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Stability Analysis of Threshold Autoregressive Distributed Lag (TARDL(1,1,1)) Models Using Dynamical Approach: Numerical Examples and Practical Application

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ABSTRACT

This research aims to study the stability condition of the Threshold Autoregressive Distributed Lag [TARDL(1,1,1)] model using dynamic Approach. The limit cycle condition was applied to the model, where the stability of the model was analysed through differential equations that reflect oscillations and cycles, firstly will prove and present the limit cycle condition for this model. Two examples were presented: the first one satisfies the stability of limit cycle and stabilizes at a constant value, while the second one does at satisfy the condition and continuous to oscillate. MATLAB programing was used to plot the model trajectories and explain whether it is stable or not. The results showed that the model stabilizes when the condition is satisfied, while not satisfying it leads to continuous oscillations. Finally a practical application was conducted on real data to determine the stability or not of the models at the limit cycle.

1. Introduction

With the rapid developments in the fields of time series analysis, nonlinear models have become pivotal in understanding the complex relationships between economic and financial variables. Among these models, the Threshold Autoregressive Distributed Lag [TARDL(1,1,1)] model stands out as powerful tool for capturing nonlinear behaviour in data, especially when there are sudden changes or critical thresholds that affect the relationship between variables.

The threshold models were first developed by Tong (1978), who introduced the idea of using thresholds to represent sudden changes in time series. Since then, these models have been greatly developed, especially in field of econometric, there they have been used to analyse nonlinear relationships between economic variables [1]. Threshold models have been used in many studies, such as the relationships between inflation and economic growth, or between interest rates and investment. For examples, Hansen 1999 analysed the effect of thresholds in economic models and how they can be used to analyse nonlinear data [2], there are many studies that have employed this model, including: used to study the impact of regulation on innovation in telecom [3], applied to hypergraph to analyse the complex dynamics of diffusion [4], used to select target groups in social networks with effect [5], and employed to estimate and analyze internal thresholds in energy prices [6].

The TARDL(1,1,1) model is an extension of the classic ARDL model, incorporating multiple thresholds that allow the model to adapt to changes in the economic or financial system. This model is able to represent nonlinear relationships between

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variables, making it suitable for analysing data affected by changing conditions, such as changes in economic policies or financial crises [7].

Ozaki developed a dynamical method for analysing the stability of models using local linearization approximation. This method is based on the analysis of oscillations and cycle in dynamic models (transform the nonlinear system to function with constant coefficients), where the nonlinear model is approximated by linear differential equation to facilitate the analysis. This method has been used in many studies to stability of many time series models [8], and then many developments were presented on this method such as Tong 1990 who discussed the applications of the dynamical approach in the analysis of nonlinear time series [9], and there are many other models that follow the same methodology as example Logistic Autoregressive model [10], Cauchy Autoregressive model [11], GARCH models [12], exponential (GARCH) models [13], GJR-GARCH (Q, P) [14], Exponential Double Autoregressive model [15], Logarithmic Double Autoregressive [16], Smooth Transition Hyperbolic Tangent Autoregressive [17], and Gompertz Autoregressive model [18].

One of the important aspects in the analysis of nonlinear models is studding their stability, especially in the presence of oscillations or cycle. Here comes the importance of the limit cycle condition, which determines whether the model will stabilize at a constant value or will continue to oscillate. This condition was developed using the dynamic method presented by Ozaki, which relies on approximating nonlinear models with linear differential equations to facilitate stability analysis.

This paper aims to study the stability of TARDL (1,1,1) model using the dynamic method, focusing on the analysis of the limit cycle condition. Also will apply the model to fictitious data to illustrate how the condition is met or not, and will use MATLAB to draw the model paths and analyse the oscillations. The results of this paper will contribute to deeping our understanding of the stability of nonlinear models and provide new insights into their applications in time series.

The presented study including five parts. The first part consists of the introduction, which included the importance and objective the model of the study. The second part included the model used and method used, the third part included proof of the stability condition of limit cycle, the fourth part included numerical examples that illustrate the application of the stability condition and a practical application was conducted on real data to determine the stability or not of the models at the limit cycle, and the last part included an interpretation of the results of the stability conditions.

2. The model and method

2.1 TARDAL(p,q)

Let $\{y_t\}$ is a time series with $t = 1,2,\dots,n$, and n is the number of observation, The ARDL model can be expressed by the following equation [19-20]:

$$y_t = \theta + \sum_{i=1}^{p} \lambda_i y_{t-i} + \sum_{j=1}^{q} \gamma_j x_{t-j} + \varepsilon_t \quad (1)$$

Where θ is the constant term, ε_t is the random error term $\varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2)$, λ_i and γ_j are the long-run and short-run relationship parameters, respectively, and p and q represent the lag periods or the order of the model for the variables y_t and x_t, respectively.

As a development of the ARDL model in Equation 1, which is incorporates regime-switching behaviour, also allows for different dynamics across regimes, and Captures nonlinear relationships, the threshold model of ARDL was introduced, which can be expressed by the equation [21]:

$$y_t = \begin{cases} \theta_1 + \sum_{i=1}^{p} \lambda_{i\tau} y_{t-i} + \sum_{j=1}^{q} \gamma_{j1} x_{t-j} + \varepsilon_{t1} \\ \theta_2 + \sum_{i=1}^{p} \lambda_{i\tau} y_{t-i} + \sum_{j=1}^{q} \gamma_{j2} x_{t-j} + \varepsilon_{t2} \\ \vdots \\ \theta_\tau + \sum_{i=1}^{p} \lambda_{i\tau} y_{t-i} + \sum_{j=1}^{q} \gamma_{j\tau} x_{t-j} + \varepsilon_{t\tau} \end{cases} \quad (2)$$

Where τ is the number of threshold value.

Or Multiple Threshold ARDL Model by form [22]:

$$y_t = \sum_{k=1}^{m+1}\left[\theta_k + \sum_{i=1}^{p}\lambda_{ki}y_{t-i} + \sum_{j=1}^{q}\gamma_{kj}x_{t-j}\right] \times I_{(m_{k-1}<z_t\leq m_k)} + \varepsilon_t \tag{3}$$

Where ε_t is the threshold value, m is the number of threshold value, and $I_{(*)}$ is indicator function.

