	0.000840331099444	0.010599104828767	0.000168332543788 e-03	0.000006400767500 e-03	0.000000000000220 e-011	0.003614959035095
	MAPE	0.093582847440191	0.158989205724074	0.045378003618792	0.158950111690899	0.635339372296995 e-03	0.123942839373753 e-03	0.246923992364145 e-011	0.096616350418889
100,25	Bias	0.003914742641723	0.065446270789539	0.009839826580879	0.065696785555720	0.000125414989553 e-03	0.020279137868482 e-03	0.113417009117711 e-011	0.003693775478355
	MSE	0.008278474954434	0.026407565883218	0.001892948659740	0.026429917619827	0.000000560284133 e-03	0.000020199887446 e-03	0.000000000000178 e-011	0.008582574256340
	MAPE	0.146819428015730	0.247481231439898	0.068340631037750	0.247607948433912	0.001081207070527 e-03	0.206206129256669 e-03	0.229082930403024 e-011	0.150385624041409
100,50	Bias	0.001975749635875	0.026277549269866	0.003871015085971	0.026542844866425	0.050045365664480 e-03	0.006646654718408 e-03	0.116166398722584 e-011	0.003401351423664
	MSE	0.004541049172392	0.013746357746611	0.001082868600819	0.013757264890708	0.000241837615477 e-03	0.000008960204664 e-03	0.000000000000189 e-011	0.004884972889116
	MAPE	0.107249576106333	0.179860673554308	0.050882415219034	0.179919528403303	0.747691954360643 e-03	0.144407283191705 e-03	0.234900476847599 e-011	0.111506576482270
100,75	Bias	0.000334397528438	0.009246582965376	0.001183128370041	0.009516893295113	0.017258045981495 e-03	0.001665210438735 e-03	0.113351594777100 e-011	0.000377056998814
	MSE	0.003237583396096	0.009256741404604	0.000794976596810	0.009261071855086	0.000155484451686 e-03	0.000005926805701 e-03	0.000000000000177 e-011	0.003473205493826
	MAPE	0.090679927094881	0.150530964391730	0.044350792945036	0.150554218125002	0.611036185047656 e-03	0.119363637754404 e-03	0.228223884235490 e-011	0.093859183422012
100,100	Bias	0.000368944973779	0.000674810639131	0.000957804099883	0.000946256526851	0.003244776272700 e-03	0.000642435090761 e-03	0.114118825500498 e-011	0.000101312692805
	MSE	0.003141185853906	0.008293849820843	0.000690988565040	0.008294564475500	0.000144582403450 e-03	0.000005655250100 e-03	0.000000000000180 e-011	0.003327910403114
	MAPE	0.088928900732648	0.144723886188726	0.041906034862249	0.144745158183586	0.590994904203817 e-03	0.116867117352972 e-03	0.229764163250934 e-011	0.092221200109280

Table 5 : Estimation value of R_s= 0.388000000000000, $\beta_1 = 2.00;\ \beta_2 = 3.00$

n ,m	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Sh3}$	$\widehat{R}_{PMM}$
25,25	0.399392591667706	0.415165903469038	0.387299046726260	0.414931183616341	0.388012808891229	0.387966373148975	0.387955582474544	0.386921443641996
25,50	0.393095753527415	0.433754096786221	0.397278565194282	0.433506553663972	0.388007678622592	0.387964714048722	0.387955582474413	0.381518815413343
25,75	0.398308888942998	0.460036529245101	0.396203407000468	0.459787276997069	0.388077171464115	0.387976733142624	0.387955582474546	0.384484666252557
25,100	0.397607888380053	0.465770407730316	0.396863596613691	0.465516400505268	0.388082114186508	0.387976105021713	0.387955582474509	0.384174001335941
50,25	0.385177164378018	0.348894344182954	0.385401623103544	0.348664251582219	0.387857908971992	0.387937732864672	0.387955582474275	0.373150483226647
50,50	0.387700686417676	0.388477151587286	0.389885556747100	0.388228299791038	0.387933075018681	0.387951261479801	0.387955582474294	0.375375215373454
50,75	0.391481087993851	0.408963346293170	0.391021482972031	0.408706490077792	0.387985679792923	0.387960149464246	0.387955582474294	0.379119653742908
50,100	0.390210833237867	0.419858493988805	0.390995224389104	0.419598627916421	0.387987291459206	0.387958917930882	0.387955582474242	0.377241667882054
75,25	0.383662098014724	0.339347277314195	0.381512567159222	0.339116679697512	0.387833281152886	0.387934159907792	0.387955582474205	0.371663943668492
75,50	0.388296302110138	0.374336731318860	0.387045973898834	0.374088077572412	0.387922986079450	0.387950473849729	0.387955582474201	0.375207989002646
75,75	0.389782843195973	0.389477973836579	0.388696117384672	0.389221894691928	0.387955455737923	0.387955543944041	0.387955582474215	0.376597239303068
75,100	0.391772021796480	0.403554645661850	0.388645078006202	0.403294720202536	0.387982713588286	0.387959270768335	0.387955582474218	0.377792023592981
100,25	0.385280017216515	0.328471820594903	0.381688842298283	0.328243595367799	0.387825462224171	0.387934314235883	0.387955582474229	0.373441201953997
100,50	0.386666413933572	0.364694891343477	0.385563738345803	0.364447919655841	0.387900513515649	0.387947777833665	0.387955582474205	0.373401184801054
100,75	0.392556547417774	0.389097445099421	0.386109706381156	0.388840662598076	0.387963327860032	0.387958737589256	0.387955582474242	0.380219087008036
100,100	0.390281944647803	0.393201557110140	0.387649551228382	0.392942737074490	0.387962376887600	0.387956916188234	0.387955582474220	0.377067839919151

Table 6: Bias, MSE and MAPE of R_s= 0.388000000000000, $\beta_1 = 2.00;\ \beta_2 = 3.00$

n ,m	Criteria	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Shr1}$	$\widehat{R}_{Shr2}$	$\widehat{R}_{Shr3}$	$\widehat{R}_{PMM}$
25,25	Bias	0.011392591667706	0.027165903469038	0.000700953273740	0.026931183616341	0.000012808891229	0.033626851025191 e-03	0.044417525456202 e-03	0.001078556358004
	MSE	0.010099389537867	0.033485538144689	0.002393660988164	0.033466536353379	0.000000657500544	0.000024712193793 e-03	0.000001972916568 e-03	0.012335698696338
	MAPE	0.207330119355884	0.366569166293586	0.100389085908402	0.366456049049965	0.001565081237268	0.305790483841516 e-03	0.114478158392274 e-03	0.229740823039608
25,50	Bias	0.005095753527415	0.045754096786221	0.009278565194282	0.045506553663972	0.000007678622592	0.035285951277601 e-03	0.044417525587057 e-03	0.006481184586657
	MSE	0.007853624936824	0.028299805943024	0.002242309277636	0.028274590469809	0.000000460422300	0.000017831372037 e-03	0.000001972916579 e-03	0.009317981784290

