IRAQI STATISTICIANS JOURNAL

https://isj.edu.iq/index.php/isj

ISSN: 3007-1658 (Online)

Extension of the Weibull Survival Regression Based on Sine Type II Topp-Leone-G Family of Distribution: Properties and Application

A. M. Isa[1], S. I. Doguwa[2], B. B. Alhaji[3] and H. G. Dikko[4]

[1]Department of Mathematics and Computer Science, Borno State University, Maiduguri, Nigeria
[2,4]Department of Statistics, Ahmadu Bello University Zaria, Kaduna State, Nigeria
[3]Nigerian Defense Academy, Kaduna State, Nigeria

[1]alhajimoduisa@bosu.edu.ng :
[2]sidoguwa@gmail.com :
[3]hgdikko@gmail.com :
[4]bbukar@nda.edu.ng :

ARTICLE INFO

Article history:

Received	29 January 2025
Revised	29 January 2025
Accepted	07 March 2025
Available online	07 March 2025

Keywords:
Sine G Family
Type II Topp-Leone G
Survival Regression
Weibull Distribution
Maximum Likelihood

ABSTRACT

This study introduces the Sine Type II Topp-Leone Weibull (STIITL-W) distribution, a novel statistical model designed to enhance flexibility in data modeling. The proposed distribution extends the classical Weibull distribution by incorporating the Sine Type II Topp-Leone family, offering improved adaptability for complex datasets. Key mathematical properties, including moments, moment generating functions, and reliability measures, are derived. Parameter estimation is performed using the maximum likelihood estimation (MLE) method, and a simulation study is conducted to evaluate the consistency and efficiency of the estimates. Additionally, a Sine Type II Topp-Leone Weibull Survival Regression (STIITL-WSR) model is developed for survival data analysis. The model's performance is assessed using real-world datasets, including transect stake distance measurements and liver cancer survival data. Comparative analysis based on the Akaike Information Criterion (AIC) demonstrates that the STIITL-W distribution outperforms competing models, such as the Topp-Leone Modified Weibull, Exponential, and Generalized Weibull distributions. Furthermore, the log-STIITL-Weibull survival regression model achieves a lower AIC compared to the log-TLG-Weibull regression model, confirming its superior performance in survival data modeling.

1. Introduction

Probability distributions are fundamental tools in statistical modelling, enabling the description, analysis, and prediction of real-world phenomena across diverse fields such as reliability engineering, medicine, and finance. The flexibility of these distributions is crucial for extracting accurate insights from complex datasets. However, classical probability distributions often fail to adequately capture multimodal behaviours, heavy-tailed data or extreme variations, highlighting the need for more robust and adaptable models.

To address these limitations, researchers have developed generalized families of distributions, often referred to as generators. These generators enhance the flexibility of existing distributions by introducing additional parameters, such as scale or shape parameters.

Corresponding author E-mail address: alhajimoduisa@bosu.edu.ng
https://doi.org/10.62933/z1y1sr28

Notable examples include the Marshall-Olkin Weibull generalized family [1], the Marshall-Olkin exponentiated half logistic-generalized family [2], and the Secant Kumaraswamy family [3].
Among these, trigonometric families of distributions have gained significant attention due to their ability to improve the flexibility of existing distributions without introducing additional parameters. This simplification facilitates parameter estimation and the derivation of distribution properties. A prominent example is the Sine-G family of distributions, introduced by [4], which has been used to extend various existing distributions. Models derived from this family, such as the Sine Exponential distribution [5], Sine Lomax distribution [6], Sine Burr XII [7], Sine Weibull distribution [8], Sine Type II Topp-Leone Burr XII [9], Cosine Marshal-Olkin distribution [10], and Cosine Lomax [11], have demonstrated superior performance compared to their parent distributions.
Recently, [12] introduced the Sine Type II Topp-Leone generator, which combines the sine family of distributions with the Type II Topp-Leone family developed by [13]. This generator enhances the flexibility of existing distributions, enabling them to better capture the complexities of real-world phenomena. For instance, [12] demonstrated the effectiveness of this generator by compounding it with the Lomax distribution, resulting in the Sine Type II Topp-Leone Lomax distribution, which outperformed the baseline Lomax distribution and other existing models when applied to real-world datasets.
The Weibull distribution is widely recognized for its versatility in modelling reliability and lifetime data. However, its limitations in accommodating complex data behaviours have led to numerous extensions, such as the Topp-Leone Weibull model [14], the Topp-Leone modified Weibull model [15], the Cosine-Weibull model [16], and the extended Weibull model [17]. These extensions introduce additional features, enabling the Weibull distribution to effectively capture diverse hazard rate shapes and complex patterns.
In this study, we propose a novel extension of the Weibull distribution based on the Sine Type II Topp-Leone family, referred to as the Sine Type II Topp-Leone Weibull (STIITL-W) distribution. This model leverages the unique properties of the Sine Type II Topp-Leone family to address the limitations of the classical Weibull distribution. We derive key properties of the proposed distribution, including moments, quantile functions and hazard rate behaviour. The performance of the model is evaluated using real-world datasets, demonstrating its superiority in fitting complex data.
The development of this model is motivated by the growing demand for robust and adaptable statistical tools capable of addressing the increasingly complex and heterogeneous nature of data across various fields. By extending the classical Weibull distribution, this study contributes to advancing statistical methodologies and provides a practical tool for researchers and practitioners working with intricate datasets.

2. Methodology

2.1 Sine G Family of Distribution

Let H(x) be the CDF of a Univariate continuous distribution and h(x) be the PDF, then, the Sine-G family of distributions according to [4] is defined by the CDF given by:

$$F(x,\xi)=\int_0^{\frac{\pi}{2}H(x,\xi)}\cos(t)dt=\sin\left\{\frac{\pi}{2}H(x,\xi)\right\} \tag{1}$$

and its corresponding PDF given by:

$$f(x,\xi)=\frac{\pi}{2}h(x;\xi)\cos\left\{\frac{\pi}{2}H(x,\xi)\right\} \tag{2}$$

Where $h(x;\xi)$ and $H(x;\xi)$ are the PDF and CDF of any baseline distribution and ξ is a vector parameter of the baseline distribution.

2.2 Sine Type II Topp-Leone G Family of Distribution

The CDF of the Sine Type II Topp Leone G family of distribution proposed by [12] is defined by:

$$F(x,\alpha,\xi)=\sin\left\{\frac{\pi}{2}\left[1-(1-G^2(x;\xi))^{\alpha}\right]\right\} \tag{3}$$

the *PDF corresponding to equation (3) is given by;*

$$f(x;\alpha,\xi)=\frac{\pi}{2}2\alpha g(x;\xi)G(x;\xi)\left[1-G^2(x;\xi)\right]^{\alpha-1}\cos\left\{\frac{\pi}{2}\left[1-(1-G^2(x;\xi))^{\alpha}\right]\right\} \tag{4}$$

where α is a shape parameter which controls the skewness and kurtosis of the distribution. When α is too large, the distribution may become too skewed thereby making it less suitable for skewed data. $G(x;\xi)$ *and* $g(x;\xi)$ *are the CDF and PDF of the baseline distribution*.