In order to find the stability condition of the model TARDL(1,p,q) in terms of the model parameters and using the local linear approximation technique by using equation (3), we first look at the model formula:

$$y_t = \left[\theta_1 + \sum_{i=1}^{p}\lambda_{1i}y_{t-i} + \sum_{j=1}^{q}\gamma_{1j}x_{t-j}\right] \times I_{(\tau_0<z_t\leq\tau_1)} + \left[\theta_2 + \sum_{i=1}^{p}\lambda_{2i}y_{t-i} + \sum_{j=1}^{q}\gamma_{2j}x_{t-j}\right] \times I_{(\tau_1<z_t\leq\tau_2)} + \varepsilon_t \tag{4}$$

And ε_t is the white noise such that $\varepsilon_t \sim iid\ N(0,1)$ but when the model is of the first order TARDL(1,1,1):

$$y_t = [\theta_1 + \lambda_{11}y_{t-1} + \gamma_{11}x_{t-1}] \times I_{(\tau_0<z_t\leq\tau_1)} + [\theta_2 + \lambda_{21}y_{t-1} + \gamma_{21}x_{t-1}] \times I_{(\tau_1<z_t\leq\tau_2)} + \varepsilon_t \tag{5}$$

Since the indicator function takes values $I = \{0,1\}$, we adopt the model formula as:

$$y_t = [\theta_1 + \lambda_{11}y_{t-1} + \gamma_{11}x_{t-1}] \times I_{(\tau_0<z_t\leq\tau_1)} + [\theta_2 + \lambda_{21}y_{t-1} + \gamma_{21}x_{t-1}] \times \left(1 - I_{(\tau_1<z_t\leq\tau_2)}\right) + \varepsilon_t \tag{6}$$

2.2 Dynamical Approach

The concepts of stability of dynamical systems consisting of a system of differential equations is that system is stable when the solution path approaches a single point or a closed trajectory called limit cycle. In this case that the time series model is nonlinear, this model contains a singular zero points and non-singular zero points ξ, so stability in this case in achieved when the solution path approaches to this points wherever value of time t increasing without limit, i.e. when $t \to \infty$. However, if the non-zero singular point is unstable for the time series model, it moves on to finding the stability of the stability of model's limit cycle. The most important stability techniques for dynamical systems used to approximate for dynamical systems used to approximate nonlinear systems to local linear systems near the singular point representing the fixed solution for systems is a local linear approximation technique, which is a method followed in this study. In order to give a clear impression of work this technique, assume that we have a nonlinear differential equation and it is converted to a linear differential equation according following steps:

1. Reducing the order of differential equation.
2. Writing the equation in the form of state space.
3. Find the fixed point and the fixed solution for system.

The following chart showing the stages of this technique.

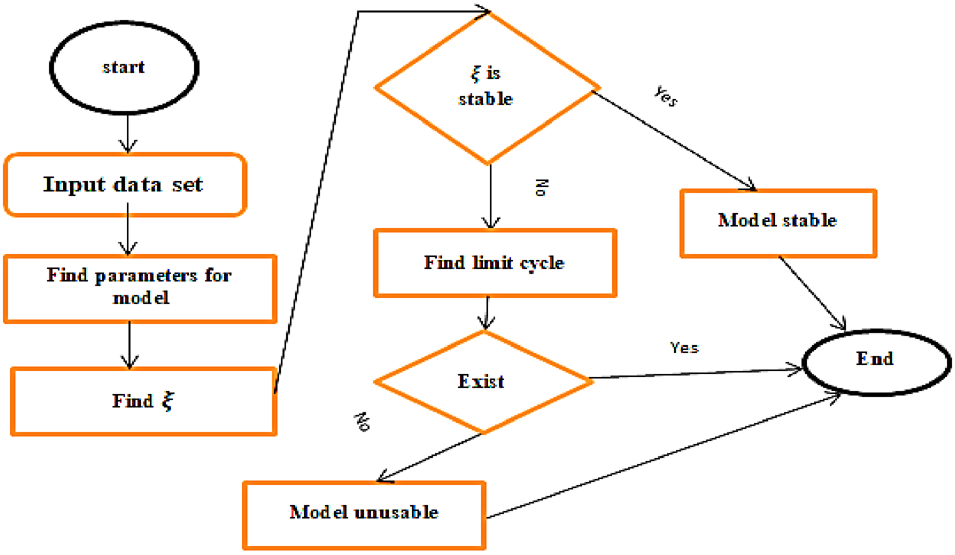


Figure 1: stages of applied the dynamic technique

The above diagram represents the stages of applying the dynamic technique, and describes the process of evaluating the stability of the model based on the data set used and the value of non-zero singular point ξ if it stable, then the model is considered stable, and the process is terminated. If not, a limit cycle is searched for. If it exist, the model is stable, and the processes is terminated. If it doesn't exist, the model is unstable. This diagram defines the stages of evaluating the stability of the dynamic model and decides whether it can be used or not based on the stability of parameters and the detection of limit cycle.

3. Limit cycle stability condition of TARDL(1,1,1) model

Proposition: If the TARDAL(1,1,1) model has a terminal period at period $d > 0$, then that terminal period is orbitally stable if:

$$\left|\frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2}\right|^{d} < 1$$

The proof*:*

Assume that the model has an end cycle at cycle d called

$$y_t, y_{t-1}, y_{t-2}, \dots, y_{t-k} = y_t \qquad d > 1$$

Nearby to each point of the terminus with a small enough radius ξ_t such that $\xi_t^n \to 0$ for every $n \geq 2$ by setting $E(y_t) = y_t + \xi_t$, $E(y_{t-1}) = y_{t-1} + \xi_{t-1}$, $E(x_{t-1}) = y_{t-1} + \xi_{t-1}$

The TARDAL(1,1,1) model is given by:

$$y_t = [\theta_1 + \lambda_{11} y_{t-1} + \gamma_{11} x_{t-1}] \times I_{(\tau_0 < z_t \leq \tau_1)} + [\theta_2 + \lambda_{21} y_{t-1} + \gamma_{21} x_{t-1}] \times \left(1 - I_{(\tau_1 < z_t \leq \tau_2)}\right) + \epsilon_t \tag{7}$$

$E(\epsilon_t) = 0$

$$E(y_t) = [\theta_1 + \lambda_{11} E(y_{t-1}) + \gamma_{11} E(x_{t-1})] \times E\left(I_{(\tau_0 < z_t \leq \tau_1)}\right) + [\theta_2 + \lambda_{21} E(y_{t-1}) + \gamma_{21} E(x_{t-1})] \times \left(1 - E\left(I_{(\tau_0 < z_t \leq \tau_1)}\right)\right) + E(\epsilon_t) \tag{8}$$

$$\mathrm{E}\left(I_{\tau_0 < z_t \leq \tau_1}\right) = \int_{-\infty}^{\infty} I_{\tau_0 < z_t \leq \tau_1} f(z_t) dz_t$$

$$= \int_{-\infty}^{0} 1\, f(z_t)dz_t + \int_{0}^{\infty} 0\, f(z_t)dz_t$$

$$= \frac{1}{2} + 0 = \frac{1}{2}$$

$$y_t + \xi_t = \frac{\theta_1 + \lambda_{11}(y_{t-1} + \xi_{t-1}) + \gamma_{11}(y_{t-1} + \xi_{t-1}}{2} + \frac{\theta_2 + \lambda_{21}(y_{t-1}y_{t-1} + \xi_{t-1}) + \gamma_{21}(y_{t-1} +}{2}$$

$$2y_t + 2\xi_t = \theta_1 + \lambda_{11}(y_{t-1} + \xi_{t-1}) + \gamma_{11}(y_{t-1} + \xi_{t-1}) + \theta_2 + \lambda_{21}(y_{t-1}y_{t-1} + \xi_{t-1}) + \gamma_{21}(y_{t-1} + \xi_{t-1})$$

$$2y_t + 2\xi_t = \theta_1 + \lambda_{11}y_{t-1} + \lambda_{11}\xi_{t-1} + \gamma_{11}y_{t-1} + \gamma_{11}\xi_{t-1} + \theta_2 + \lambda_{21}y_{t-1} + \lambda_{21}\xi_{t-1} + \gamma_{21}y_{t-1} + \gamma_{21}\xi_{t-1}$$