	MAPE	0.181969991254774	0.326758602508994	0.093074489032126	0.326621672760026	0.001324747703356	0.267750605522028 e-03	0.114478158729529 e-03	0.200373268270926
25,75	Bias	0.010308888942998	0.072036529245102	0.008203407000468	0.071787276997069	0.000077171464115	0.023266857376037 e-03	0.044417525453449 e-03	0.003515333747443
	MSE	0.007057354963503	0.031453510405279	0.001748919407820	0.031423249750630	0.000000502046133	0.000018504278407 e-03	0.000001972916567 e-03	0.007944748085226
	MAPE	0.164753428801007	0.323014757427543	0.083068384243156	0.322817119201932	0.001232648384521	0.252813232254359 e-03	0.114478158385178 e-03	0.182261330517819
25,100	Bias	0.009607888380053	0.077770407730315	0.008863596613692	0.077516400505268	0.000082114186508	0.023894978286711 e-03	0.044417525490465 e-03	0.003825998664059
	MSE	0.006712577753441	0.028052278871531	0.001847892872902	0.028014278945209	0.000000423978385	0.000015723130390 e-03	0.000001972916571 e-03	0.007667943091501
	MAPE	0.165920397814053	0.326443072040155	0.084032380978133	0.326175141639004	0.001208104135297	0.244457232266805 e-03	0.114478158480580 e-03	0.182008955674365
50,25	Bias	0.002822835621982	0.039105655817046	0.002598376896456	0.039335748417781	0.000142091028008	0.062267135327678 e-03	0.044417525724572 e-03	0.014849516773353
	MSE	0.007323318175193	0.022839670391601	0.001791189428459	0.022842942960627	0.000000506780266	0.000021431035056 e-03	0.000001972916591 e-03	0.009595657172456
	MAPE	0.178411211742260	0.305810267209591	0.086639934729119	0.305861965246747	0.001352254834961	0.277646431896747 e-03	0.114478159083949 e-03	0.206110738810902
50,50	Bias	0.000299313582324	0.000477151587287	0.001885556747100	0.000228299791038	0.000066924981319 e-03	0.048738520198789 e-03	0.044417525705999 e-03	0.012624784626546
	MSE	0.005451017478640	0.015093722295169	0.001211297876919	0.015087872373167	0.000000299689686 e-03	0.000013137388435 e-03	0.000001972916590 e-03	0.006945418892142
	MAPE	0.150378568400177	0.246062965375567	0.070987266879350	0.246029463373697	0.001088367432930 e-03	0.233638141695996 e-03	0.114478159036079 e-03	0.174930333952211
50,75	Bias	0.003481087993851	0.020963346293170	0.003021482972031	0.020706490077792	0.014320207076476 e-03	0.039850535754105 e-03	0.044417525706304 e-03	0.008880346257092
	MSE	0.004244833327635	0.012817103741155	0.001017306179859	0.012802679231716	0.000214674417095 e-03	0.000009488041390 e-03	0.000001972916590 e-03	0.004944354434335
	MAPE	0.134331357350808	0.229311963493667	0.064868130192135	0.229196557514400	0.936023060671306 e-03	0.202984011141927 e-03	0.114478159036867 e-03	0.147099392671286
50,100	Bias	0.002210833237867	0.031858493988806	0.002995224389104	0.031598627916422	0.012708540794174 e-03	0.041082069117959 e-03	0.044417525757712 e-03	0.010758332117946
	MSE	0.003817001951500	0.012706347023578	0.000923394399668	0.012687519346306	0.000188661175123 e-03	0.000008719407090 e-03	0.000001972916594 e-03	0.004796276578076
	MAPE	0.126119123286486	0.222764095364358	0.062413991712737	0.222610577316338	0.866036559419864 e-03	0.192737371084021 e-03	0.114478159169360 e-03	0.142818249952929
75,25	Bias	0.004337901985276	0.048652722685805	0.006487432840778	0.048883320302488	0.000166718847114 e-03	0.065840092208002 e-03	0.044417525794607 e-03	0.016336056331508
	MSE	0.006805869978257	0.020137121070295	0.001677931804988	0.020146033871908	0.000000475642170 e-03	0.000020559247507 e-03	0.000001972916598 e-03	0.008642208945440
	MAPE	0.172497449339552	0.291274808263617	0.083589333128699	0.291358685861320	0.001298568333087 e-03	0.268765435695233 e-03	0.114478159264452 e-03	0.192680295664994
75,50	Bias	0.000296302110138	0.013663268681139	0.000954026101166	0.013911922427588	0.077013920549438 e-03	0.049526150271299 e-03	0.044417525798715 e-03	0.012792010997354
	MSE	0.004106134684099	0.012079987298896	0.001006010536351	0.012080131312358	0.000216931633804 e-03	0.000010224562430 e-03	0.000001972916598 e-03	0.005218779570787
	MAPE	0.131313079166275	0.225068358515685	0.066111982129353	0.225078283153514	0.917115578129329 e-03	0.200730085887796 e-03	0.114478159275038 e-03	0.149833878032688
75,75	Bias	0.001782843195973	0.001477973836579	0.000696117384672	0.001221894691928	0.044544262077026 e-03	0.044456055958732 e-03	0.044417525784896 e-03	0.011402760696932
	MSE	0.003468694996058	0.009105009818468	0.000740844395080	0.009100149395155	0.000169829980174 e-03	0.000008274554365 e-03	0.000001972916597 e-03	0.004391959368770
	MAPE	0.120085305690516	0.194241736705027	0.055059270928268	0.194190120017101	0.824915186498977 e-03	0.187035161510559 e-03	0.114478159239422 e-03	0.137736239985074
75,100	Bias	0.003772021796480	0.015554645661850	0.000645078006202	0.015294720202536	0.017286411714083 e-03	0.040729231665001 e-03	0.04441752578211 e-03	0.010207976407019
	MSE	0.002915437860227	0.008984036812184	0.000707629519511	0.008972736209386	0.000133022836367 e-03	0.000006731740701 e-03	0.000001972916597 e-03	0.003488795155764
	MAPE	0.111593256732321	0.191015730145837	0.054512621097849	0.190894425339609	0.739715703924065 e-03	0.168264261016830 e-03	0.114478159232257 e-03	0.121976264839639
100,25	Bias	0.002719982783485	0.059528179405097	0.006311157701717	0.059756404632201	0.000174537775829 e-03	0.065685764116535 e-03	0.044417525771147 e-03	0.014558798046003
	MSE	0.006744887027068	0.019688529227286	0.001533488457528	0.019702102789270	0.000000497248194 e-03	0.000021306582223 e-03	0.000001972916596 e-03	0.008586320481633
	MAPE	0.169271969573460	0.284358958052880	0.079318817405135	0.284488840862336	0.001277638320451 e-03	0.263382210235307 e-03	0.114478159203986 e-03	0.195757290625956
100,50	Bias	0.001333586066428	0.023305108656522	0.002436261654197	0.023552080344159	0.099486484351277 e-03	0.052222166334904 e-03	0.044417525794852 e-03	0.014598815198946
	MSE	0.004071613048046	0.011338277633434	0.000904862655477	0.011343253668399	0.000218080987469 e-03	0.000010487624905 e-03	0.000001972916598 e-03	0.005420898781970
	MAPE	0.1319205275	0.2198655557	0.0620113400	0.2199202630	0.9293037194	0.2052408397	0.1144781592	0.1538330329