2.3 *Weibull* Distribution

Supposed that the baseline distribution is Weibull with CDF and PDF given respectively by:

$$F(x;\gamma,\lambda)=1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}} \tag{5}$$

and

$$f(x;\gamma,\lambda)=\frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}},\quad x>0,\lambda>0,\gamma>0 \tag{6}$$

Where $x>0$ and γ is a shape parameter and λ is a scale parameter. If λ is too large or too small, the distribution may become too spread or too concentrated which will limit its applicability to certain datasets.

2.4 The Proposed Sine Type II Topp-Leone Weibull Distribution

The CDF of Sine Type II Topp Leone-Weibull distribution was obtained by inserting the CDF of the Weibull distribution in equation (5) into to the CDF of the Sine-Type II Topp-Leone G family in equation (3) as follows;

$$F(x;\alpha,\gamma,\lambda)=\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^2\right]^{\alpha}\right]\right\} \tag{7}$$

Where

and the corresponding pdf is also obtained by substituting the PDF and CDF of the Weibull distribution in equation (5) and (6) respectively into the PDF of the Sine-Type II Topp-Leone G family in equation (4) as follows:

$$f(x;\alpha,\gamma,\lambda)=\frac{\pi}{2}\frac{2\alpha\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^2\right]^{\alpha-1}\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^2\right]^{\alpha}\right]\right\} \tag{8}$$

where α and γ are shape parameters while λ is a scale parameter. These parameters control the flexibility and the behaviour of the distribution and $x > 0$. The survival function. S(x), hazard function $h(x)$ and the quantile function Q(u) are given in equation (9), (10) and (11) respectively:

$$S(x) = 1 - \sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]\right\} \quad (9)$$

The survival function represents the probability that the event of interest has not occurred by time t.

$$h(x) = \frac{\frac{\pi}{2}\frac{2\alpha\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1} e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha-1}\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]\right\}}{1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]\right\}} \quad (10)$$

$$Q(u) = \lambda^{\gamma}\left\{-\log\left[1-\left[1-\left[1-\frac{\sin^{-1}(u)}{\pi/2}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2}}\right]\right\}^{\frac{1}{\gamma}} \quad (11)$$

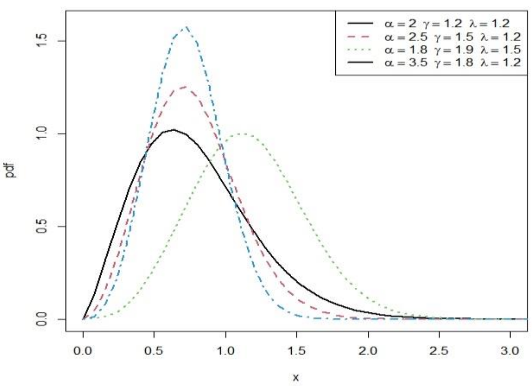

Figure 1: PDF plot of STIITLW distribution

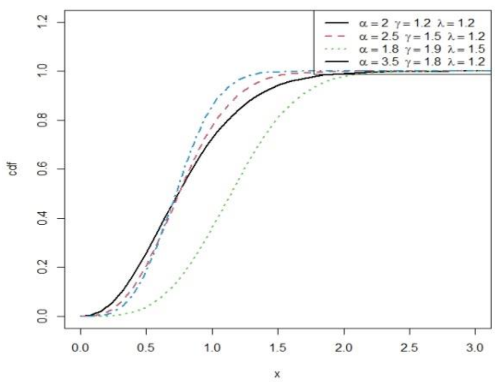

Figure 2: CDF plot of STIITLW distribution

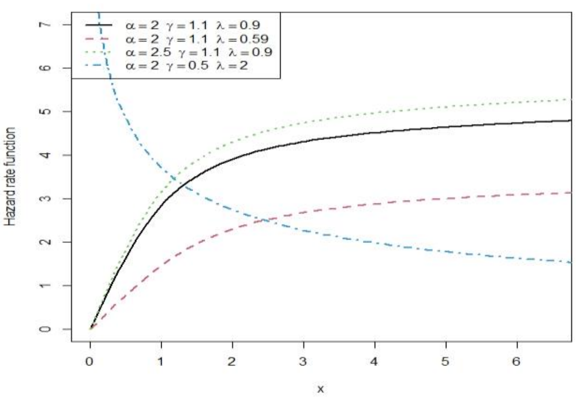

Figure 3: HRF plot of STIITLW distribution

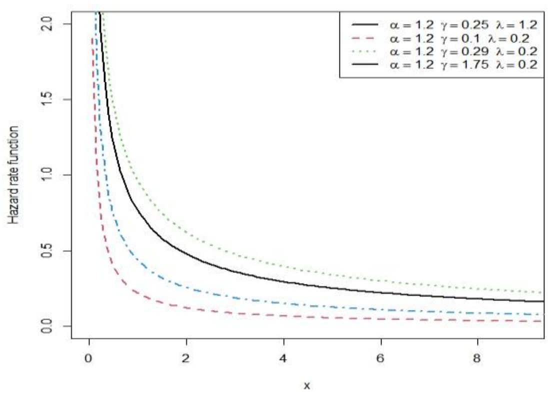

Figure 4: HRF plot of STIITLW distribution

Figure 1 and 2 represents the PDF of the STIITL-W distribution for different parameter values. The plot demonstrates how the shape of the distribution changes with varying α, γ and λ while figure 3 and 4 display the hazard rate function of the STIITL-W distribution, illustrating how the risk of failure changes over time.