Since $y_t = \frac{\theta_1+\theta_2}{2-(\lambda_{11}+\gamma_{11}+\lambda_{21}+\gamma_{21})y_{t-1}}$

Then

$$2\xi_t = (\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21})\xi_{t-1} \tag{9}$$

If the solution is convergent, then the terminal period is orbitally stable if [15]:

$$\left|\frac{\xi_{t+d}}{\xi_t}\right| < 1 \tag{10}$$

After repeating equation (10) for d times, we get:

$$\xi_{t+1} = \frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2} \cdot \xi_t$$

$$\xi_{t+d} = \frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2} \cdot \xi_{t+d-1}$$

$$\xi_{t+d} = \underbrace{\frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2} \cdot \frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2} \cdots \frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2}}_{d-times} \cdot \xi_t$$

$$\xi_{t+d} = \prod_{j=1}^{d} \frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2} \xi_t$$

$$\frac{\xi_{t+d}}{\xi_t} = \prod_{j=1}^{d} \left(\frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2}\right)^j$$

According to the convergence condition shown in relationship

$$\mu^1 - \left(\frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2}\right)\mu^{1-\mathrm{i}} = 0$$

The final cycle of d cycles is orbitally stable if the previous condition is met and the model is orbitally stable such that:

$$\left|\frac{\xi_{t+k}}{\xi_t}\right| = \left|\prod_{j=1}^{d} \left(\frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2}\right)^j\right| < 1$$

$$\therefore \left(\frac{\lambda_{11} + \gamma_{11} + \lambda_{21} + \gamma_{21}}{2}\right)^d < 1 . \blacksquare \tag{11}$$

To apply the stability condition proven above, examples are given to illustrate the stability condition at the orbitally stable (limit cycle) of the model's as follows:

4. Numerical Examples

In this section, two data series generated using simulation are divided into two parts. The simulation in this section aims to generate data generated following TARDL model to test the stability properties and analyse the effect of different thresholds on the temporal relationships. This is done through several steps that include generating values for the independent and dependent variables, adding random noise, and tuning the system based on the chosen threshold.

Example 1. If we take the number of observation 400 and $m = 0.7$, then such that figure 2, show the plot of dependent variable y_t, independent variable x_t, and threshold variable z_t.

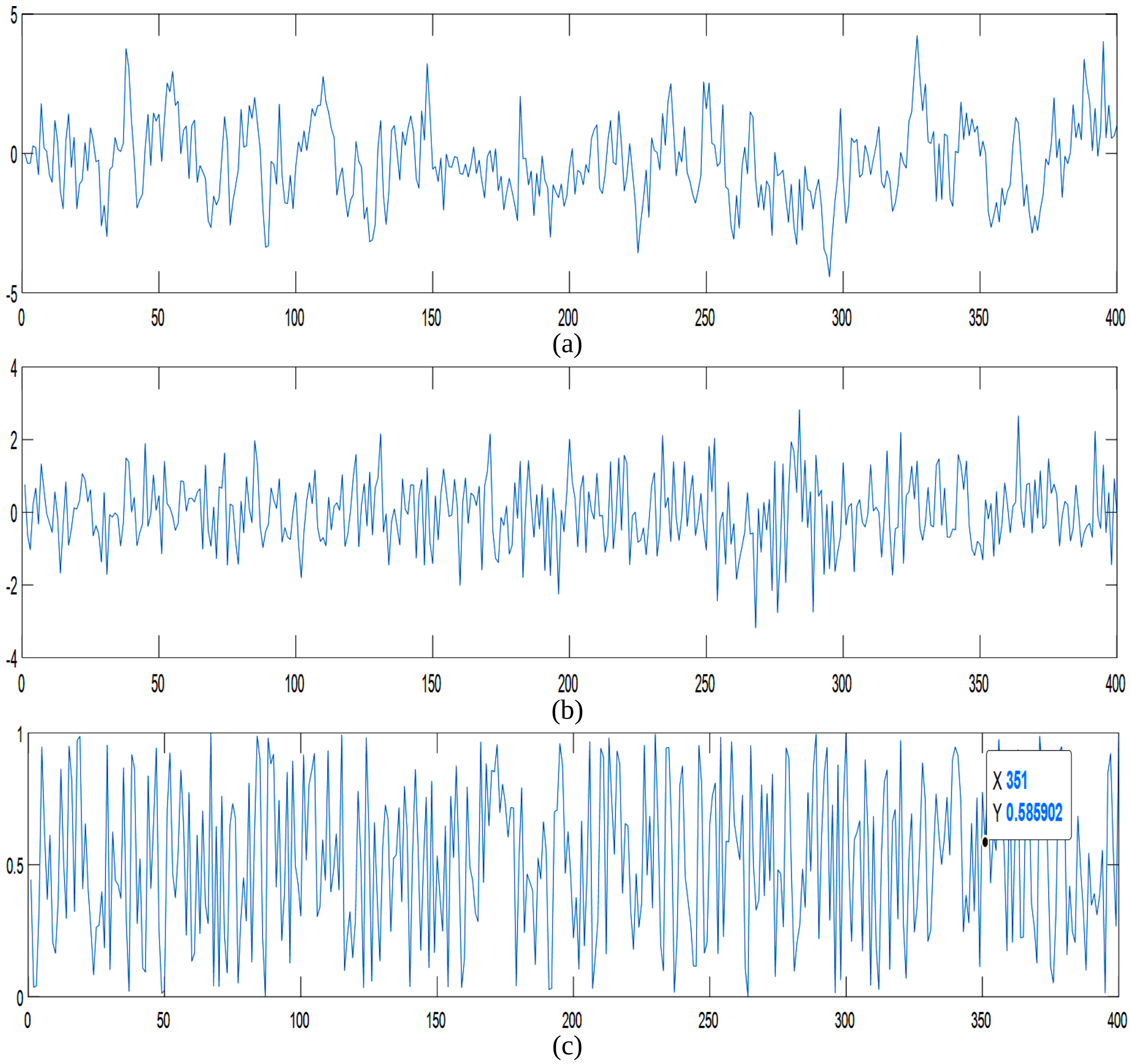


Figure 2: plot (a) dependent variable y_t, (b) independent variable x_t, (c) threshold variable z_t at 400 observation

For this series the value of parameters were: $\theta_1 = 0.5, \theta_2 = 0.5, \lambda_{11} = 0.3, \lambda_{21} = 0.2$, $\gamma_{11} = 0.2, \gamma_{21} = 0.1$, with initial values for $y_0 = 0.5$, and $x_0 = 0.4$, with number of iteration equal 1000.