		68765	53568	71414	55741	18338 e-03	79839 e-03	65083 e-03	36348
100,75	Bias	0.004556547417774	0.001097445099421	0.001890293618844	0.000840662598076	0.036672139968086 e-03	0.041262410743534 e-03	0.044417525757739 e-03	0.007780912991963
	MSE	0.002971814795992	0.008416370043016	0.000688722268294	0.008412106745090	0.000137872283513 e-03	0.000006917727798 e-03	0.000001972916594 e-03	0.003770826380326
	MAPE	0.111553094290613	0.187100215408562	0.053375525793687	0.187092067086579	0.743632409327781 e-03	0.168632768350615 e-03	0.114478159169431 e-03	0.126960102357179
100,100	Bias	0.002281944647803	0.005201557110140	0.000350448771618	0.004942737074490	0.037623112400288 e-03	0.043083811766215 e-03	0.044417525779503 e-03	0.010932160080848
	MSE	0.002562222415031	0.007603795942232	0.000602311418973	0.007597510307315	0.000118981387280 e-03	0.000006454953196 e-03	0.000001972916596 e-03	0.003036555311457
	MAPE	0.103217910400943	0.179137487108749	0.050560536299402	0.179075917271552	0.690503306235730 e-03	0.164898006607232 e-03	0.114478159225522 e-03	0.113181332163727

Table 7 : Estimation value of R_{Fs}= 0.916666666666667, $k = 1.00, \beta_1 = 1.00;\ \beta_2 = 2.0$

n ,m	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Sh3}$	$\widehat{R}_{PMM}$
25,25	0.958860930161543	0.957920800390718	0.928788758440327	0.917000800257557	0.916572432043478	0.916471927872152	0.830267481829936	0.959845619053222
25,50	0.959845619053222	0.959845619053232	0.956773426978625	0.684606707674358	0.916967545010690	0.916566424504769	0.916471927872179	0.835068227856051
25,75	0.959870252072223	0.959870253072263	0.956307936238580	0.573009079774016	0.916941053115493	0.916561930045095	0.916471927872164	0.833654580203620
25,100	0.960138207407014	0.960138207407314	0.956379332763140	0.698591770521791	0.916921628278139	0.916559039932012	0.916471927872125	0.835935732632930
50,25	0.959139544741611	0.959139544741535	0.956808074643154	0.272622395850188	0.916958207566313	0.916564590743519	0.916471927872051	0.830634565847964
50,50	0.960992056543322	0.960992056543632	0.955795150977699	0.824409209409088	0.916918100744442	0.916557278198469	0.916471927872058	0.838434542229170
50,75	0.960413549896908	0.960413549895448	0.955698297401747	0.628462214489926	0.916900023979840	0.916554421173435	0.916471927872055	0.836803466595389
50,100	0.960585414739472	0.960585414715472	0.955639407603647	0.658952881565733	0.916888437344690	0.916552981571263	0.916471927872060	0.838225107542355
75,25	0.958135158108039	0.958135158108479	0.956889881618369	0.186303165597128	0.916927167007896	0.916559212262897	0.916471927872015	0.826392451920966
75,50	0.960464706046321	0.960464706046321	0.95582376032546	0.553513383924681	0.916901319633755	0.916554621611815	0.916471927872029	0.835321566348593
75,75	0.960794919830393	0.960794919946393	0.955265397370218	0.799622064611469	0.916890945650900	0.916553243345041	0.916471927872029	0.838647664117079
75,100	0.960479866916322	0.960479866915391	0.954969007331873	0.650465133403650	0.916869956712983	0.916549988368436	0.916471927872011	0.837116278860759
100,25	0.958617032827777	0.958617032828543	0.956743296202500	0.974731821365327	0.916915403892052	0.916557627026392	0.916471927871987	0.826855696206979
100,50	0.959756309411621	0.959756309457321	0.955655954065332	0.519028894162916	0.916880240599444	0.916551277588215	0.916471927871998	0.834039086398334
100,75	0.960306102266058	0.960306102267128	0.955221335984401	0.748535035501628	0.916870514901431	0.916550017164972	0.916471927872004	0.836457896591090
100,100	0.959643103327163	0.959643103328741	0.954932185471270	0.682443406436508	0.916849931813830	0.916546793182342	0.916471927871974	0.832899407447335

Table 8: Bias, MSE and MAPE of R_{Fs}= 0.916666666666667, $k = 1.00, \beta_1 = 1.00;\ \beta_2 = 2.0$

n ,m	Criteria	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Shr1}$	$\widehat{R}_{Shr2}$	$\widehat{R}_{Shr3}$	$\widehat{R}_{PMM}$
5,25	Bias	0.042194263494876	0.042194263494876	0.041254133724052	0.000121220917737 e+02	0.334133590890166 e-03	0.094234623188942 e-03	0.194738794514686 e-03	0.086399184836731
	MSE	0.002490611641199	0.002490611641199	0.001871954150041	1.119269407442329 e+02	0.000304134427112 e-03	0.000015909317386 e-03	0.000037923198089 e-03	0.019984511275484
	MAPE	0.049872774539153	0.049872774539153	0.045004509517147	0.012575124900232 e+02	0.433166410209614 e-03	0.121115492133305 e-03	0.212442321288748 e-03	0.106705523944261
5,50	Bias	0.043178952386556	0.043178952386556	0.040106760311959	0.232059958992308 e+02	0.300878344023250 e-03	0.100242161898346 e-03	0.194738794487244 e-03	0.081598438810616
	MSE	0.002374685254036	0.002374685254036	0.001735206769820	2.096005602831891	0.000226046308489 e-03	0.000015000736416 e-03	0.000037923198078 e-03	0.015550207656131
	MAPE	0.049111576728888	0.049111576728888	0.043752829431228	0.523380442062563	0.381634906808452 e-03	0.120502590770801 e-03	0.212442321258812 e-03	0.098053274677636
5,75	Bias	0.043203585405556	0.043203585405556	0.039641269571913	0.343657586892651	0.274386448826743 e-03	0.104736621571449 e-03	0.194738794502067 e-03	0.083012086463047
	MSE	0.002303105622886	0.002303105622886	0.001675438567249	2.810064018014246	0.000184446623283 e-03	0.000014976460450 e-03	0.000037923198084 e-03	0.014327007333471
	MAPE	0.048667266811733	0.048667266811733	0.043245021351178	0.397906369595794	0.348061626279456 e-03	0.121269811545225 e-03	0.212442321274982 e-03	0.096685819497357
5,100	Bias	0.043471540740348	0.04347154074458	0.039712666096473	0.218074896144876	0.254961611472455 e-03	0.107626734654584 e-03	0.194738794541799 e-03	0.080730934033737
	MSE	0.002277912710018	0.002277912775418	0.001674718297254	1.598820662413243	0.000146300154989 e-03	0.000014581029169 e-03	0.000037923198100 e-03	0.012776645408461
	MAPE	0.048700312060509	0.048700312045609	0.043322908468880	0.330105151509644	0.320093636222226 e-03	0.120562937553074 e-03	0.212442321318326 e-03	0.092482908031249
0,25	Bias	0.042472878074869	0.04247287868769	0.040141407976487	0.011892890625169 e+02	0.291540899646274 e-03	0.102075923147708 e-03	0.194738794616013 e-03	0.086032100818703
	MSE	0.002416417743888	0.00241641776798	0.001739269513805	3.492792842742765 e+02	0.000212096495406 e-03	0.000015052339882 e-03	0.000037923198128 e-03	0.017449473784325
	MAPE	0.049395902431903	0.049395902453203	0.043790626883441	0.021264066906995 e+02	0.379722275887328 e-03	0.120045553777811 e-03	0.212442321399287 e-03	0.102972637713944
0,50	Bias	0.044325389876965	0.044325389878329	0.039128484311032	0.092257457257579	0.251434077775226 e-03	0.109388468198251 e-03	0.194738794608496 e-03	0.078232124437496
	MSE	0.002241566453500	0.002241566459636	0.001614194704386	26.288252646605748	0.000125803948829 e-03	0.000014269265685 e-03	0.000037923198126 e-03	0.011165172976786
	MAPE	0.048788933679415	0.048788933674795	0.042685619248398	0.830633686365762	0.302973837649659 e-03	0.120540577892705 e-03	0.212442321391086 e-03	0.089106112664317
0,75	Bias	0.043746883230241	0.04374688325796	0.039031630735080	0.288204452176740	0.233357313174316 e-03	0.112245493231646 e-03	0.194738794611656 e-03	0.079863200071278