2.5 Density Expansion

The PDF and CDF of the proposed STIITLW distribution can be expanded using Taylor's Series Expansion and Binomial Expansion as follows:

$$\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]\right\}=\sum_{m=1}^{\infty}\frac{(-1)^{m}}{(2m)!}\frac{\pi^{2m}}{2^{2m}}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]^{2m}$$

$$\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]^{2m}=\sum_{n=0}^{\infty}(-1)^{n}\binom{2m}{n}\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha n}$$

$$\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{(\alpha n+\alpha-1)}=\sum_{p=0}^{\infty}(-1)^{p}\binom{(\alpha n+\alpha-1)}{p}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{p}$$

$$\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{p+1}=\sum_{q=0}^{\infty}(-1)^{q}\binom{p+1}{q}\left(e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{q}$$

$$\therefore\quad f(x)=\frac{2\alpha\gamma}{\lambda}\sum_{m,n,p,q=0}^{\infty}\frac{(-1)^{m+n+p+q}\pi^{2m+1}}{(2m)!2^{2m+1}}\binom{2m}{n}\binom{\alpha n+\alpha-1}{p}\binom{p+1}{q}\left(\frac{x}{\lambda}\right)^{\gamma-1}\left[e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{q+1}$$

$$f(x)=\sum_{i,j,p,q}^{\infty}\kappa_{m,n,p,q}\left[e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{q+1}$$

where $$\kappa_{m,n,p,q}=\frac{2\alpha\gamma}{\lambda}\frac{(-1)^{m+n+p+q}\pi^{2m+1}}{(2m)!2^{2m+1}}\binom{2m}{n}\binom{\alpha n+\alpha-1}{p}\binom{p+1}{q}\left(\frac{x}{\lambda}\right)^{\gamma-1} \qquad (12)$$

The CDF can also be expressed as follows:

$$\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]\right\}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{(2i+1)!}\frac{\pi^{2i+1}}{2^{2i+1}}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha}\right]^{(2i+1)}$$

$$\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2i}\right]^{\alpha}\right]^{(2i+1)} = \sum_{j=0}^{\infty}(-1)^{j}\binom{2i+1}{j}\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha j}$$

$$\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}\right]^{\alpha j} = \sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha j}{k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2}$$

$$\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{2k} = \sum_{l=0}^{\infty}(-1)^{l}\binom{2k}{l}\left(e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{l}$$

$$F(x) = \sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}}{(2i+1)!}\frac{\pi^{2i+1}}{2^{2i+1}}\binom{2i+1}{j}\binom{\alpha j}{k}\binom{2k}{l}\left(e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{l}$$

Let $\phi_{i,j,k,l} = \frac{(-1)^{i+j+k+l}}{(2i+1)!}\frac{\pi^{2i+1}}{2^{2i+1}}\binom{2i+1}{j}\binom{\alpha j}{k}\binom{2k}{l}$

Hence $F(x) = \sum_{i,j,k,l=0}^{\infty}\phi_{i,j,k,l}\left(e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{l}$ (13)

2.6 Mathematical Properties of STIITL-W Distribution

The mathematical properties of the STIIL-Weibull distribution such as the moment, moment generating function, entropy and order statistics are derive:

2.6.1 Moment of STIITL-Weibull Distribution

The moment of the STIITL-Weibull distribution is obtained as follows:

$$\mu_r^{'} = \sum_{i,j,p,q}^{\infty}\kappa_{m,n,p,q}\int_0^{\infty}x^r\left[e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{q+1}dx$$

$$\mu_r^{'} = \sum_{i,j,p,q}^{\infty}\frac{\lambda^r\kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}}\Gamma\left(\frac{r}{\gamma}+1\right) \quad (14)$$

The first, second, third and the forth moments can be obtained by substituting r=1, 2, 3, and 4 into equation (4.36) as follows:

$$\mu_1^{r} = \sum_{i,j,p,q}^{\infty}\frac{\lambda^r\kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}}\Gamma\left(\frac{1}{\gamma}+1\right) \quad (15)$$

$$\mu_2^{'} = \sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{2}{\gamma}+1\right) \tag{16}$$

$$\mu_3^{'} = \sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{3}{\gamma}+1\right) \tag{17}$$

$$\mu_4^{'} = \sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{4}{\gamma}+1\right) \tag{18}$$

2.6.2 Skewness and Kurtosis of STIITL-W Distribution

Skewness is the measure of extent that the distribution leans to the one side of the mean and kurtosis is used to measure the flatness or peakedness of the probability curve. The skewness and kurtosis are denoted by β_1 and β_2 respectively and expressed in the following form:

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} \tag{19}$$

and

$$\beta_1 = \frac{\mu_4}{(\mu_2)^2} \tag{20}$$

The skewness of the STIITL-W distribution is obtained by substituting equation (16) and (17) into equation (19) as follows:

$$\beta_1^{'} = \frac{\left(\sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{4}{\gamma}+1\right)\right)^2}{\left(\sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{4}{\gamma}+1\right)\right)^3} \tag{21}$$

and the kurtosis can be obtained by substituting equation (16) and (18) into equation (20):

$$\beta_2^{'} = \frac{\left(\sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{4}{\gamma}+1\right)\right)}{\left(\sum_{i,j,p,q}^{\infty} \frac{\lambda^r \kappa_{m,n,p,q}}{(q+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{2}{\gamma}+1\right)\right)^2} \tag{22}$$

2.6.3 Moment Generating Function of STIITL-W Distribution

The moment generating function of a random variable x can be obtaied as follows:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Therefore, the moment generating function of the STIITL-W distribution is given by:

$$M_x(t) = \sum_{i,j,p,q}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \kappa \; \Gamma\left(\frac{k}{\gamma} + 1 \right) \tag{23}$$

2.7 Maximum Likelihood Estimate of STIITL-W Distribution

The likelihood function of the STIITL-W Distribution is given by:

$$\ell = n\log(\frac{\pi}{2}) + n\log 2 + n\log\alpha + n\log\gamma + n\log\lambda - (\gamma-1) + \sum_{i=1}^{n} \log x_i - n(\gamma-1)\log\lambda - \gamma\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)$$

$$+\sum_{i=1}^{n}\log\left(1-e^{\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-\left[1-e^{\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right]^2\right] + \sum_{i=1}^{n}\log\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right)^2\right]^{\alpha}\right]\right\}$$

(24)

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=0}^{n}\log\left[1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right]^2\right] + \sum_{i=1}^{n}\frac{\pi}{2}\tan\left\{\left[1-\left[1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right]^2\right]^{\alpha}\right]\right\}$$

$$\times\left[1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right]^2\right]^{\alpha}\log\left[1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^{\gamma}}\right]^2\right] \tag{25}$$

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} x - n\log\lambda - \sum_{i=1}^{n}\left(\frac{x}{\lambda}\right) + \sum_{i=1}^{n}\frac{\left(\frac{x}{\lambda}\right)^{\gamma}\log\left(\frac{x}{\lambda}\right)e^{\left(\frac{x}{\lambda}\right)^{\gamma}}}{\left(1-e^{\left(\frac{x}{\lambda}\right)^{\gamma}}\right)} + \sum_{i=1}^{n}\frac{2(\alpha-1)\left(\frac{x}{\lambda}\right)^{\gamma}\log\left(\frac{x}{\lambda}\right)e^{-\left(\frac{x}{\lambda}\right)^{\gamma}\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]}}{\left[1-\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^2\right]}$$

$$+\sum_{i=1}^{n}\frac{\pi}{2}2\alpha\left(\frac{x}{\lambda}\right)^{\gamma}\log\left(\frac{x}{\lambda}\right)e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\left(1-e^{\left(\frac{x}{\lambda}\right)^{\gamma}}\right)\left[1-\left(1-e^{\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha-1}\right]\sum_{i=1}^{n}\tan\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{\left(\frac{x}{\lambda}\right)^{\gamma}}\right)\right]^{\alpha}\right]\right\} \tag{26}$$

$$\frac{\partial\ell}{\partial\lambda} = -\frac{n}{\lambda} - \frac{n(\gamma-1)}{\lambda} - \gamma\sum_{i=1}^{n}\frac{x}{\lambda^2} + \sum_{i=1}^{n}\frac{\frac{\gamma}{\lambda^2}\left(\frac{x_i}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}}{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)} + \frac{2(\alpha-1)\gamma}{\lambda^2}\left(\frac{x_i}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]\sum_{i=1}^{n}\left[1-\left(e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^2\right]^{-1}$$

$$+\frac{\pi}{2}\frac{2\alpha\gamma}{\lambda^2}\left(\frac{x_i}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^2\right]^{\alpha-1}\tan\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^2\right]^{\alpha}\right]\right\} \tag{27}$$