Using equation (7) and take $I = 0.5 + 0.3\sin(0.1t)$ is indicator function is an oscillating function to obtain high oscillations. Then the model has form:

$$y_1 = [0.5 + 0.3 \times 0.5 + 0.2 \times 0.4] \times I + [0.5 + 0.2 \times 0.5 + 0.1 \times 0.4] \times (1 - I)$$

$$y_1 = 0.73I + 0.64(1 - I)$$

Repeat the process to calculate $y_2, y_3, \ldots$

Now the stability condition is applied in equation 11, to get:

$$\therefore \left(\frac{0.3 + 0.2 + 0.2 + 0.1}{2}\right)^{1000} = 0.4^{1000}$$

$$= 1.6069 \times 10^{-602} < 1$$

Therefore, the stability condition is met, as shown in the following figure.

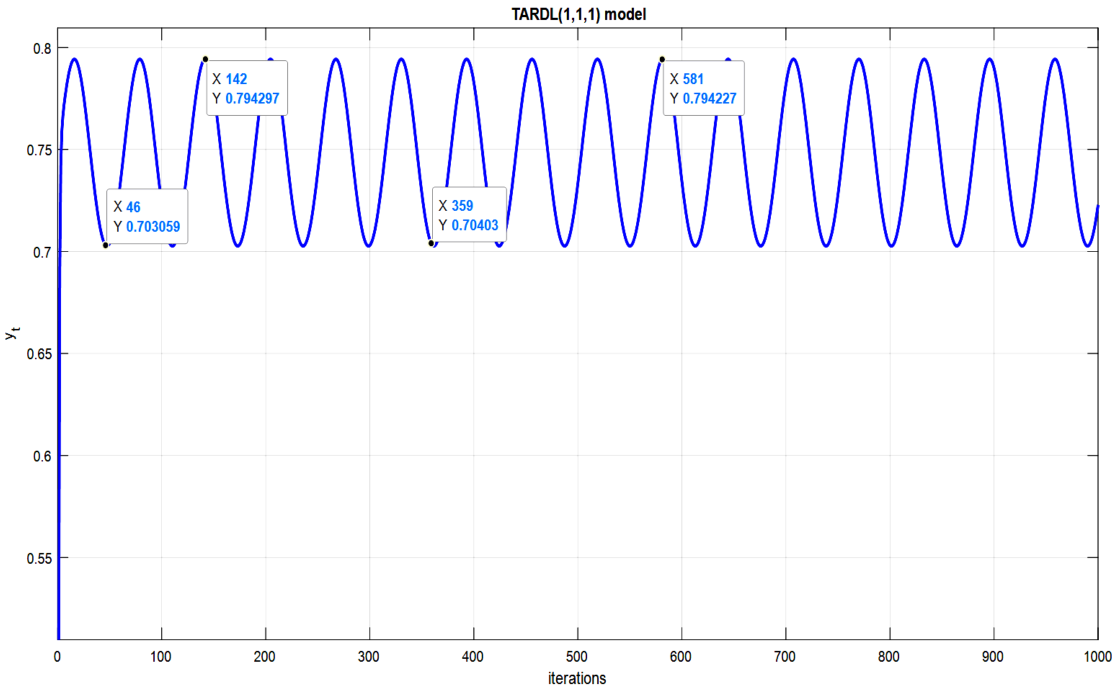
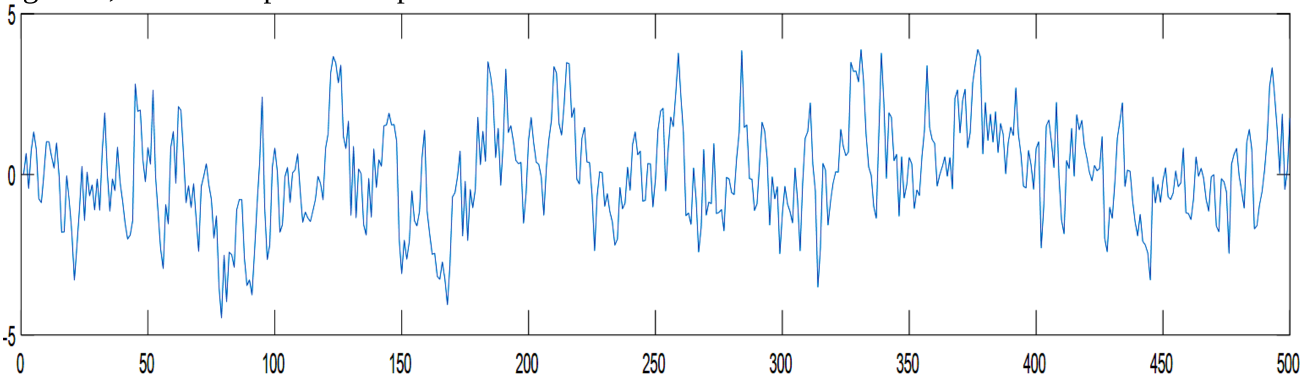
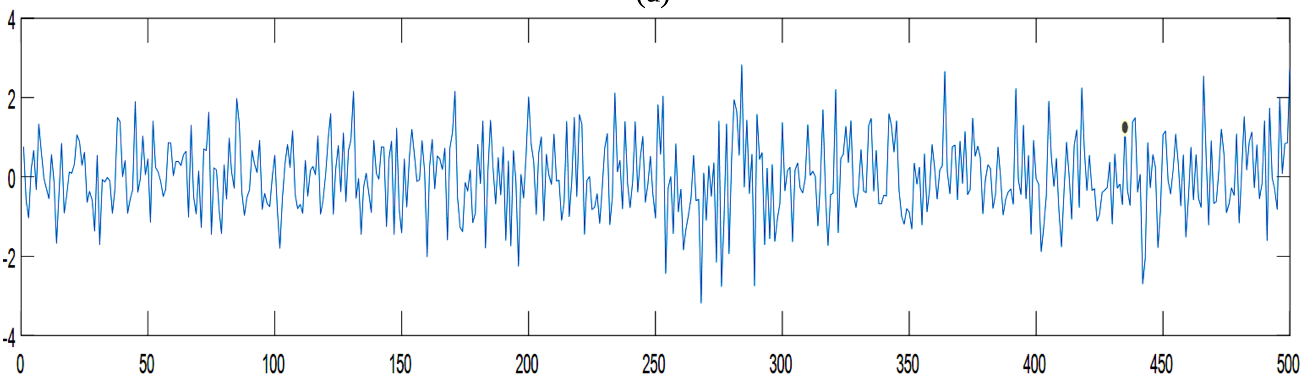


