Logical Twofold Integral

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Abstract

The aim of this paper is to present a new framework for studying capacities and twofold integral from a point of view of two-valued logic. In this framework, we propose equivalent definitions of capacities and twofold integral that may be more easily interpretable. First, we define a logical capacity, then, we propose definition of logical twofold integral with respect to logical capacities based on the two-valued logic.

Keywords: Capacities, Fuzzy integrals, Twofold integral, Fuzzy logic, Two-valued logic.

التكامل المضاعف المنطقي

الخلاصة

الغرض من هذا البحث هو تقديم أطار جديد لدراسة القدرات والتكامل المضاعف من وجهة نظر المنطق الرياضي ذو القيمتيين. في هذا الأطار نقترح تعاريف بديله للقدرات وللتكامل المضاعف التي ربما تكون قابل للتفسير بسهوله ال حد ما. أولاً نعرف القدرات المنطقيه ثم نقترح تعريف التكامل المضاعف المنطقي المتعلق بالقدرات المنطقيه مبني على المنطق الرياضي ذو القيمتيين.

1. Introduction

Aggregation operators are used for a large variety of purposes (see e.g., [5], [14]). Due to these needs, different families of aggregation operators have been defined. Choquet integral and Sugeno integral are powerful aggregation operators that are specially appropriate when criteria are not independent but there exist some interaction between them. These operators combine the information taking into account a capacity (or fuzzy measure). This capacity permits to express the interaction between the criteria.

Recently, Torra [13] proposed the twofold integral that generalizes both Choquet and Sugeno integrals. This new operator combines the evaluation of each criteria with respect to two capacities. One capacity corresponds to the Choquet integral and the other corresponds to the Sugeno integral.

The aim of this paper is to construct a different approach from [13] for introducing alternative definitions of capacities and twofold integral from a point of view of two-valued logic by considering the fact that the degree of membership in sense of fuzzy logic can be taken only two truth values, 1 to be true and 0 to be false. Accordingly, we

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define a logical capacity, then, we propose definition of logical twofold integral with respect to logical capacities based on the two-valued logic.

This paper is organized as follows. In the next section we recall basic definitions of usual capacities and twofold integral. In Section 3 we present logical capacities. Then in Section 4, we propose our alternative approach logical twofold integral based on the two-valued logic. The paper finishes with some conclusions. Throughout the paper, we denote R as the set of all real numbers, R_+ the set of all non-negative real numbers, $\overline{R} = R \cup \{-\infty, \infty\}$ is the extended real line, and $\overline{R}_+ = \{x \in \overline{R} \mid x \geq 0\}$; the universal set X denotes a finite set of n elements (states of nature, criteria, individuals, etc), and $2^X = \{A \mid A \subset X\}$.

2. Usual capacities and twofold integral

In this section we give some basic definitions of usual capacities and twofold integral.

Usual capacities on some finite universe are special monotone set functions defined in the following way (see, e.g. [2], [5]).

Definition 1, [5]: A capacity on X is a set function $m : 2^X \to [0,1]$ satisfying the following requirements

(i) $m(f) = 0$ and $m(X) = 1$.

(ii) $\forall A, B \in 2^X$, $A \subset B$ implies $m(A) \leq m(B)$.

The twofold integral was introduced by Torra in [13]. Its definition of a finite universal set is as follows (see, [13]).

Definition 2, [13]: Let m_C and m_S be two capacities on X, then the logical twofold integral of a function $h : X \to [0,1]$ with respect to the capacities m_C and m_S is defined by:

$$T_{m_C,m_S}(h) = \sum_{i=1}^{n} [\bigvee_{j=1}^{i} (h(x_{s(j)})) \wedge m_S(A_{s(j)})) (m_C(A_{s(i)}) - m_C(A_{s(i+1)}))] \quad (1)$$

where $h(x_{s(i)})$ indicates that the indices have been permuted so that $0 \leq h(x_{s(1)}) \leq ... \leq h(x_{s(n)}) \leq 1$, $\vee$ is the maximum, $\wedge$ is the minimum, and $A_{s(i)} = \{x_{s(i)}, ..., x_{s(n)}\}$ with the convention $A_{s(n+1)} = f$.