3.1 STIITL-W Survival Regression

The STIITL-W Survival Regression (STIITL-WSR) was obtained using the one-to-one transformation defined in equation (28) is used in this section to change the probability density function of X directly in terms of the probability density function of $y = \log(x), x > 0$

$$g(y) = f(x)\left|\frac{dx}{dy}\right| \tag{28}$$

Exposure variables such as sex, age and tumor size at a start of a treatment have impact on survival time in a variety of real world application. Investigating the casual relationship between survival time and exposure factors is crucial. It is possible to employ a survival regression model to capture the effect of these factors on the survival of the subjects. We take into account a group of location scale model, where the vector $X = (x_1, x_2, x_3, \dots, x_p)^\tau$ of covariates is associated with the response variable $Y = \log(x)$ using a regression framework.

3.1.1 Sine Type II Topp-Leone Weibull Survival Regression

Let X be a random variable with pdf STIITL-W as defined in equation (28). By replacing the scale parameter $\lambda = e^\mu$ and the shape parameter $\gamma = 1/\sigma$ and using the log transformation yield the following:

$$g(y) = \frac{\pi}{2}\frac{2\alpha}{\sigma}e^{-\mu}\left(\frac{e^y}{e^\mu}\right)^{\frac{1}{\sigma}-1} e^{-\left(\frac{e^y}{e^\mu}\right)^{\frac{1}{\sigma}}}\left(1-e^{-\left(\frac{e^y}{e^\mu}\right)^{\frac{1}{\sigma}}}\right)\left[1-\left(1-e^{-\left(\frac{e^y}{e^\mu}\right)^{\frac{1}{\sigma}}}\right)^2\right]^{\alpha-1} \cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{e^y}{e^\mu}\right)^{\frac{1}{\sigma}}}\right)^2\right]^\alpha\right]\right\}\left|e^y\right|$$

$$g(y) = \frac{\pi}{2}\frac{2\alpha}{\sigma}e^{\left(\frac{y-\mu}{\sigma}\right)}e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\left[1-e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\right]\left[1-\left[1-e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\right]^2\right]^{\alpha-1}\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\right]^2\right]^\alpha\right]\right\} \tag{29}$$

Where $-\infty < y < \infty$ and μ is a location parameter, equation (5.2) is referred to as log STIITL-W distribution and $Y \sim LSTIITL - W(\alpha, \sigma, \mu)$. The survival function of the transformed pdf is presented in equation (30).

$$S(y) = 1 - \sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\right]^2\right]^\alpha\right]\right] \tag{30}$$

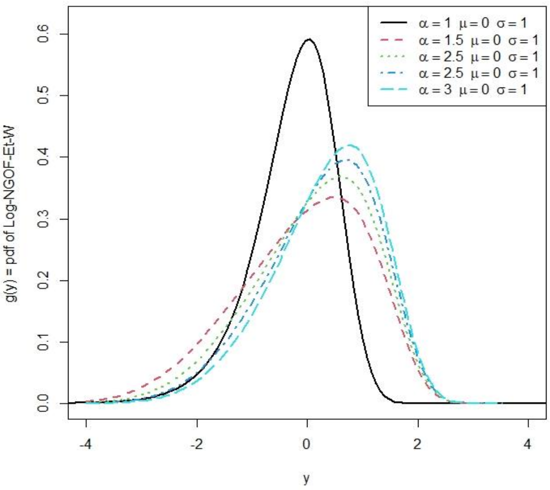

Figure 5: Pdf plot of log-STIITL-W SR

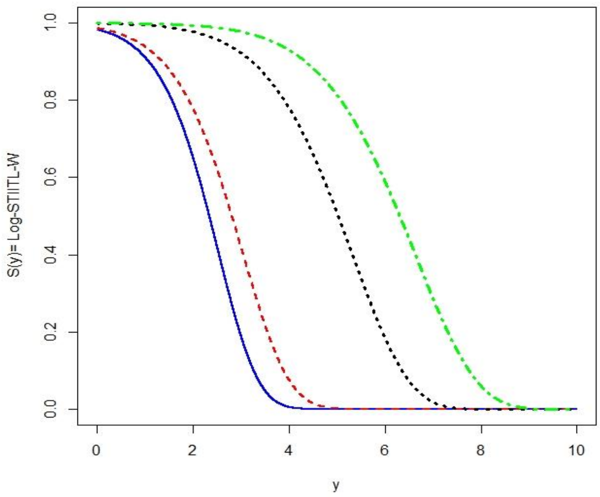

Figure 6: Survival plot of log-STIITL-W SR

Equation (29) can be written as a log-linear model:

$$Y = \rho + \sigma Z \tag{31}$$

Let $z = \left(\dfrac{\mu - y}{\sigma}\right)$ be the standardized random variable, then, the random variable z has pdf given by:

$$g(y) = \frac{\pi}{2}\frac{2\alpha}{\rho}e^{z}e^{-e^{z}}\left[1-e^{-e^{z}}\right]\left[1-\left[1-e^{-e^{z}}\right]^{2}\right]^{\alpha-1}\cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-e^{z}}\right]^{2}\right]^{\alpha}\right]\right] \tag{32}$$

Where $-\infty < z < \infty$ Now, considering the framework of regression model based on the Log-STIITL-W distribution in equation (28). Linking the response variable Y to the vector of explanatory variables X, and letting $\mu_i = X_i^{\tau}\pi$ in the Log-Linear model of equation (31) so that the model of Y given X can be represented by:

$$y_i = X_i^{\tau}\pi + z_i \qquad i = 1,2,3,...,n \tag{33}$$

Where z_i are random errors which follows the Log-STIITL-W distribution in equation (29), $\alpha, \sigma > 0$, and $\pi = (\pi_1, \pi_2, ..., \pi_p)$ are unknown parameters, therefore, equation (33) is referred to as the Log-STIITL-W survival regression model, with survival function given in equation (30). This transformation allows the model to be expressed as a log-linear regression model, making it easier to interpret the effects of covariates on survival time.