Figure 3. limit cycle trajectories are orbitally stable at initial values

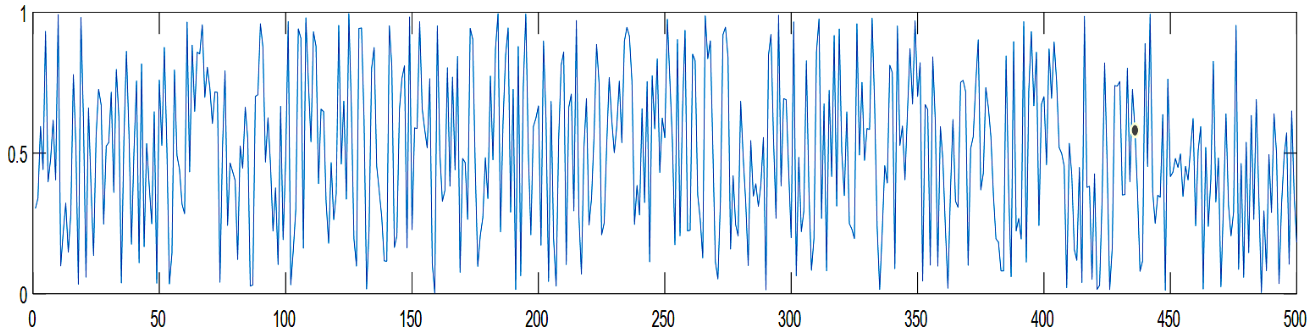
Example 2. If we take the number of observation 500 and $m = 0.9$, then such that figure 4, show the plot of dependent variable y_t, independent variable x_t, and threshold variable z_t.



(a)



(b)



(c)

Figure 4: plot (a) dependent variable y_t, (b) independent variable x_t, (c) threshold variable z_t at 500 observation

For this series the value of parameters were: $\theta_1 = 0.5, \theta_2 = 0.5, \lambda_{11} = 0.8, \lambda_{21} = 0.7$, $\gamma_{11} = 0.6, \gamma_{21} = 0.5$, with initial values for $y_0 = 1$, and $x_0 = 1$, with number of iteration equal 1000.

Using equation (7) and take $I = 0.5 + 0.4\sin(0.2t)$ is indicator function is an oscillating function to obtain high oscillations. Then the model has form:

$$y_1 = [0.5 + 0.8 \times 1 + 0.6 \times 1] \times I + [0.5 + 0.7 \times 1 + 0.5 \times 1] \times (1 - I)$$

$$y_1 = 1.9\, I + 1.7\,(1 - I)$$

Repeat the process to calculate $y_2, y_3, \ldots$

Now the stability condition is applied in equation 11, to get:

$$\therefore \left(\frac{0.8 + 0.6 + 0.7 + 0.5}{2}\right)^{1000} = 1.3^{1000} > 1$$

Therefore, the stability condition is not holding, as shown in the following figure.

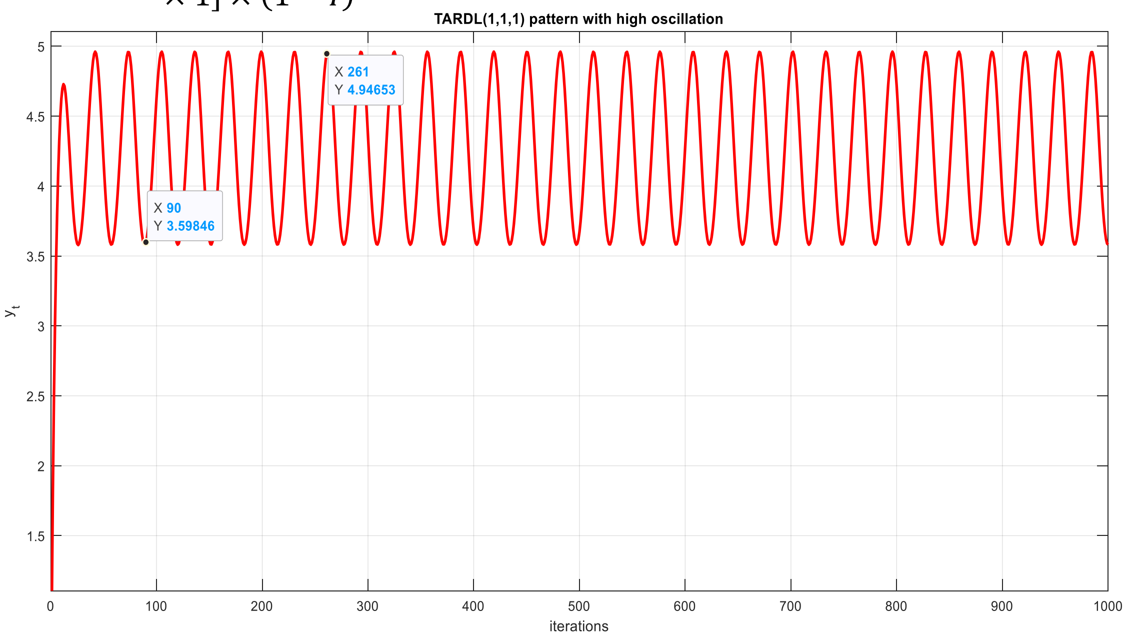


Figure 5. limit cycle trajectories are orbitally stable at initial values for TARDL(1,1,1) pattern with high oscillation

5. Practical Application

The data used in applied aspect of the study consisted of 401 observations of the weekly average silver price per ounce in dollars at the close of the weekly market for period (1-january 2015 to 18-septemper 2022). This data obtained from the website: https://sa.investing.com/commodities/silver

The data analysis process begins by creating a time series. We enter the data into Matlab programing and plot it. Figure 6 represent the time series graph.

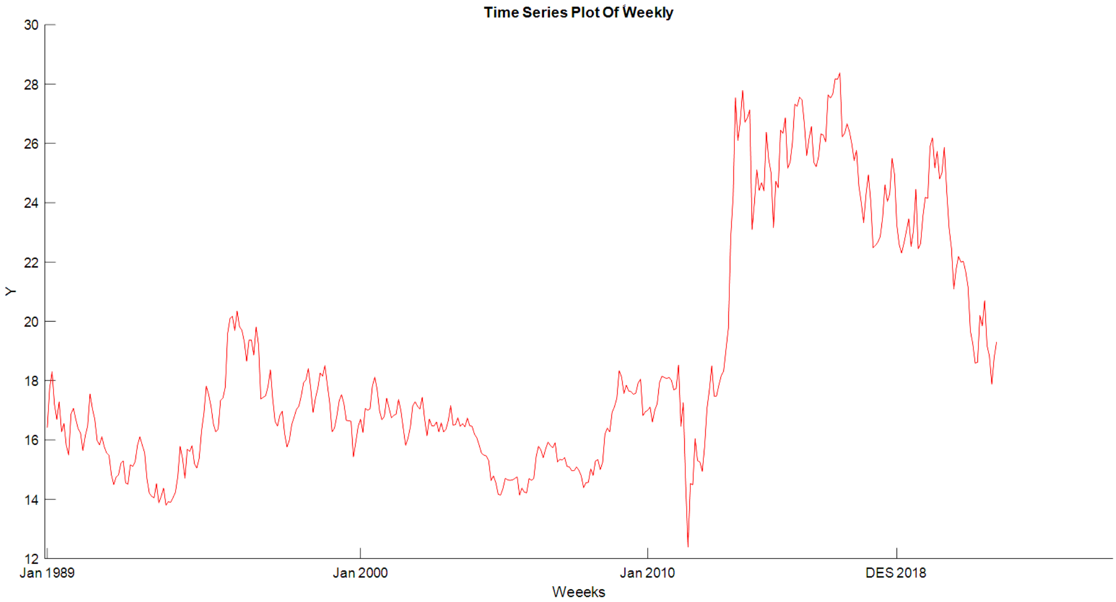


Figure 6. weekly average silver price per ounce in dollars at the close of the weekly market for period (1-january 2015 to 18-septemper 2022)

When observing the time series in above figure, it is noticeable that the series follows irregular components, meaning it's not defined by a specific pattern. This means that the series lacks a specific shape or trend.

