# Local and Global Uniqueness Theorems of the N-th Order Differential Equations

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**Abstract**

Local and Global uniqueness theorem of solutions of the differential equation

$$x^{n}(t)=f(t,x(t),x^{'}(t),\dots,x^{(n-1)}(t)) \qquad ,0<t<a \;,\; a>0$$

have been obtained, which are applications of Bihari's and Gronwall's inequalities.

# نظريتي الوحدانية العامة والمحلية للمعادلات التفاضلية من الرتبة النونية

**الخلاصة**

لقد تم الحصول على نظرية الوحدانية العامة والمحلية لحل المعادلة التفاضلية

$$x^{n}(t)=f(t,x(t),x^{'}(t),\dots,x^{(n-1)}(t)) \qquad ,0<t<a \;,\; a>0$$

وهي تطبيق لمتراجحات بايهاري وكرونول

**Introduction**

Consider the differential equation of the type

$$x^{(n)}(t)=f(t,x(t),x^{'}(t),\dots,x^{(n-1)}(t)) \qquad t\in J$$

with $x^{(j)}(0)=x_0^{(j)}$

$j=0,1,\dots,n-1$ ….(1)

where *J=[0,a)* ,a>0 , $f\in[J\times R^{n},R]$ and $R^{n}$ denotes the real n-dimensional Euclidean space, $x_0^{(j)}, j=0,1,,\dots,n-1$ is a real positive constants, *Liu* and *Ge[2]* based on the coincidence degree method of Gaines and Mawhin [3]. Proved that (1) has at least one solution ***U. Elias*** [4] proved the existence of global at least one solution to (1).

In this paper Bihari's inequality is applied to obtain local uniqueness and Gronwall's inequality to obtain global uniqueness of solution to (1).

It is important mentioning that Baihov D. and Simeonov [1] showed it that the solution of (1) is of the form:

$$x(t)=\sum_{j=0}^{n-1}\frac{x^{(j)}(0)}{j!}t^{j}+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}.$$
$$f(s,x(s),x^{'}(s),\dots,x^{n-1}(s))ds$$
….(2)

with *0<t<a* *,a>0* and

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$x^{(j)}(0)$ are initial constants .

**2- Local Uniqueness:**
In this section, a local uniqueness result is proved by applying Bihari's inequality theorem.

***Bihari's Inequality Theorem [1]***
Suppose the following conditions holds:
1. $a(t)$ is positive continuous function in $J=[a,b)$
2. $K_j(t,s),\ j=1,2,3,\dots,n$ ,are non negative continuous functions for $a\le s\le t\le b$ which are no decreasing in **t** for any fixed **s.**
3. $g_j(u),\ j=1,2,\dots,n$ are non decreasing continuous functions in $R_+$ ,with $g_j(u)>0$ for *u>o* and $g(au)\le r(a)w(u)$ ,for $a>0, u\ge 0$ were r(a)is non negative continuous function in $R_+$ ,which is positive for *u>0.*
4.u(t) is non negative continuous function in *J* and

$$u(t)\le a(t)+\sum_{j=1}^{n}\int_{\alpha}^{t}k_j(t,s)g_j(u(s))ds,\ t\in J$$

then

$$u(t)\le a(t)y_{n-1}(t)G_n^{-1}(G_n(1)+\frac{u_n(a(t))y_{n-1}(t)}{a(t)})\int_0^t k_n(t,s)ds$$

where

$$G_n(u)=\int_{u_n}^{u}\frac{dx}{g_n(x)}\quad ,u>0,(u_n>0)$$

***Theorem 1: (Local Uniqueness)***
The initial value problem (1) has a unique solution on the interval *0<t<a,* if the function f is continuous in the region

$$0<t<a,$$
$$\left|(x,x',\dots,x^{(n-11)})-(x_0,x_0',\dots,x_0^{(n-1)})\right|\le b$$

and such that

$$\left|f(t,x,x',\dots,x^{(n-11)})-f(t,y,y',\dots,y^{(n-11)})\right|\le\sum_{j=0}^{n-1}f_j(\left|x^{(j)}-y^{(j)}\right|)$$

where $f$ *(u)*is a continuous non decreasing function on 0<u<A, with $f$ *(0)=0* , b>0 and A is a positive constant .