3.1.2 Simulation Study of STIITL-Weibull Distribution

A Monte Carlo simulation study was conducted to evaluate the performance of the newly proposed STIITL-Weibull distribution.

The study involved generating data using the quantile function of the distribution for various sample sizes (n = 20, 50, 100, 150, 200), with parameters set at α = 2.0, γ = 0.5, and λ = 1.1. Each sample size was replicated 1000 times. The mean, bias, and root mean square error (RMSE) of the maximum likelihood estimates were computed. The results, presented in Table 1, demonstrate the accuracy and consistency of the parameter estimates for different sample sizes.

Table 1: Estimate, Bias and Root Mean Square Error of the STIITL-W distribution

N	Properties	$\alpha = 2.0$	$\gamma = 0.5$	$\lambda = 1.1$
	Estimate	2.2446	0.6231	0.4591
20	Bias	0.1446	0.0231	0.0591
	RMSE	0.9516	0.1232	0.2369
	Estimate	2.1695	0.5934	0.4535
50	Bias	0.0695	-0.0066	0.0535
	RMSE	0.6700	0.0715	0.1580
	Estimate	2.1203	0.5850	0.4467
100	Bias	0.0203	-0.0150	0.0467
	RMSE	0.3850	0.0506	0.1042
	Estimate	2.1003	0.5814	0.4449
150	Bias	0.0003	-0.0186	0.0449
	RMSE	0.3025	0.0439	0.0906
	Estimate	2.1058	0.5800	0.4453
200	Bias	0.0058	-0.0200	0.0453
	RMSE	0.2661	0.0387	0.0824

Table 1 represents the results of the Monte Carlo simulation study, showing the bias and RMSE of the parameter estimates for different sample sizes. The results shows that, the values of biases and RMSEs tend to zero as shown in table and the estimates tend to the true parameter values as the sample size increases, indicating that the estimates are efficient and consistent.

4.1 Application

4.1.1 Fitting STIITL-W Distribution to Transect stake Distance Measurements

This data set, obtained from [18], represents the distances from the transect line for the 68 stakes detected in walking L = 1000 m and searching w = 20 m on each side of the line. The measurements are:
2.0, 0.5, 10.4, 3.6, 0.9, 1.0, 3.4, 2.9, 8.2, 6.5, 5.7, 3.0, 4.0, 0.1, 11.8, 14.2, 2.4, 1.6, 13.3, 6.5, 8.3, 4.9, 1.5, 18.6, 0.4, 0.4, 0.2, 11.6, 3.2, 7.1, 10.7, 3.9, 6.1, 6.4, 3.8, 15.2, 3.5, 3.1, 7.9, 18.2, 10.1, 4.4, 1.3, 13.7, 6.3, 3.6, 9.0, 7.7, 4.9, 9.1, 3.3, 8.5, 6.1, 0.4, 9.3, 0.5, 1.2, 1.7, 4.5, 3.1, 3.1, 6.6, 4.4, 5.0, 3.2, 7.7, 18.2

Table 2: The MLEs, Log-Likelihoods and Goodness of Fits Statistic of Transect stake Distance Measurements

Model	α	γ	λ	LL	AIC
STIITLW	1.8560	0.5783	6.6905	-186.081	376.162
TIITLW	3.6326	0.1912	16.433	-189.422	384.845
TLMW	0.4845	0.5949	1.0467	-186.368	378.735
W	1.2238	6.2368	-	-186.170	378.340
GW	1.1000	1.1000	0.2000	-187.779	381.546
TW	1.1385	5.4002	0.2984	-186.041	378.083

Table 2 presents the results of the analysis of STIITL-Weibull distribution. The analysis compared the performance of the Sine-Topp-Leone Exponentiated Weibull by [20] distribution against several other distributions, namely the Topp-Leone Exponentiated Weibull distribution, Topp Leone Modified Weibull distribution, the Weibull distribution, Generalized Weibull and Transmuted Weibull distributions.
The results indicated that the proposed STIITL-Weibull distribution outperformed some competing distributions, as it exhibits the lowest AIC value. The visual assessment of the

goodness of fit, as depicted in Figures 1, further validates the superiority of the proposed distribution when compared to other competing distributions. Therefore, it can be concluded that the proposed family of distributions is the most suitable choice for modelling distance transect line for the stakes detected in walking L = 1000 m and searching w = 20 m on each side of the line.

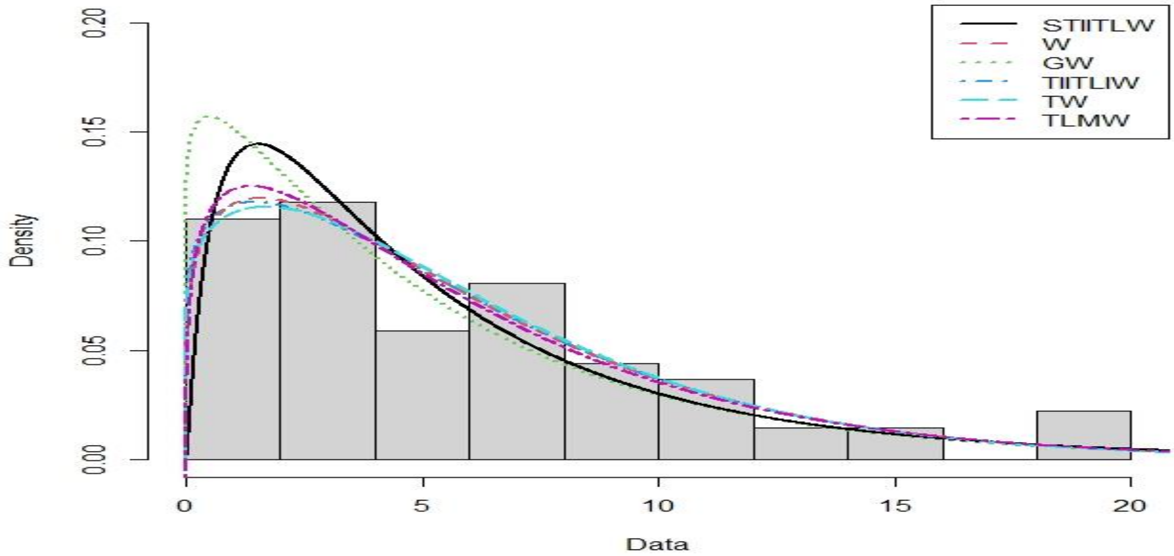

Figure 7: Fitted pdfs for the STIITLW, W, GW, TITLIW, TW and TLMW on distance transect line

4.1.2 Fitting STIITL-W Survival Regression Model on Liver Cancer Dataset

The liver cancer survival time dataset, sourced from the Real Statistics website, contains information on 40 liver cancer patients, with 27.5% of the cases censored and 72.5% uncensored. The primary response variable, denoted as y, represents the observed survival time in weeks, while the censoring indicator d is used to differentiate between patients who were still alive or lost to follow-up (coded as 0) and those who died due to liver cancer (coded as 1). The dataset includes survival-related variables such as age, gender, blood levels, and tumor size. The age is categorized into three groups: below 35 years (x_1), 35 to 50 years (x_2), and above 50 years (x_3). Gender is recorded as x_4 for male and x_5 for female. Blood levels of Alpha-fetoprotein are classified into two levels: level 1 (x_6) and level 2 (x_7). Tumor size is grouped into three stages: stage 1 (x_8), stage 2 (x_9), and stage 3 (x_{10}).