By convert the series plotted above into return series, by conducting returns transform, where figure 7 represents the returns series graph, while figure 8 represents Total autocorrelation functions (ACF) and partial autocorrelation functions (PACF) for returns series, respectively:

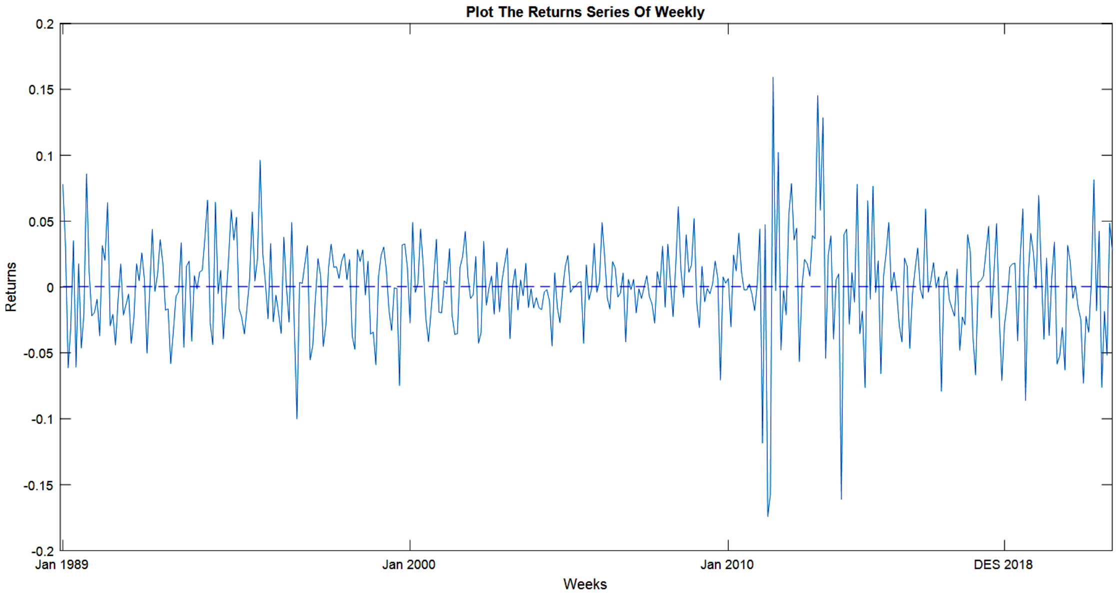


Figure 7. plot returns series

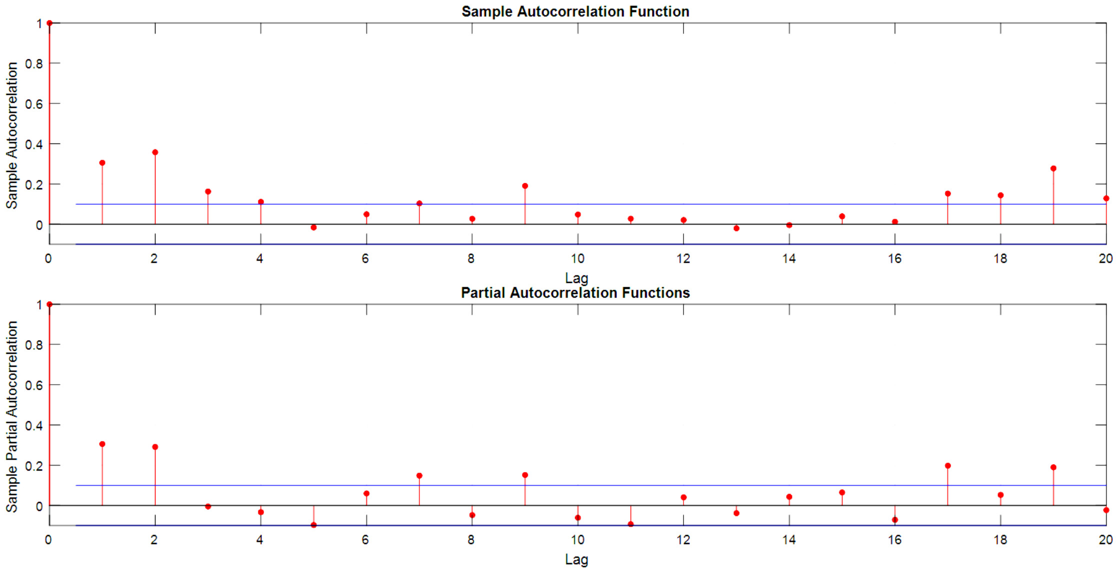


Figure 8. plot ACF and PACF for returns series

From figure 8 the ACF that there are lags outside the confidence intervals limits, which at $\text{lags} = 1, 2, 3, 4, 9, 17, 18, 19,$ and $20,$ while PACF are at $\text{lags} = 1, 2, 7, 4, 9, 17,$ and $19,$ which is greater than the acceptable percentage of 5%.

After that covert the returns series to an square error series, where figure 9 represents the square error series graph, while figure 10 represents ACF and PACF for error series, respectively:

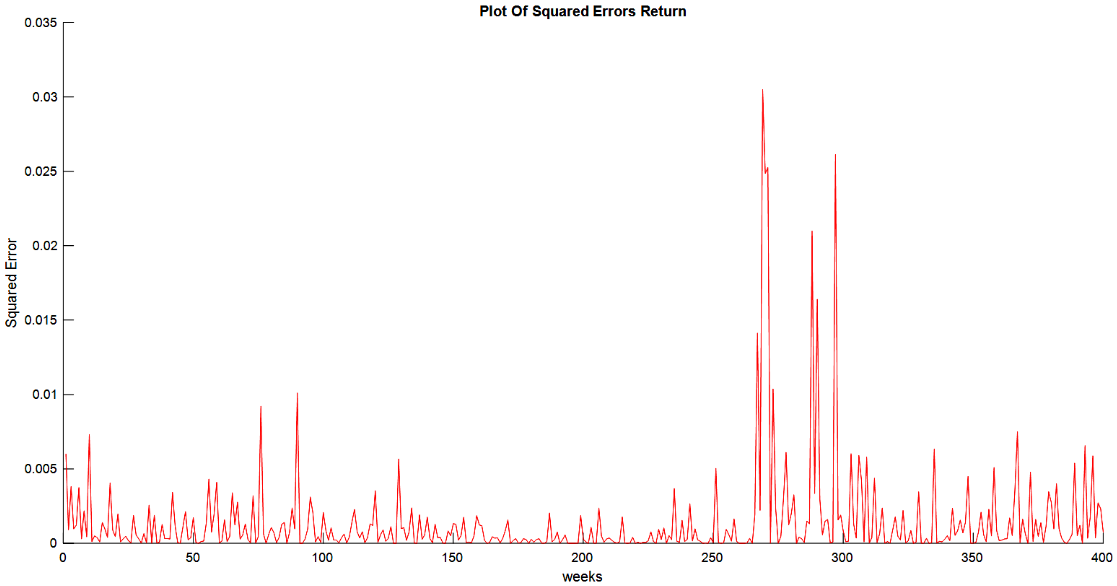
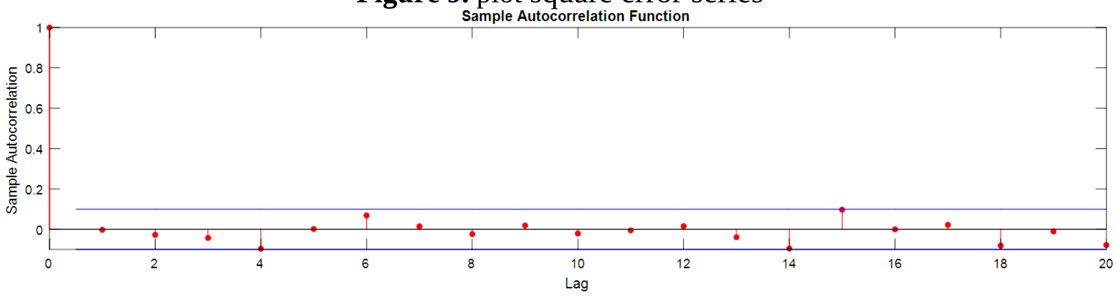


Figure 9. plot square error series



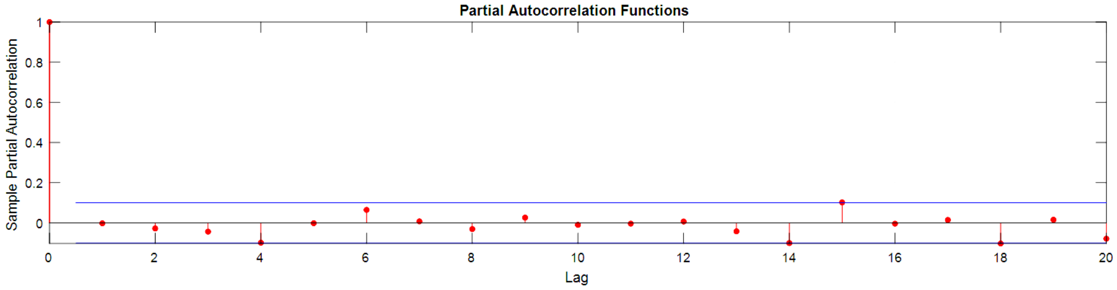


Figure 10. plot ACF and PACF for square error series

From figure 10 the ACF that all lags inside the confidence intervals limits we notice that the time series has gained stability, and then the threshold values for the model are drawn.

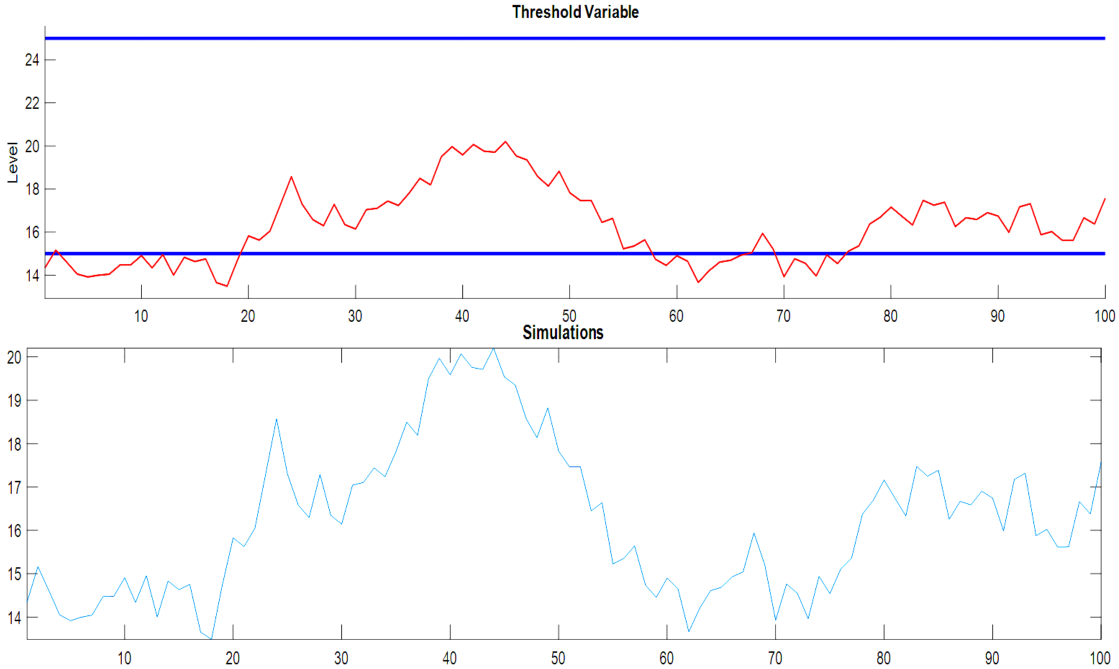


Figure 11. plot threshold values for model TARDL

New we fit and estimate the parameters of TARDL model. The best way to estimate parameter of the TARDL model using Maximum likelihood method. This step involves choosing the best arrangement of the model using AIC and BIC information criteria. Table 1 represent the AIC and BIC [23-24] in formation criteria.

Table 1 AIC and BIC in formation criteria

TARDL$(p, q, 1)$	AIC	BIC
TARDL(1,1,1)	-1.4802e+03	-1.4683e+03
TARDL(1,2,1)	-1.4786e+03	-1.4626e+03
TARDL(1,3,1)	-1.4773e+03	-1.4574e+03
TARDL(2,1,1)	-1.4794e+03	-1.4555e+03
TARDL(2,2,1)	-1.4774e+03	-1.4495e+03
TARDL(2,3,1)	-1.4772e+03	-1.4452e+03
TARDL(3,1,1)	-1.4752e+03	-1.4393e+03
TARDL(3,2,1)	-1.4736e+03	-1.4337e+03
TARDL(3,3,1)	-1.4719e+03	-1.4280e+03

From table above its clear that the best model is TARDL(1,1,1) according to both criteria.