3 Logical capacities

Remark: Hereafter, to simplify notation, we assume a finite universe of discourse $X = \{1,...,i,...,n\}$ instead of $X = \{x_1,...,x_i,...,x_n\}$.

In classical set theory, the notions “element”, “set” and the relation “is an element of” are well-described concepts. Thus, a simple statement describing whether a particular element having a certain property belongs to a particular set defines set. The characteristic function $\mathrm{M}_B(i)$ (equation (2)) of a crisp set B assigns a value of either 1 or 0 to each individual in the universal set $X = \{1,...,i,...,n\}$ thereby discriminating between members and nonmembers of the crisp set under consideration.

$$\mathrm{M}_B(i) = \begin{cases} 1 & \text{iff } i \in B, \\ 0 & \text{iff } i \notin B. \end{cases} \qquad (2)$$

On the other hand, if B is a fuzzy subset of X, then $\mathrm{M}_B(i)$ is usually called the membership function of B, and the values it assumes are grades of membership of an element i in B. We will say that [0, 1] is the degree of membership, where 1 is the full membership and 0 represents the zero degree of membership of being member. If we consider "the degree of membership" to be "the degree of truth" and takes only two truth values, 1 to be "true" and 0 to be "false", then this characteristic function $\mathrm{M}_B(i)$ (equation 2) is based on a two-valued logic.

The interpretation and notation of the elements and sets of the universal set are as follows.

Let B be subset of the universal set.

- If i is member of B ($\mathrm{M}_B(i)$ =1), then the element i is interpreted as the true element and represented as i^T in B.
- If i is nonmember of B ($\mathrm{M}_B(i) = 0$), then the element i is interpreted as the false element (i.e., represents the zero degree of truth of being true) and represented as i^F in B.
- A set for which $\mathrm{M}_B(i) = 0$, $\forall i$ is represented as $\{1^F,\ldots,n^F\}$. This corresponds to the empty set f in usual set theory.
- A set for which $\mathrm{M}_B(i) = 1$, $\forall i$ is represented as $\{1^T,\ldots,n^T\}$. This corresponds to the universal set X in usual set theory.

For example, suppose the universal set containing three elements, then, in its subset $B = \{1^T, 2^F, 3^T\}$ which is concedes with ({1, 3}) in the usual notation, the meaning of each element is as follows: The element 1 is member of B, the element 2 is non-member of B, and the element 3 is member of B. Hereafter, for convenience, we denote the universal set which is based on two-valued logic by $X^* = \{1^T,\ldots,n^T,1^F,\ldots,n^F\}$.

Definition 3: Let $X^* = \{1^T,\ldots,n^T,1^F,\ldots,n^F\}$ be the universal set which is based on two-valued logic. A complementary set $C \in X^*$ is set which have complementary elements (A complementary element for i^T is i^F and vise wise).

A non-complementary set $D \in X^*$ is a set which does not have complementary elements.

For example, if $n = 3$ then $C = \{1^T, 1^F, 2^T, 3^T\}$ is complementary set while $D = \{1^T, 2^F\}$ is non-complementary set.

Definition 4: A basic set $B \in X^*$ is a set which is non-complementary set with the n elements ($|B| = n$)). That is, a basic set is a set which contains only i^T or i^F for all i, (i = 1,…,n).

In other words, a basic set is a set of binary alternative.

For example, if $n = 4$ then $\{1^T, 2^F, 3^T, 4^T\}$ is basic set.

The set of all Complementary sets of X^* is denoted by Λ and the set of all basic sets of X^* is denoted by B.

Definition 5: Let B be the set of all basic sets of X^* and $B_1, B_2 \in \mathrm{B}$. Then, $B_1 \subseteq B_2$ holds iff $i^F \in B_1$ implies i^F or $i^T \in B_2, \forall i$.

Now, based on the definitions and notations mentioned above, we give the definition of logical capacity from a point of view of two-valued logic as follows.

Definition 6 : (Logical capacity) Let B be the set of all basic sets of X^*. A set function $m : \mathrm{B} \rightarrow [0,1]$ is called logical capacity if it satisfies the following requirements:

(i) $m(f) = m(1^F,..., n^F) = 0$ and $m(f) = m(1^T,..., n^T) = 1$.

(ii) $\forall B_1, B_2 \in \mathrm{B}$, $B_1 \subseteq B_2$ implies $m(B_1) \leq m(B_2)$.