Table 3: Parameter estimates, log likelihood, AIC of log-STIITL-Weibull and TLG-Weibull Regression Models

Parameters	log-STIITL-WSR		log -TLG-WSR	
α	1.6458	<0.001	0.0273	0.0273
σ	–	–	0.1262	0.1262
θ	4.4471	<0.001	-8.6188	<0.001
β_1	7.2598	<0.001	-1.5763	0.0516
β_2	7.4355	<0.001	-1.3236	0.0896
β_3	7.0589	<0.001	-1.6512	0.0472
β_4	-3.4873	<0.001	-8.2837	<0.001
β_5	-2.9618	<0.001	-8.0825	<0.001
β_6	1.1040	<0.001	13.860	<0.001
β_7	1.3329	<0.001	13.758	<0.001

β_8	-3.4468	<0.001	-2.2247	0.0039
β_9	-3.5911	<0.001	-2.1271	0.0092
β_{10}	-3.0626	<0.001	-2.0519	0.0140
LL	**-102.5833**		**-103.3930**	
AIC	**128.5833**		**133.3930**	

Table 4 compares the performance of the log-STIITL-Weibull survival regression model with the log-TLG-Weibull survival regression model for the liver cancer dataset based on the AIC statistics. The results revealed that, log-STIITL-WSR has AIC value of 128.5833 while the comparator model log-TLG-WSR has AIC of 133.3930. This indicated that, the proposed models is more suitable to model liver cancer dataset. The results for the best-fitted survival regression model shows that, all the exposure variables are statistically significant at 0.05 percent level of significant. While log-STLE-WSR also shows that, all the exposure variables are significant except for age group below 35 and age group above 30-50 years.

4.2 Cox-Snell Residuals Analysis for Liver Cancer

An essential step after constructing a survival regression model is analyzing residuals to assess the adequacy of the model. This involves comparing actual survival outcomes with those predicted by the model. The research focuses on the Cox-Snell residual method, introduced by [20], which is calculated as the negative natural logarithm of the estimated survival probability, comparing observed and estimated survival times. More precisely, it is defined as the negative natural logarithm of the estimated survival probability [21], as outlined below:

$$e_i = -\log[S(y_i \mid x_i)], \quad i = 1, 2, \ldots, n$$

where $S(t_i)$ is the survival function obtained from the fitted survival regression models. The Cox-Snell residuals of the proposed log-STIITL-W survival regression models is as follows;

$$e_i = -\log\left[1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-e^{\left(\frac{y-x_i^T\pi}{\rho}\right)}}\right]^2\right]^{\alpha}\right]\right\}\right], \quad i = 1, 2, \ldots, n \tag{34}$$

If the fitted models are adequate, the residuals will follow standard exponential distribution.

Table 4: Parameter estimates, log likelihood, AIC of log-STIITL-Weibull and TLG-Weibull Regression Models

Parameters	**log-STIITL-WSR**		**log -TLG-WSR**	
α	15.6767	0.1804	16.5597	0.5339
σ	2.0363	<0.001	7.7708	0.3131
θ	-	-	4.0132	0.4222
β_1	6.5146	<0.001	8.7767	0.0754
β_2	0.2809	0.3536	-2.1205	0.1948
β_3	-0.1229	0.7231	-2.5233	0.1399
β_4	-0.7629	0.0761	-1.3923	0.5627
β_5	-0.6029	0.2984	-1.9509	0.4650
β_6	1.0076	6.0076	-14.8104	<0.001

β_7	1.1176	0.0221	-14.2886	<0.001
β_8	-4.1206	<0.001	1.0647	0.5102
β_9	-3.6304	<0.001	-1.9509	0.4650
β_{10}	-3.5428	<0.001	0.9010	0.6269
LL	**-112.5062**		**120.1437**	
AIC	**138.5062**		**148.1437**	

The Cox-Snell residuals analysis evaluates the fit of the regression models by comparing the observed survival data with the expected survival data under the assumed model. The proposed model and the competing model were assessed and the results revealed that, the proposed STIITL-WSR yielded a lower AIC value (138.51) compared to the log-TLG-Weibull model (148.14), indicating that the log-STIITL-Weibull model provides a better fit to the liver cancer survival data. Significant parameter estimates, particularly for variables $\beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}$, in the log-STIITL-WSR model, suggest these variables strongly influence survival time.

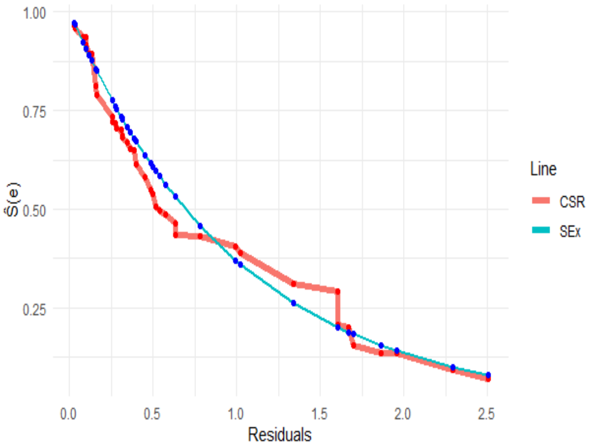

Figure 8: Cox Residual plot of STIITL-WSR

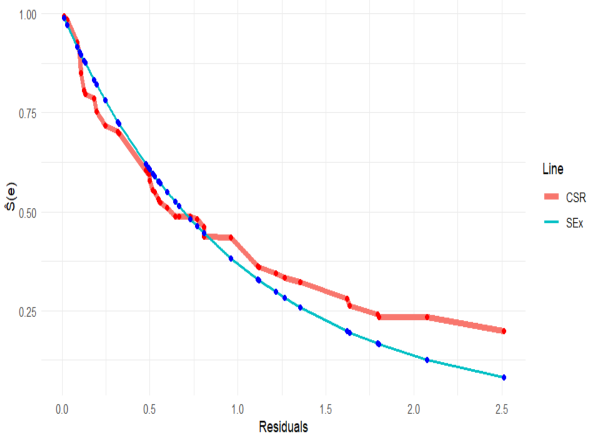

Figure 9: Cox Residual plot of TLG-WSR

Using the Kolmogorov-Smirnov one-sample test, the D_n statistic yields values of 0.0972, and 0.117 for the log-STIITL-WSR and log-TLG-WSR models, respectively. These values are less than the table value (0.215) at 0.05 significance level. This suggests that the Cox-Snell residuals for these models follow the standard exponential distribution. Correspondingly, Figures 8, and 9 exhibit Cox-Snell residuals plots for the log-STIITL-WSR and log-TLG-WSR models respectively. From these plots, it is evident that the estimated standard exponential curve closely aligns with the theoretical survival curve both each model.