Now the stability condition is applied in equation 11, to get:

$$\therefore \quad \frac{-0.0285049 - 0.002805 + 0.00280507 - 0.0285049}{2} = \frac{-0.05700973}{2} = -0.028504865 < 1$$

Form above equation, its clear that the model is stable. Therefore, the inferred value of conditional mean for the data series that explain in figure 12. Also the stability of the model is then plotted in figure 13, respectively:

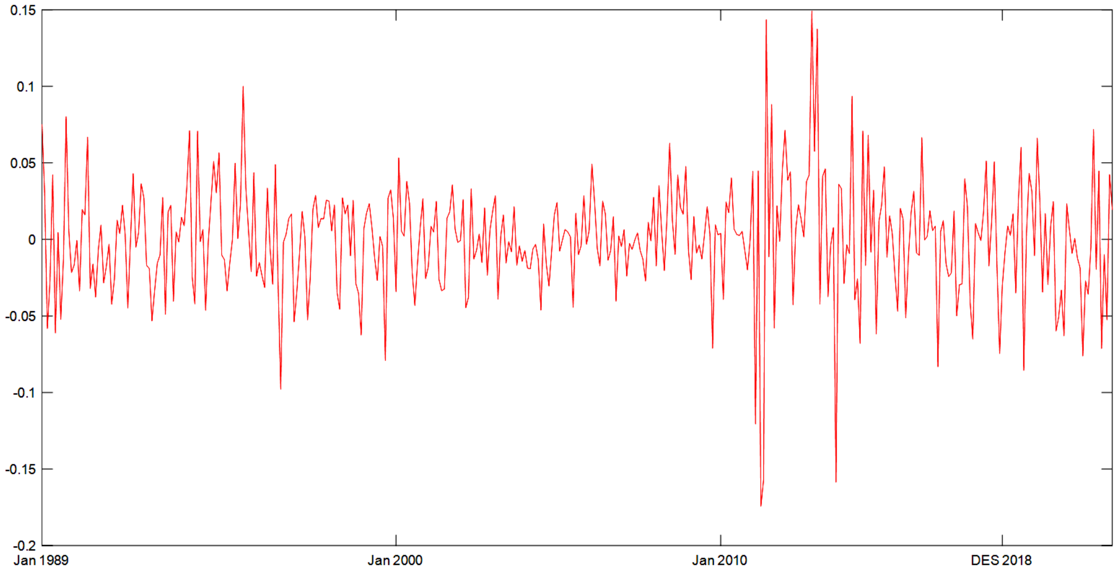


Figure 12. plot inferred conditional mean

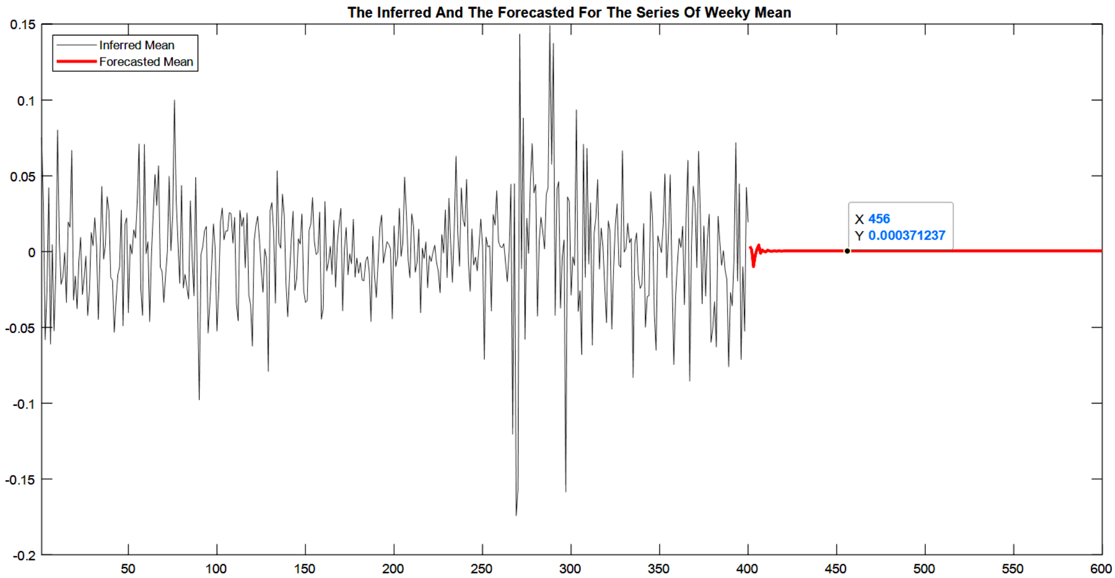


Figure 13. plot inferred and forecasting for the series or weekly mean

6. Results

In figure 2 shows three components for variables of the time series used in example1: 400 observation were generated, where the model parameters were adjusted to generate data following the TARDL(1,1,1) model. The graph showed oscillatory behaviour on the early stages, reflecting the initial instability before the values began to stabilize over time. This behaviour indicates that is model subject to limit cycle stability condition, which allows it to stabilize at a constant value after certain number of iterations, while in figure 3 the stability condition was applied through equation (11), and result was that model is orbitally stable, meaning that over time, the oscillation fade away and model reaches a constant value. This result was verified using MATLAB, where showed that generated trajectories were constant with the specified stability conditions.

From figure 4 at 500 observation using same model in example 1 but with a different set of initial values and parameters. In contrast to example 1, the results show constant oscillations and instability in values, indicating that model doesn't meet limit cycle stability condition. While figure 5 which show that oscillations continue without reaching stable value when applied equation (11) the model doesn't meet limit cycle stability condition resulting in continued random changes over time include that model suffers from orbital instability means it will not reach a stabile equilibrium even with large number of iterations.

Therefore, the results indicate that:

- In Example 1, the model meets stability condition as oscillations fade away with time, and model reaches a stable state after a certain number of iterations.
- In Example 2, the model is unstable as the oscillations continue without reaching a stable value, indicating that the orbital stability condition isn't met.
- The difference between the two examples lies in the initial values and parameters used, as slight differences in the threshold variable or other parameters lead to a significant change in the stability of the model.

The practical aspect of real data, confirmed the model's stability when the limit cycle condition is met. The inferred conditional mean and predicted series demonstrated the effectiveness of the TARDL(1,1,1) model in capturing the underlying dynamics and achieving series stability.

Conclusions

The limit cycle stability condition of TARDL(1,1,1) model is proven, which depends on the values of the model coefficients. If this condition is met, the model stable at constant value, and if condition isn't met, the model continuous to oscillate unstably. The main difference between the two examples in the values of initial coefficients and threshold variables. If example 1, the coefficients and thresholds variable stabilize the model, while in example 2 small changes in these values led to instability model. This illustrates the sensitivity of TARDL(1,1,1) model to change coefficients and threshold variables, as small changes can lead to completely different results in terms of stability. Practical application to real data further supports these results, demonstrating that the model can be stabilized effectively when the appropriate conditions for limit cycle stability are met.

The paper presents an in-depth analysis of stability of TARDL(1,1,1) using dynamic approach, with a practical application that illustrates the importance of stability conditions in determining the behavior of model, which can be a powerful tool for analyzing nonlinear data, but only if the necessary conditions for stability are met.

Conflicts of interest

Non

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