	MSE	0.002189308175290	0.00218930815796	0.001590935468247	0.797449101358205	0.000109760277532 e-03	0.000014658412698 e-03	0.000037923198127 e-03	0.011238392261131
	MAPE	0.048299300210647	0.04829930025886	0.042579960801906	0.352904087655261	0.286617025601962 e-03	0.123704862672042 e-03	0.212442321394534 e-03	0.089466845629444
	Bias	0.043918748072805	0.043918748058786	0.038972740936980	0.257713785100934	0.221770678023415 e-03	0.113685095403263 e-03	0.194738794606257 e-03	0.078441559124312
,100	MSE	0.002148405169259	0.00214840514212	0.001577954397494	0.357299411112513	0.000096132360943 e-03	0.000014694271764 e-03	0.000037923198125 e-03	0.009662728569494
	MAPE	0.048039370722150	0.048039370727979	0.042515717385797	0.291954147904792	0.265229537255636 e-03	0.124571781557720 e-03	0.212442321388644 e-03	0.087389806356689
	Bias	0.041468491441372	0.04146849147588	0.007303635010695	0.007303635010695 e+02	0.260500341228994 e-03	0.107454403769565 e-03	0.194738794651684 e-03	0.090274214745701
5,25	MSE	0.002242778025203	0.002242778058642	4.348595518134172	4.348595518134172 e+02	0.000192278564964 e-03	0.000016113059580 e-03	0.000037923198142 e-03	0.017872658812020
	MAPE	0.047597696519315	0.04759769653553	0.014396857191087	0.014396857191087 e+02	0.343417051224227 e-03	0.125432623672401 e-03	0.212442321438201 e-03	0.105275666669416
	Bias	0.043986955723781	0.043986955757868	0.038697436085837	0.261389897315164	0.240818747512684 e-03	0.110866382419070 e-03	0.194738794644418 e-03	0.079745125635925
5,50	MSE	0.002192185558989	0.002192185557907	0.001572725771435	26.444133586056822	0.000119236863344 e-03	0.000014567381232 e-03	0.000037923198140 e-03	0.010903660261994
	MAPE	0.048054958842751	0.04805495884686	0.042215384820913	0.796928312745496	0.290751644029922 e-03	0.122693038689910 e-03	0.212442321430275 e-03	0.090435213578596
	Bias	0.044128253163727	0.04412825315869	0.038598730703551	0.117044602055197	0.224278984232996 e-03	0.113423321625865 e-03	0.194738794637346 e-03	0.078019002549588
5,75	MSE	0.002166448274020	0.00216644825779	0.001549020342775	15.436659434890453	0.000095562209420 e-03	0.000014578753706 e-03	0.000037923198137 e-03	0.009819384803885
	MAPE	0.048317605513790	0.04831760551896	0.042107706222056	0.578221846945401	0.268516793209791 e-03	0.124418353492315 e-03	0.212442321422559 e-03	0.086719861369352
	Bias	0.043813200249656	0.043813200247575	0.038302340665206	0.266201533263016	0.203290046316023 e-03	0.116678298230842 e-03	0.194738794655883 e-03	0.079550387805908
,100	MSE	0.002102043611650	0.002102043614681	0.001511485964015	0.377013972404046	0.000074024109021 e-03	0.000014875536099 e-03	0.000037923198144 e-03	0.009365067303862
	MAPE	0.047849425382919	0.047849425382426	0.041784371634771	0.309725745713855	0.243070140313042 e-03	0.127349968639181 e-03	0.212442321442782 e-03	0.087565768363069
	Bias	0.041950366161111	0.041950366165979	0.040076629535834	0.058065154698661	0.248737225384796 e-03	0.109039640274835 e-03	0.194738794679734 e-03	0.089810970459688
00,25	MSE	0.002178898932673	0.002178898934879	0.001707310570799	59.261266792497189	0.000172376693758 e-03	0.000015959063479 e-03	0.000037923198153 e-03	0.016052781933276
	MAPE	0.047079633593971	0.047079633595879	0.043719959493637	1.043688193840492	0.320141757681674 e-03	0.126286937106772 e-03	0.212442321468801 e-03	0.104296668729585
	Bias	0.043089642744954	0.043089642747588	0.038989287398666	0.397637772503750	0.213573932777303 e-03	0.115389078452262 e-03	0.194738794668350 e-03	0.082627580268332
00,50	MSE	0.002108911481409	0.002108911442679	0.001586547855367	12.810150466291846	0.000094824909587 e-03	0.000015158190552 e-03	0.000037923198149 e-03	0.011258782065915
	MAPE	0.047346537467693	0.047346537496421	0.042533768071272	0.689025007001058	0.264482293178022 e-03	0.127028354042787 e-03	0.212442321456382 e-03	0.091847800838972
	Bias	0.043639435599391	0.04363943558075	0.038554669317734	0.168131631165039	0.203848234764771 e-03	0.116649501694783 e-03	0.194738794662202 e-03	0.080208770075576
00,75	MSE	0.002093872563341	0.00209387254758	0.001538813918874	8.449061016817485	0.000078120274638 e-03	0.000015013490082 e-03	0.000037923198146 e-03	0.009626486993435
	MAPE	0.047716169291573	0.04771616924758	0.042059639255710	0.434686356091339	0.242955425649882 e-03	0.127416672888187 e-03	0.212442321449675 e-03	0.088696383822364
	Bias	0.042976436660496	0.04297643666690	0.038265518804603	0.234223260230159	0.183265147163140 e-03	0.119873484324736 e-03	0.194738794692781 e-03	0.083767259219331
0,100	MSE	0.002009151573816	0.00200915157806	0.001507719517375	0.343750092258382	0.000062311230982 e-03	0.000015501796324 e-03	0.000037923198158 e-03	0.009898025689160
	MAPE	0.046948067456153	0.04694806745708	0.041744202332294	0.308249395892118	0.218781032597863 e-03	0.130808712382544 e-03	0.212442321483033 e-03	0.091903095128038