4 logical twofold integral

In this section, we propose twofold integral model based on the two-valued logic of a measurable function (h).The basic idea underlying this model is each input value of a measurable function $h(i)$ expressed by the true value $h(i^T)$ (itself) and the false value $h(i^F)$ (its complement) for all i, and we treat the two values as different variables in the representation of twofold integral.

4.1 Input Values of h ($h(i^T), h(i^F)$)

Let X is the universal set and h a measurable function (alternatives, acts, etc.) on X. Then, for $i \in X$, $h(i)$ is the utility or score of h with respect to criterion i. The representation of twofold integral depends on determination $h(i)$ for $i \in X$. To determine the measurable function $h(i)$ in the new framework "logical twofold integral", we consider each input value of $h(i)$ becomes a true input $h(i^T) = h(i)$ and its complement $h(i^F)$, and we treat the two input $h(i^T)$ and $h(i^F)$ as different variables in the representation of logical twofold integral.

Mathematically we translate this by: Let the universal set of logical twofold integral inputs be $X^* = \{1^T,...,n^T,1^F,...,n^F\}$ and $h : X^* \rightarrow [0,1]$. The input value is converted from $h(i) \in [0,1]$ into $h(i^T) \in [0,1]$ and $h(i^F) \in [0,1]$ such that $h(i^T) + h(i^F) = 1$ using "conversion functions":

$$h(i^T) = h(i) \text{ and } h(i^F) = 1 - h(i^T) \quad \ldots (3)$$

4.2 Assignment of logical capacity values (m)

In the assignment of logical capacity over the universal set, there are 2^n sets and all these sets are basic sets. Hence, the logical capacity on the set of all basic sets (B) of X^* generally require 2^n parameters. The logical capacity (m) for sets not containing basic sets and complementary sets containing basic sets (which are described in section 3) are assigned as follows:

1. Sets not containing the basic set (i.e. not containing the binary alternative) are assigned 0. For example, when $n = 3$, $m(\{1^T, 2^F\}) = 0$.

2. Assignment of logical capacity value of complementary set C which include basic sets is sum of assignment

of true basic set B_1 and false basic set B_2 as follows

- $\forall C \in \Lambda$, $\exists B_1, B_2 \subset C$ such that:
  - $C = B_1 \cup B_2$.
  - B_1 and B_2 are basic sets.
  - If $i^T \in C$ and $i^F \in C$ then $i^T \in B_1$ and $i^F \in B_2$.
  - If $i^T \in C$ and $i^F \notin C$ then $i^T \in B_1$ and $i^T \in B_2$.
  - If $i^T \notin C$ and $i^F \in C$ then $i^F \in B_1$ and $i^F \in B_2$.
- Assignment of logical capacity value of complementary set C equals to Sum of assignment of basic set B_1 and assignment of basic set B_2:

$$m(C)=m(B_1)+m(B_2). \qquad (4)$$

For example, if $n=3, C=\{1^T,2^T,2^F,3^F\}$ then $B_1=\{1^T,2^T,3^F\}$, $B_2=\{1^T,2^F,3^F\}$, and

$m(\{1^T,2^T,2^F,3^F\})=$
$m(\{1^T,2^T,3^F\})+\ m(\{1^T,2^F,3^F\})$.

4.3 Definition of logical twofold integral

Here, we propose the definition of logical twofold integral, which is equivalent expression of usual twofold integral (Definition 2), although being based on a different structure. The logical twofold integral of a measurable function h with respect to logical capacities m_C and m_S we will denote by $T_{m_C,m_S}(h)$, and define as follows.

Definition 7: Let m_C and m_S be two logical capacities on X, then the logical twofold integral of a function $h: X \to [0,1]$ with respect to the logical capacities m_C and m_S is defined by:

$$T_{m_C,m_S}(h)=\sum_{j=1}^{2n}\Big[\bigvee_{k=1}^{j}\big(h(s_k^{p_k})\wedge m_S(A_{s(k)})\big)\\ \big(m_C(A_{s(j)})-m_C(A_{s(j+1)})\big)\Big] \qquad (5)$$

where, $h(s_1^{p_1}),\ldots,h(s_j^{p_j}),\ldots,h(s_{2n}^{p_{2n}})$ are the input values of h (after conversion) such that $h(s_1^{p_1})\le\ldots\le h(s_{2n}^{p_{2n}})$ for all $s_1^{p_1},\ldots,s_{2n}^{p_{2n}}\in X^*$, $p_1,\ldots,p_{2n}\in\{T,F\}$, and $A_{s(j)}=\{s_j^{p_j},\ldots,s_{2n}^{p_{2n}}\}$ with the convention $A_{s(2n+1)}=f$.