4.3 Liver Cancer Kaplan-Meier Survival Probability

The results of the Kaplan-Meier survival probability analysis for the liver cancer dataset are represented in table 5 while the 95% confidence interval for the survival probability is shown in figure 10. The proposed and the competing models are also plotted with the non-parametric Kaplan-Meier estimator in figure 11 and 12.

Table 5: Kaplan Meier Survival Probabilities for the Liver Cancer Data Set

Time (in week)	No at risk	No of event	Survival prob.	SE	95% LCI	95% UCI
2	40	2	0.950	0.0345	0.8848	1.000
6	38	4	0.850	0.0565	0.7462	0.968
7	34	2	0.800	0.0632	0.6852	0.934
9	31	1	0.774	0.0663	0.6546	0.916
11	30	3	0.697	0.0732	0.5672	0.856
13	25	1	0.669	0.0754	0.5364	0.834
14	23	1	0.640	0.0775	0.5046	0.811

15	22	1	0.611	0.0792	0.4736	0.788
16	21	1	0.582	0.0806	0.4433	0.763
17	19	2	0.520	0.0830	0.3808	0.711
18	17	1	0.490	0.0835	0.3506	0.684
19	16	2	0.429	0.0836	0.2925	0.628
37	12	1	0.393	0.0839	0.2585	0.597
41	11	2	0.321	0.0834	0.1944	0.531
51	8	1	0.281	0.0813	0.1596	0.496
52	7	1	0.241	0.0790	0.1268	0.458
67	4	2	0.121	0.0721	0.0373	0.389
80	1	1	0.000			

Table 5 represents Kaplan-Meier survival probabilities for liver cancer patients over time, showing how survival rates decrease as time progresses. It can be observed that, as more events (deaths) occur over time, the survival probability continues to decline. The table shows how survival trend and uncertainty (captured by SE and CIs) at various time points.

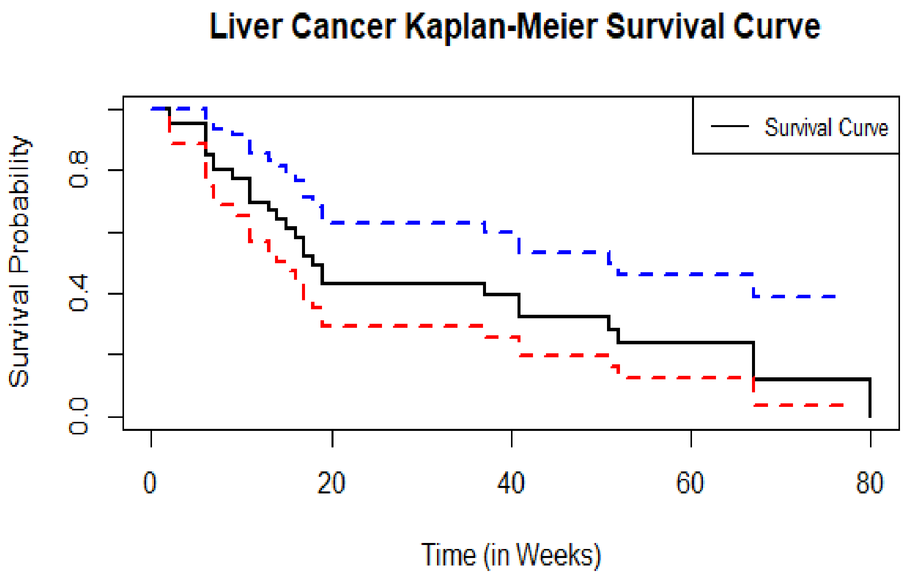

Figure 10: Kaplan Meier Survival Curve for the Liver Cancer Data

The exposure variable had improved the survival probabilities for these patients who experience the event (died) before week twenty, but worsened their survival probability. The plot in figure 10 shows the best fitted survival regression model together with the non-parametric Kaplan-Meier survival probability. The effect of the covariate plays a significant role in describing the distribution of the event time and the existence of the association between the explanatory variables and the event time distributions. This made the best fitted model more flexible compared to the Kaplan-Meier survival probability model.

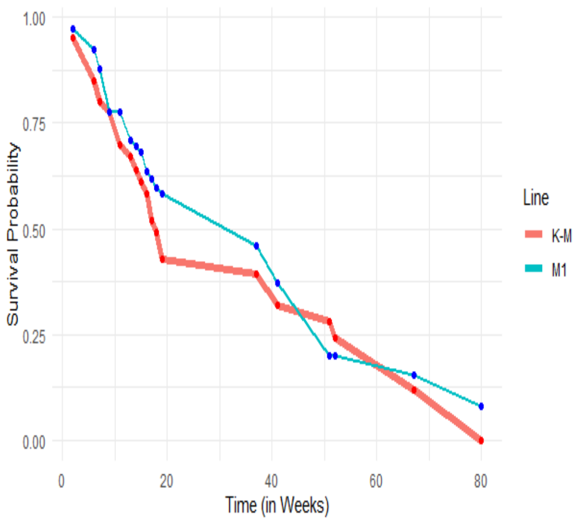

Figure 11: K-M and log-STIITL-WSR survival curve

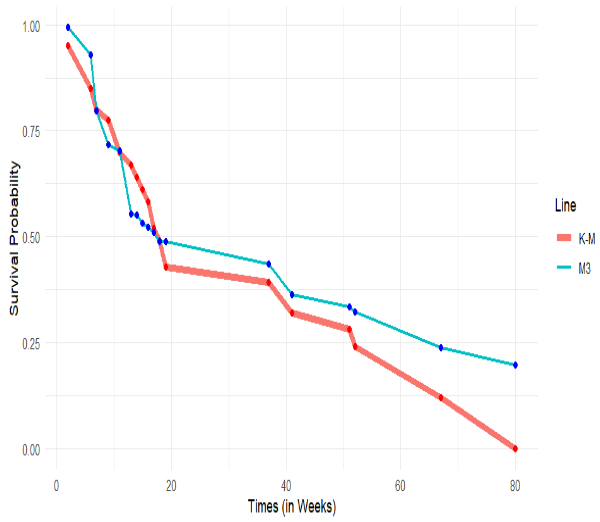

Figure 12: K-M and log-TLG-WSR survival curve

5. Conclusion

In this study, the STIITL-Weibull distribution was developed and fitted to the transect stake distance measurements obtained from [21] Patil and Rao (1994). Maximum likelihood estimation (MLE) was used to estimate the parameters of the proposed model, and its goodness-of-fit was evaluated using log-likelihood and AIC. The STIITL-Weibull distribution achieved the lowest AIC value (376.162) compared to other competing models, such as the TIITLW, TLMW, W, GW, and TW models, indicating that the STIITL-Weibull model provides the best fit for the transect stake distance data.