Table 9 : Estimation value of R_{Fs}= 0.750000000000000, $k = 1, \beta_1 = 1;\ \beta_2 = 1$

n ,m	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Sh3}$	$\widehat{R}_{PMM}$
25,25	0.876130045576672	0.876130045575321	0.869841493953149	0.793908739603168	0.750759077914343	0.749535474219693	0.749248560248825	0.292345410864888
25,50	0.875825784668818	0.875825784456118	0.868268357273841	0.613388384998694	0.750638383346315	0.749513404979440	0.749248560248887	0.296142506054012
25,75	0.875293389082993	0.872893389082993	0.865777316225495	0.590526252605898	0.750591952457992	0.749506166540036	0.749248560248839	0.293870908256326
25,100	0.876158449354111	0.876158449355853	0.864487683519989	0.595453626049238	0.750539698186515	0.749498360324041	0.749248560248767	0.294504577196231
50,25	0.876525458549468	0.876525458547538	0.866032238761985	0.470784823756056	0.750619934605099	0.749509850331711	0.749248560248552	0.303397382048772
50,50	0.877327748556136	0.877327748558431	0.864702315893071	0.592762511906833	0.750521413871181	0.749492038299761	0.749248560248548	0.306999410322922
50,75	0.876954209810609	0.876954209819742	0.861455734110751	0.574656141595571	0.750463849057448	0.749482655205876	0.749248560248497	0.295381127232299
50,100	0.874775988160989	0.874775988142619	0.862731026541263	0.568834585650954	0.750405477547994	0.749473567542910	0.749248560248472	0.290608508058459
75,25	0.873879413772196	0.873879413772316	0.864739021562005	0.322181086183430	0.750540610701285	0.749496275585716	0.749248560248414	0.283652708130970
75,50	0.875472073473324	0.875472073679924	0.863548356883993	0.684442530257923	0.750439132285764	0.749477858762873	0.749248560248413	0.297197913706709
75,75	0.875110591300740	0.875110591382424	0.862176470164835	0.574732079250101	0.750394251081301	0.749470834558033	0.749248560248376	0.295038970748665
75,100	0.877804050164925	0.877804050153126	0.861952894366709	0.592675581335380	0.750386489300714	0.749471540398208	0.749248560248442	0.305782061629255
100,25	0.875539960814589	0.875539960869421	0.865515537376310	0.530696254019707	0.750514398017786	0.749493297191925	0.749248560248346	0.288300292881454
100,50	0.879870456232397	0.879870456284912	0.860102542254484	0.647759075496760	0.750481557275057	0.749488260577141	0.749248560248410	0.316848407696908
100,75	0.877309055258957	0.877309055246819	0.861520593390155	0.571631176893329	0.750380628065938	0.749470441490013	0.749248560248393	0.301742441811295
100,100	0.878002865459351	0.878002865480421	0.860574133644320	0.586279456330039	0.750367009859826	0.749470061028117	0.749248560248379	0.307670942755149

Table 10: Bias, MSE and MAPE of R_{Fs}= 0.750000000000000, $k = 1, \beta_1 = 1;\ \beta_2 = 1$

n ,m	Criteria	$\hat{R}_{MLE}$	$\hat{R}_{PE}$	$\hat{R}_{ALS}$	$\hat{R}_{WLS}$	$\hat{R}_{Shr1}$	$\hat{R}_{Shr2}$	$\hat{R}_{Shr3}$	$\hat{R}_{PMM}$
25,25	Bias	0.126130045576672	0.126130045578596	0.119841493953148	0.043908739603168	0.000759077914343 e-03	0.464525780306977 e-03	0.000751439751175 e-03	0.457654589135112
	MSE	0.021348156320768	0.021348156326907	0.015730946751180	73.761427557998715	0.000002023444484 e-03	0.000268519096255 e-03	0.000000564661700 e-03	0.389210045345073
	MAPE	0.178694184211994	0.178694184214800	0.159788658604198	0.966586585750143	0.001352221993478 e-03	0.642381314742553 e-03	0.001001919668234 e-03	0.623985827703249
25,50	Bias	0.125825784668817	0.12582578466858	0.118268357273842	0.136611615001306	0.000638383346315	0.486595020559811 e-03	0.000751439751113 e-03	0.453857493945987
	MSE	0.020055666615673	0.02005566663268	0.015014826793436	0.742776118239413	0.000001463791962	0.000275397772117 e-03	0.000000564661700 e-03	0.335904057630161
	MAPE	0.174297522839046	0.17429752283379	0.157691143031789	0.292617287102858	0.001156252827559	0.660069405677130 e-03	0.001001919668151 e-03	0.610277264755347
25,75	Bias	0.125293389082993	0.125293389084789	0.115777316225495	0.159473747394103	0.000591952457992 e-03	0.493833459964124 e-03	0.000751439751162 e-03	0.456129091743674
	MSE	0.019542932962574	0.019542932963785	0.014317754314207	0.040886805407466	0.000001302703428 e-03	0.000278754182453 e-03	0.000000564661700 e-03	0.325796379673251
	MAPE	0.172328473969520	0.172328473964869	0.154369754967327	0.216536833114733	0.001077306493539 e-03	0.666430418290545 e-03	0.001001919668215 e-03	0.613256197792760
25,100	Bias	0.126158449354110	0.126158449347899	0.114487683519989	0.154546373950761	0.539698186514754 e-03	0.501639675958464 e-03	0.000751439751233 e-03	0.455495422803769
	MSE	0.019133124773729	0.019133124773684	0.013971848892316	0.041003745951876	0.001070509933564 e-03	0.000280337154986 e-03	0.000000564661700 e-03	0.308003508464730
	MAPE	0.171220291896718	0.171220291893795	0.152650244693318	0.210162293187881	0.989816230873965 e-03	0.674666526713475 e-03	0.001001919668311 e-03	0.611478117806184
50,25	Bias	0.126525458549468	0.126525458546907	0.116032238761985	0.279215176243944	0.000619934605100	0.490149668289021 e-03	0.000751439751448 e-03	0.446602617951228
	MSE	0.019886911259432	0.019886911255087	0.014613162142936	4.228995732653211	0.000001278227488	0.000272825872766 e-03	0.000000564661700 e-03	0.310413176461597
	MAPE	0.173437146683115	0.173437146687632	0.154709651682646	0.531654847041114	0.001119600104004	0.659278738168114 e-03	0.001001919668597 e-03	0.600109085764050
50,50	Bias	0.127327748556136	0.127327748555890	0.114702315893071	0.157237488093167	0.521413871181247 e-03	0.507961700239534 e-03	0.000751439751452 e-03	0.443000589677078
	MSE	0.019105108268124	0.01910510826379	0.013855405604612	1.404286116351568	0.000835305769930 e-03	0.000278752581147 e-03	0.000000564661700 e-03	0.282065782700487
	MAPE	0.172739010262538	0.172739010264790	0.152936421190762	0.319386305857573	0.934173919066009 e-03	0.677978298062058 e-03	0.001001919668603 e-03	0.592189664028150
50,75	Bias	0.126954209810609	0.126954209816900	0.111455734110751	0.175343858404428	0.463849057447420 e-03	0.517344794123430 e-03	0.000751439751503 e-03	0.454618872767701
	MSE	0.018419974776439	0.01841997477680	0.013065962532169	0.054945793184944	0.000657225400681 e-03	0.000284089168468 e-03	0.000000564661700 e-03	0.275235774980003
	MAPE	0.170733943133452	0.17073394313575	0.148607645481002	0.239728873288454	0.821723758451429 e-03	0.689918440651368 e-03	0.001001919668671 e-03	0.606457620175297
50,100	Bias	0.124775988160989	0.124775988166806	0.112731026541263	0.181165414349046	0.405477547994388 e-03	0.526432457090354 e-03	0.000751439751528 e-03	0.459391491941541
	MSE	0.017745228937693	0.017745228934685	0.013278550377682	0.526971321994357	0.000592018949780 e-03	0.000293244973618 e-03	0.000000564661700 e-03	0.274691341627938
	MAPE	0.167553994406717	0.167553994404685	0.150308035388351	0.242276862083871 e+02	0.758871722937896 e-03	0.704048920835014 e-03	0.001001919668704 e-03	0.613241190103721
75,25	Bias	0.123879413772195	0.123879413772195	0.114739021562005	0.010721810861834 e+02	0.000540610701285	0.503724414284439 e-03	0.000751439751586 e-03	0.466347291869030
	MSE	0.019024365542241	0.019024365542241	0.014108944473918	8.798551419323029 e+02	0.000001087213727	0.000282828119770 e-03	0.000000564661700 e-03	0.325251269194194
	MAPE	0.169666942362843	0.169666942362843	0.152985362082674	0.020487025619659 e+02	0.001025889326195	0.676189094418703 e-03	0.001001919668782 e-03	0.624728152829981
75,50	Bias	0.125472073473324	0.125472073475990	0.113548356883993	0.065557469742077	0.439132285764472 e-03	0.522141237127160 e-03	0.000751439751587 e-03	0.452802086293291
	MSE	0.018020462008001	0.018020462003684	0.013501053312622	19.654112341093306	0.000631728274228 e-03	0.000288955339646 e-03	0.000000564661700 e-03	0.273280833792699
	MAPE	0.168522040029038	0.168522040037864	0.151397809178658	0.468499311599807	0.798072174548618 e-03	0.696188316169546 e-03	0.001001919668783 e-03	0.604473286929758
75,75	Bias	0.125110591300740	0.125110591309779	0.112176470164835	0.175267920749899	0.394251081300824 e-03	0.529165441966474 e-03	0.000751439751624 e-03	0.454961029251335
	MSE	0.017674942985082	0.01767494295869	0.013093147124500	0.051609332845689	0.000496194208198 e-03	0.000292901469889 e-03	0.000000564661700 e-03	0.264912799952036
	MAPE	0.167738555083045	0.16773855504796	0.149568626886447	0.233918750734782	0.731925074040850 e-03	0.705553922621966 e-03	0.001001919668832 e-03	0.606695719251302
75,100	Bias	0.127804050164925	0.127804050158997	0.111952894366710	0.157324418664621	0.386489300713596 e-03	0.528459601791570 e-03	0.000751439751558 e-03	0.444217938370745
	MSE	0.017787545245566	0.017787545258709	0.012971920968466	0.030636569468076	0.000420452412712 e-03	0.000289686681005 e-03	0.000000564661700 e-03	0.240695224891098
	MAPE	0.170770239098516	0.17077023904879	0.149270525822279	0.209787318397466	0.658487302021324 e-03	0.704612802388759 e-03	0.001001919668745 e-03	0.592290584494326
100,25	Bias	0.125539960814589	0.125539960814790	0.115515537376309	0.219303745980293	0.514398017786175 e-03	0.506702808074912 e-03	0.000751439751654 e-03	0.461699707118546
	MSE	0.018859413889982	0.018859413883787	0.014219674083134	1.668957184168698	0.000952029904698 e-03	0.000281957770465 e-03	0.000000564661700 e-03	0.305875213715596
	MAPE	0.170334614451997	0.170334614453705	0.154020716501746	0.491195758244416	0.948583081398640 e-03	0.678399039633870 e-03	0.001001919668872 e-03	0.617110490196119