The following numerical example illustrates the logical twofold integral with respect to logical capacities m_C and m_S.

Example 1: We consider the universal set containing two elements and let $m_S(\{1^F,2^F\})=m_C(\{1^F,2^F\})=0$, $m_S(\{1^T,2^F\})=0.8$, $m_C(\{1^T,2^F\})=0.4$, $m_S(\{1^F,2^T\})=0.6$, $m_C(\{1^F,2^T\})=0.7$, and $m_S(\{1^T,2^T\})=m_C(\{1^T,2^T\})=1$.

Then calculation of the output values for $h(1)=0.3$ and $h(2)=0.8$ is as follows.

From the conversion functions (Equation (3)),
$h(1^T)=0.3$, $h(1^F)=0.7$, $h(2^T)=0.8$, $h(2^F)=0.2$.

Then, after ordering of values in increasing order and applying the formula (5) we get

$$
\begin{aligned}
T_{m_C,m_S}(h) = {} & (h(2^F) \wedge m_S(\{1^T,2^T,2^F,1^F\})) \\
& (m_C(\{1^T,2^T,2^F,1^F\}) - m_C(\{1^T,2^T,1^F\})) \\
& + (h(2^F) \wedge m_S(\{1^T,2^T,2^F,1^F\}) \vee \\
& (h(1^T) \wedge m_S(\{1^T,2^T,1^F\})) \\
& (m_C(\{1^T,2^T,1^F\}) - m_C(1^F,2^T\})) + \\
& (h(2^F) \wedge m_S(\{1^T,2^T,2^F,1^F\}) \vee \\
& (h(1^T) \wedge m_S(\{1^T,2^T,1^F) \vee (h(1^F) \wedge \\
& m_S(\{1^F,2^T\}))) (m_C(\{1^F,2^T\}) - m_C(\{2^T\})) \\
& + (h(2^F) \wedge m_S(\{1^T,2^T,2^F,1^F\}) \vee (h(1^T) \wedge \\
& m_S(1^T,2^T,1^F\}) \vee (h(1^F) \wedge m_S(\{1^F,2^T\}) \vee \\
& (h(2^T) \wedge m_S(2^T)))(m_C(\{2^T\}) - m_C(\{f\}))
\end{aligned}
$$

$$
\begin{aligned}
= {} & (0.2 \wedge (m_S(\{1^T,2^T\}) + m_S(\{1^F,2^F\})) \\
& (m_C(\{1^T,2^T\}) + m_C(2^F,1^F\}) - (m_C(\{1^T,2^T\}) \\
& + m_C(1^F,2^F\})) + (0.2 \wedge (m_S(\{1^T,2^T\}) + \\
& m_C(1^F,2^F\}))) \vee (0.3 \wedge (m_S(\{1^T,2^T\}) + \\
& m_S(1^F,2^F\}))) (m_C(\{1^T,2^T\}) + m_C(1^F,2^F\})) \\
& - m_C(1^F,2^T\})) + (0.2 \wedge (m_S(\{1^T,2^T\}) + \\
& m_S(1^F,2^F\}))) \vee (0.3 \wedge (m_S(\{1^T,2^T\}) + \\
& m_C(1^F,1^T))) \vee (0.7 \wedge (m_S(\{1^F,2^T\}))) \\
& ((m_C(\{1^F,2^T\}) - m_C(\{2^T\})) + 0 \\
= {} & 0.58.
\end{aligned}
$$

5. Conclusions

In this work we have studied the capacities and twofold integral from a point of view of two-valued logic. We have translated the vagueness of fuzzy logic (in fact, two-valued logic) to the ambiguity of capacity, and then proposed logical capacity and logical twofold integral. Translation from ambiguity to vagueness is important because vagueness is an easier fuzziness to handle than ambiguity when taking technological applications into consideration [8]. This fact is evidence for clarifying the usefulness of the proposed framework.

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