Furthermore, the STIITL-W was transformed into log-STIITL-WSR and was also compared with the log-TLG-Weibull regression model based on their AIC values. The log-STIITL-WSR yielded a lower AIC (128.5833) compared to the log-TLG-WSR (133.3930), confirming that it is a more suitable model for analyzing liver cancer dataset. Also, significant parameter estimates were observed for all exposure variables in the log-STIITL-WSR, indicating strong relationships between the predictors and the outcome variable.

Cox-Snell residuals analysis was conducted to further assess the fit of the regression models. The analysis revealed that the log-STIITL-WSR model outperformed the log-TLG-Weibull model, with a lower AIC value of 138.51 as compared to 148.14. The significant parameters in the proposed model (specifically, $\beta_6, \beta_7, \beta_8, \beta_9$, and β_{10}) demonstrated that these variables have significant influence on liver cancer patients survival time. Therefore, the results demonstrate that the STIITL-Weibull distribution and its associated survival regression model provide a robust and reliable framework for modelling transect stake distance measurements, offering improved accuracy compared to existing models. This study contributes to the field by highlighting the effectiveness of the STIITL-Weibull distribution in real-world data modelling and encourages further applications of the model in similar datasets.

References

[1] Klakattawi, H., Alsulami, D., Elaal, M. A., Dey, S., & Baharith, L. (2023). Correction: A new generalized family of distributions based on combining Marshal-Olkin transformation with T-X family. *PLoS ONE*, *18*(10), e0293100.

[2] Gabanakgosi, M., & Otlaadisa, M. (2024). A new family of generalized-G distributions with properties and applications. *Scientific African*, *26*, e02344.

[3] Nanga, S., Sayibu, S. B., Angbing, I. D., Alhassan, M., Benson, A.-M., Abubakari, A. G., Nasiru, S., & Suleman, S. (2024). Secant Kumaraswamy family of distributions: Properties, regression model, and

applications. *Computational and Mathematical Methods*, 8925329.

[4] Kumar, D., Singh, U., & Singh, S. K. (2015). A new distribution using sine function: Its application to bladder cancer patients' data. *Journal of Statistics Applications & Probability, 4*(3), 417-425.

[5] Isa, A. M., Bashiru, S. O., Ali, B. A., Adepoju, A. A., & Itopa, I. I. (2022). Sine-exponential distribution: Its mathematical properties and application to real dataset. *UMYU Scientifica*, *1*(1), 127-131.

[6] Mustapha B. A., Isa A. M., Sule O. B., & Itopa I. I. (2023). Sine-Lomax distribution: properties and applications to real data sets. *FUDMA Journal of Sciences*, *7*(4), 60 - 66.

[7] Isa, A. M., Ali, B. A., & Zannah, U. (2022). Sine burr XII distribution: Properties and application to real data sets. *Arid. Zone J. Basic Appl. Res*, *1*, 48-58.

[8] Faruk, M. U., Isa, A. M., & Kaigama, A. (2024). Sine-Weibull Distribution: Mathematical Properties and Application to Real Datasets. *Reliability: Theory & Applications*, *19*(1), 65-72.

[9] Isa, A. M., Doguwa, S. I., Alhaji, B. B., & Dikko, H. G. (2024). Sine Type II Topp-Leone Burr XII Distribution: Mathematical Properties and Application to Biomedical and Environmental Data. *Iraqi Statisticians journal*, 30-40.

[10] Adepoju, A. A., Isa, A. M., & Bello, O. A. (2024). Cosine Marshal-Olkin-G Family of Distribution: Properties and Applications. *Reliability: Theory & Applications*, *19*(3), 408-422.

[11] Isa, A. M., Khalaf, A. A., & Bashiru, S. O. (2024). Some Properties of the Cosine Lomax Distribution with Applications. *Iraqi Journal for Applied Science 1*(2), 1-11.

[12] Isa, A. M., Doguwa, S. I., Alhaji, B. B., & Dikko, H. G. (2023). Sine Type II Topp-Leone G Family of Probability Distribution: Mathematical Properties and Application. *Arid Zone Journal of Basic and Applied Research*, *2*(4), 124-138.

[13] Elgarhy, M., Arslan Nasir, M., Jamal, F., & Ozel, G. (2018). The type II Topp-Leone generated family of distributions: Properties and applications. *Journal of Statistics and Management Systems, 21*(8), 1529-1551.

[14] Tuoyo, D. O., Opone, F. C., & Ekhosuehi, N. (2021). The Topp-Leone Weibull distribution: Its properties and application. Earthline *Journal of Mathematical Sciences*, *7*(2), 381-401.

[15] Alyami, S. A., Elbatal, I., Alotaibi, N., Almetwally, E. M., Okasha, H. M., & Elgarhy, M. (2022). Topp–Leone modified Weibull model: Theory and applications to medical and engineering data. *Applied Sciences*, *12*(20), 10431.

[16] Wu, X., Ahmad, Z., Hussam, E., Alhelali, M. H., Aldallal, R., Almuqrin, M. A., & Riad, F. H. (2023). A new Cosine-Weibull model: Distributional properties with applications to basketball and medical sectors. *Alexandria Engineering Journal, 66*, 751-767.

[17] Alshanbari, H. M., Ahmad, Z., El-Bagoury, A. A.-A. H., Odhah, O. H., & Rao, G. S. (2024). A new modification of the Weibull distribution: Model, theory, and analyzing engineering data sets. *Symmetry*, *16*, 611.

[18] Patil, G. P. & Rao, C. R. (1994). Handbook of Statistics 12: Environmental Statistics; Elsevier Science: *Amsterdam, the Netherlands. Statistics in medicine*, *34*(8), 1366-1388.

[19] Isa, A. M., Doguwa, S. I., Alhaji, B. B., & Dikko, H. G. (2024). Sine-Topp-Leone Exponentiated-G Family of Distributions: Properties, Survival Regression and Application. *Reliability: Theory & Applications*, *19*(3), 157-172.

[20] Cox, D. R., & Snell, E. J. (1968). A general definition of residuals. *Journal of the Royal Statistical Society: Series B (Methodological), 30*(2), 248-265.

[21] Kleinbaum, D. G., & Klein, M. (2012). Survival analysis: A self-learning text (3rd ed.). Springer Science & Business Media.