100,50	Bias	0.129870456232397	0.12987045623479	0.110102542254484	0.102240924503240	0.481557275057153 e-03	0.511739422858807 e-03	0.000751439751590 e-03	0.433151592303092
	MSE	0.018935892251154	0.018935892253790	0.012704349793370	11.578445597128781	0.000686989367463 e-03	0.000278968490196 e-03	0.000000564661700 e-03	0.249524926539628
	MAPE	0.173627972233886	0.173627972237686	0.146803389672645	0.433800934165399	0.835116777825743 e-03	0.683146773397016 e-03	0.001001919668786 e-03	0.578561497563300
100,75	Bias	0.127309055258958	0.12730905524868	0.111520593390155	0.178368823106671	0.380628065938521 e-03	0.529558509986912 e-03	0.000751439751607 e-03	0.448257558188705
	MSE	0.017648672026132	0.01764867202376	0.012848220902241	0.102060224254141	0.000423596531023 e-03	0.000291166331854 e-03	0.000000564661700 e-03	0.242844632835857
	MAPE	0.170060357370189	0.17006035737376	0.148694124520206	0.241399256387582	0.653823600503285 e-03	0.706078013315883 e-03	0.001001919668810 e-03	0.597802944054258
100,100	Bias	0.128002865459351	0.128002865456987	0.110574133644320	0.163720543669961	0.367009859825857 e-03	0.529938971882774 e-03	0.000751439751621 e-03	0.442329057244851
	MSE	0.017758836043645	0.017758836047901	0.012634346886196	0.033627101552871	0.000364754553214 e-03	0.000289897795988 e-03	0.000000564661700 e-03	0.235508951748752
	MAPE	0.170898024063716	0.170898024048489	0.147432178192427	0.218294058226615	0.630075562946308 e-03	0.706585295843698 e-03	0.001001919668828 e-03	0.589772076326468

Table 11 : Estimation value of R_{Fs}= $0.986111111111111, k = 1\ \beta_1 = 2.00;\ \beta_2 = 3.00$

n ,m	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Sh3}$	$\widehat{R}_{PMM}$
25,25	0.993416064607729	0.993416064697453	0.993303243833350	0.608996066915321	0.986186700251597	0.986109529274161	0.986091417522002	0.980541077463897
25,50	0.993365309216928	0.993365309222385	0.993064848641416	0.890196386992618	0.986176704024993	0.986107678695161	0.986091417521999	0.980638612227221
25,75	0.993556901167140	0.993556901159731	0.993126563557587	0.093502089568807	0.986174497691989	0.986107362019559	0.986091417521996	0.981412811065302
25,100	0.993568981527564	0.993568981595213	0.993035655176963	0.369737019864774	0.986173264625485	0.986107259535502	0.986091417522000	0.981462016822973
50,25	0.993358754841829	0.993358754811264	0.993276921538785	0.943739469627749	0.986175105961837	0.986107369791099	0.986091417521979	0.980923210671450
50,50	0.993598332416399	0.993598332417754	0.992986912096267	0.952899550932700	0.986170204470091	0.986106494630931	0.986091417521979	0.981578250073415
50,75	0.993810332286841	0.993810332289441	0.992972902474263	0.975351001902380	0.986169046455296	0.986106376468839	0.986091417521983	0.982460634458983
50,100	0.993733296708735	0.993733296768421	0.992864517739515	0.873591424025571	0.986166561407413	0.986106037539944	0.986091417521983	0.981932848071230
75,25	0.993278952872010	0.993278952896312	0.993128559641878	0.959020614488422	0.986172210931874	0.986106909564227	0.986091417521970	0.980519254132736
75,50	0.993788945754053	0.993788945525241	0.992920395966434	0.914100260250677	0.986169866941475	0.986106533854374	0.986091417521980	0.982402927336295
75,75	0.993653686579811	0.993653686586312	0.992882136333179	0.904749057666151	0.986164188038075	0.986105540969399	0.986091417521972	0.981971069693115
75,100	0.993752395864800	0.993752395742419	0.992812132940320	0.973474243337847	0.986163533932012	0.986105555309500	0.986091417521978	0.982255558256345
100,25	0.993552087507178	0.993552087956218	0.993132089253611	0.939220531372442	0.986173628746987	0.986107311592329	0.986091417521970	0.981462721695818
100,50	0.993645583731858	0.993645583778211	0.992939322757843	0.772916629624639	0.986164230930431	0.986105580431224	0.986091417521972	0.982091623213851
100,75	0.993770478084736	0.993770478896214	0.992883237292560	0.960388052652532	0.986163903400249	0.986105625620090	0.986091417521973	0.982290597577884
100,100	0.993693723525783	0.992792285524975	0.768682447774649	0.986160904105044	0.986105183375174	0.986091417521971	0.981979506938774	0.986111111111111

Table 12: Bias, MSE and MAPE of R_{Fs}= $0.986111111111111, k = 1\ \beta_1 = 2.00;\ \beta_2 = 3.00$

n ,m	Criteria	$\widehat{R}_{MLE}$	$\widehat{R}_{PE}$	$\widehat{R}_{ALS}$	$\widehat{R}_{WLS}$	$\widehat{R}_{Shr1}$	$\widehat{R}_{Shr2}$	$\widehat{R}_{Shr3}$	$\widehat{R}_{PMM}$
25,25	Bias	0.007304953496618	0.007304953447985	0.007192132722239	0.377115044195790	0.755891404858782 e-04	0.015818369495596 e-04	0.196935891095363 e-04	0.005570033647214
	MSE	0.000074911123301	0.000074911136743	0.000056580724246	91.460857371904837	0.000116418836379 e-04	0.000002186726709 e-04	0.000003878374520 e-04	0.000274741519473
	MAPE	0.008102703113974	0.008102703138666	0.007293430366214	0.699417604904533	0.831219288846580 e-04	0.115886047459884 e-04	0.199709636040368 e-04	0.010749027599030
25,50	Bias	0.007254198105817	0.007254198169657	0.006953737530304	0.095914724118493	0.655929138823246 e-04	0.034324159495661 e-04	0.196935891119804 e-04	0.005472498883890
	MSE	0.000068890063776	0.000068890064587	0.000051909972268	2.576718678373144	0.000083661985792 e-04	0.000001602755024 e-04	0.000003878374521 e-04	0.000192878002941
	MAPE	0.007757327414561	0.007757327415879	0.007051677495520	0.377067213089152	0.712890924256950 e-04	0.103475925917690 e-04	0.199709636065153 e-04	0.009688841816866
25,75	Bias	0.007445790056029	0.007445790054757	0.007015452446476	0.008926090215423 e+02	0.633865808779752 e-04	0.374909155178793 e-05	0.196935891153642 e-04	0.004698300045809
	MSE	0.000068364140975	0.000068364143756	0.000052340006287	5.950863331886557 e+02	0.000074456597343 e-04	0.000014003037485 e-05	0.000003878374522 e-04	0.000148719072039
	MAPE	0.007833110592300	0.007833110594789	0.007114261635863	0.011760863523354 e+02	0.676399032504352 e-04	0.953413612056662 e-05	0.199709636099468 e-04	0.008610534519738
25,100	Bias	0.007457870416452	0.007457870416452	0.006924544065852	0.000638362590875 e+02	0.621535143741417 e-04	0.385157560881522 e-05	0.196935891112717 e-04	0.004649094288139
	MSE	0.000067938253582	0.000067938254735	0.000051065515568	4.150607668883361 e+04	0.000071591865817 e-04	0.000013661015443 e-05	0.000003878374521 e-04	0.000145770807516
	MAPE	0.007809468886421	0.007809468884785	0.007022072855512	0.000682880366843 e+04	0.663952707407974 e-04	0.969013450698373 e-05	0.199709636057967 e-04	0.008561654705323
50,25	Bias	0.007247643730718	0.007247643730718	0.007165810423644	0.042371641483363 e+04	0.639948507258645 e-04	0.037413200121837 e-04	0.196935891322458 e-04	0.005187900439661
	MSE	0.000067566671511	0.000067566671511	0.000054904433573	2.500175529646021	0.000078257237271 e-04	0.000001500295151 e-04	0.000003878374529 e-04	0.000184312944965
	MAPE	0.007691376646783	0.007691376646783	0.007266737333736	0.339035732961923	0.692634248645140 e-04	0.101906855325591 e-04	0.199709636270662 e-04	0.009552160428448
50,50	Bias	0.007487221305288	0.007487221347846	0.006875800985156	0.033211560178411	0.590933589803925 e-04	0.461648017987448 e-05	0.196935891322276 e-04	0.004532861037696
	MSE	0.000066094547431	0.000066094548589	0.000049610782066	3.356204181387549	0.000058012753507 e-04	0.000010640364388 e-05	0.000003878374529 e-04	0.000120584523215
	MAPE	0.007732580024	0.007732580058	0.006972643252	0.319322855135	0.624700688377	0.865713258780	0.199709636270	0.007828843708

		073	427	553	703	933 e-04	937 e-05	477 e-04	471
50,75	Bias	0.007699221175730	0.007699221586484	0.006861791363152	0.089239890791268	0.579353441840964 e-04	0.473464227214915 e-05	0.196935891282862 e-04	0.003650476652128
	MSE	0.000066353182218	0.000066353137968	0.000049063053676	4.052551216351294	0.000050175488826 e-04	0.000008426248880 e-05	0.000003878374528 e-04	0.000082073157701
	MAPE	0.007842748405080	0.007842748447425	0.006958436311929	0.392372458319559	0.602385196948815 e-04	0.766681660631082 e-05	0.199709636230508 e-04	0.006536553812399
50,100	Bias	0.007622185597624	0.007622185547585	0.006753406628404	0.112519687085540	0.554502963023784 e-04	0.507357116720397 e-05	0.196935891280452 e-04	0.004178263039881
	MSE	0.000064939882108	0.000064939847688	0.000047478710942	0.232497522398103	0.000046364205326 e-04	0.000008447737202 e-05	0.000003878374527 e-04	0.000086431451829
	MAPE	0.007753601456180	0.007753601435747	0.006848525031621	0.196302235399436	0.575474485288433 e-04	0.779415105582626 e-05	0.199709636228064 e-04	0.006790613543951
75,25	Bias	0.007167841760899	0.007167841775970	0.007017448530767	0.027090496622689	0.610998207625241 e-04	0.420154688466845 e-05	0.196935891406602 e-04	0.005591856978376
	MSE	0.000066876590364	0.000066876587697	0.000052237530612	1.684592154091140	0.000073009367994 e-04	0.000014834065027 e-05	0.000003878374532 e-04	0.000197559492126
	MAPE	0.007672801538985	0.007672801570987	0.007116285834017	0.325068912623975	0.665148810528818 e-04	0.994071659258182 e-05	0.199709636355991 e-04	0.009598627988057
75,50	Bias	0.007677834642942	0.00767783468576	0.006809284855323	0.027989149139566	0.587558303638971 e-04	0.457725673674991 e-05	0.196935891318019 e-04	0.003708183774817
	MSE	0.000066767857387	0.00006676785657	0.000048579537577	4.645204262803659	0.000053732255404 e-04	0.000009247259515 e-05	0.000003878374529 e-04	0.000089454050446
	MAPE	0.007839982381513	0.00783998233567	0.006905190275820	0.402654847416204	0.612605154199359 e-04	0.799714400714230 e-05	0.199709636266160 e-04	0.006855769680520
75,75	Bias	0.007542575468700	0.007542575457399	0.006771025222068	0.081362053444960	0.530769269642781 e-04	0.557014171191117 e-05	0.196935891390889 e-04	0.004140041417997
	MSE	0.000063484227921	0.000063484236589	0.000047494076822	2.845396151840177	0.000041811527314 e-04	0.000008267744079 e-05	0.000003878374532 e-04	0.000080772253969
	MAPE	0.007699341238622	0.007699341247588	0.006866391774491	0.340131986915501	0.551709058557531 e-04	0.771643954281765 e-05	0.199709636340057 e-04	0.006563834483761
75,100	Bias	0.007641284753689	0.007641284758469	0.006701021829208	0.012636867773264	0.524228209008454 e-04	0.555580161119584 e-05	0.196935891330839 e-04	0.003855552854766
	MSE	0.000063654076006	0.000063654047468	0.000046344871356	2.797251922244012	0.000039016953264 e-04	0.000007532306348 e-05	0.000003878374529 e-04	0.000062660130689
	MAPE	0.007767680691688	0.007767680647754	0.006795402418352	0.304280094664635	0.538415295097445 e-04	0.744745935709718 e-05	0.199709636279161 e-04	0.006031190304151
100,25	Bias	0.007440976396067	0.007440976359659	0.007020978142500	0.046890579738669	0.625176358756772 e-04	0.379951878166274 e-05	0.196935891414242 e-04	0.004648389415293
	MSE	0.000067009960349	0.000067009964676	0.000052268842760	8.944434060987687	0.000075568257502 e-04	0.000014894878844 e-05	0.000003878374533 e-04	0.000149741253362
	MAPE	0.007723121110597	0.007723121147427	0.007119865158591	0.454793438724090	0.665800910357521 e-04	0.987732571299159 e-05	0.199709636363738 e-04	0.008737274850105
100,50	Bias	0.007534472620747	0.007534472658579	0.006828211646732	0.000735902774074	0.531198193195754 e-04	0.553067988698464 e-05	0.196935891389202 e-04	0.004019487897261
	MSE	0.000062880158671	0.000062880147584	0.000048581931794	4.556262840360840	0.000042928625061 e-04	0.000008590390799 e-05	0.000003878374532 e-04	0.000075620089879
	MAPE	0.007652775008086	0.007652775047479	0.006924383641756	0.000797349922657	0.548744480422624 e-04	0.793475437829533 e-05	0.199709636338345 e-04	0.006404008095442
100,75	Bias	0.007659366973625	0.007659366956965	0.006772126181449	0.025723058458579 e+04	0.527922891376956 e-04	0.548549102097950 e-05	0.196935891378374 e-04	0.003820513533227
	MSE	0.000064107808966	0.000064107804748	0.000047222482237	17.005707437803050 e+04	0.000039537347476 e-04	0.000007497115640 e-05	0.000003878374531 e-04	0.000071855165316
	MAPE	0.007790742634625	0.007790742647475	0.006867508240342	0.516892322967719 e+04	0.544962531110996 e-04	0.739846738338876 e-05	0.199709636327365 e-04	0.006171373519560
100,100	Bias	0.007582612477648	0.007582612413526	0.006681174413863	0.001782571336664 e+03	0.497929939331765 e-04	0.592773593756368 e-05	0.196935891396530 e-04	0.004131604172337
	MSE	0.000062413136787	0.000062413163957	0.000045973823891	3.693902914756908 e+03	0.000034743939046 e-04	0.000007438808102 e-05	0.000003878374532 e-04	0.000065492680148
	MAPE	0.007694814937484	0.007694814976993	0.006775275461946	0.002208877838534 e+03	0.513123915951584 e-04	0.742154696233848 e-05	0.199709636345777 e-04	0.005980987330055

6. Discussion of Results

To clarify the results shown in the tables above, they are as follows:

- Table 1, contains the estimated values for R_s using different techniques at different sample sizes, different techniques as MLE, ALS, WLS and shrinkage techniques show convergence in some cases to the true value (0.612), while there is a clear deviation some other techniques. MLE and WLS appeared to be the most stable compared to the rest of the techniques when the sample size increasing. Table 2 aims to determine which estimation technique has the lowest deviation and best accuracy. It was observed that the values for MLE and the Sh1 and Sh3 techniques have the lowest bias and error values, indicating that they are the best.
- Table 3, shows good convergence between the different techniques with some slight deviations at small sample sizes. MLE,

PMM and Sh3 provide estimates very close to the true value. Table 4, shows that MLE and PMM have the lowest error values, indicating their high accuracy. The lowest values in MSE and MAPE indicate that shrinkage techniques may be a good choice in some cases.

- Table 5, estimated values vary significantly at small sample sizes but stabilize with increasing sample size, techniques such as MLE, PMM, Sh2, and Sh3 were closet to the true value. Table 6, shows that MLE, Sh12, and Sh3 are most stable.
- Table 7, its clear that techniques as MLE, PMM, Sh2, and Sh3 achieve good convergence to the value at small sample sizes. There is some variation between the techniques, but as the samples increases, the estimated values become more stable. It is observed that ALS and WLS have larger deviations from true value. While table 8 the lower values of MSE and MAPE indicate that MLE, Sh2, and Sh3 were the best while ALS and WLS showed less accurate performance based on accuracy criteria.
- Table 9, traditional and shrinkage techniques achieve close estimates of true value, there is a slight deviation in the estimates of ALS and WLS, bit they remain within acceptable limits. The best performance techniques for large samples are MLE, Sh2, and Sh3. In table 10, shows the techniques with lowest bias and error rate values are MLE, Sh2, and Sh3, while the techniques such as ALS were not stable for all sample sizes.
- Table 11, estimated values close to 0.9 especially using MLE, Sh2, and Sh3. For small samples, there are some noticeable deviations, but for large samples the accuracy improves. PMM has acceptable stability but is not the best, while table 12 shows the lowest values for MSE and MAPE showing that MLE, Sh2, and Sh3 are most accurate. WLS and ALS have the highest error rates making them less reliable.

From the results presented above, the tables show that the performance of the techniques varies with sample sizes. The Sh2, Sh3 and MLE are the most accurate according the criteria's. at small sample sizes, some techniques tend to have large variance, but as the sample size increases, the values become more stable. The effect of fuzzy estimation was evident as some traditional techniques such as WLS and ALS showed greater bias when using fuzzy data.

7. Conclusion

In this study, classical series components systems of reliability and fuzzy series components systems of reliability were proven for the NXL distribution and the eight techniques were simulated. The different estimation techniques showed variation in performance according sample size, the Sh2, Sh3 and MLE were most accurate when using error criteria. In small samples, there were large deviations for some techniques, while the values became more stable as the sample size increased. The effect of fuzzy data was evident, as some classical techniques such as WLS and ALS showed larger deviations when dealing with fuzzy data. In contrast, techniques such as MLE, Sh2, and Sh3 provided more stable results.

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