Proof:
Let *x(t)* and *y(t)* be two solutions to (1) which are defined in neighborhood at the right of $t_o$ .That is

$$x(t)=\sum_{j=0}^{n-1}\frac{x^{(j)}(0)}{j!}t^j+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}.f(s,x(s),x'(s),\dots,x^{n-1}(s))ds$$

$$y(t)=\sum_{j=0}^{n-1}\frac{y^{(j)}(0)}{j!}t^j+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}.f(s,y(s),y'(s),\dots,y^{n-1}(s))ds$$

This leads easily to

$$|x(t)-y(t)|\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j_!}\right|\left|x^{(j)}(0)-y^{(j)}(0)\right|+\int_0^t\left|\frac{(t-s)^{n-1}}{(n-1)!}\right|.$$

$\left| f(s,x(s),x^{'}(s),x^{''}(s),...,x^{(n-1)}(s))\, ds - f(s,y(s),y^{'}(s),y^{''}(s),...,y^{(n-1)}(s))\, ds \right|$

$$|x(t)-y(t)| \le \sum_{j=0}^{n-1} |x^{(j)}(0)-y^{(j)}(0)||t^j| + \sum_{j=0}^{n-1}\int_0^t |t^{n-1}| f_j(x^j - y^j)ds$$

Let v(t)= $x^{(j)} - y^{(j)}$, then

$$|x(t)-y(t)| \le \sum_{j=0}^{n-1} |x^{(j)}(0)-y^{(j)}(0)||t^j| + \sum_{j=0}^{n-1}\int_0^t |t^{n-1}| f_j(v(s))ds$$

Let $r(t)$ be the right hand side of the above inequality, then

v(t)≤r(t) and

$$|x(t)-y(t)| \le \sum_{j=0}^{n-1} |x^{(j)}(0)-y^{(j)}(0)||t^j| + \sum_{j=0}^{n-1}\int_0^t |t^{n-1}| f_j(r(s))ds$$

Since

$x^{(j)}(0) = y^{(j)}(0) \ \ , \forall j$ ,then

$$r(t) \le \in + \sum_{j=0}^{n-1}\int_0^t |t^{n-1}| f_j(r(s))ds$$

for some $\in > 0$

By using Bihari's inequality yields

$$r(t) \le \in y_{n-1}(t) G_n^{-1}(G_n(1) + \frac{r_n(\in) y_{n-1}(t)}{\in}) \int_0^t |t^{n-1}| ds$$

If $\in \to 0$ then $r(t) \le 0$ and since r(t)≥0 then r(t)=0 and hence x(t)=y(t).

## 3 Global Uniqueness

The global uniqueness for the initial value problem (1) will be discussed with the aid of Gronwall's inequality, which seems by the following theorem.

***Gronwall's Inequality Theorem* [6]**

Let *a (t)*, *b (t)* and *u (t)* be continuous functions in *J*= $[a,b]$ and let *b(t)* be a nonnegative in $J=[a,b]$ and *a(t)* is nondecreasing in *J*= $[a,b]$ suppose

$$u(t) \le a(t) + \int_a^t b(s)u(s)\,ds \quad ,t \in J$$

Then

$$u(t) \le a(t) e^{\int_a^t b(s)ds} \quad ,t \in J$$

Theorem (2) ***(Global uniqueness theorem)***

Assume that:

1. f is a continuous function in the region R={( $t, x, x^{'}, x^{''}, ..., x^{(n-1)}$ ): $0<t<a$, $\left|\left(x, x^{'}, x^{''}, ..., x^{(n-1)} - \left(x_0, x_0^{'}, x_0^{''}, ..., x_0^{(n-1)}\right)\right)\right| \le b$ } $\subset \Omega$ where $\Omega$ is an open $\left(t, x, x^{'}, ..., x^{(n-1)}\right)$ in $R^{n+1}$ with a, b>0.
2. f satisfy Lipschitz condition with respect to $\left(x, x^{'}, x^{''}, ..., x^{(n-1)}\right)$,

$$\left| f(t,x,x^{'},...,x^{(n-11)}) - f(t,y,y^{'},...,y^{(n-11)}) \right| \le L \sum_{j=0}^{n-1} |x^j - y^j|$$

For some positive constant L, then the solution of (1) is unique.

**Proof**

Let x (t) and y (t) be two solutions to (1) then

$$x(t)=\sum_{j=0}^{n-1}\frac{x^{(j)}(0)}{j!}t^j+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}.$$
$$f(s,x(s),x'(s),...,x^{n-1}(s))ds$$
$$y(t)=\sum_{j=0}^{n-1}\frac{y^{(j)}(0)}{j!}t^j+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}.$$
$$f(s,y(s),y'(s),...,y^{n-1}(s))ds$$

From which we get

$$|x(t)-y(t)|\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j_!}\right|\left|x^{(j)}(0)-y^{(j)}(0)\right|+$$

$$\int_0^t\left|\frac{(t-s)^{n-1}}{(n-1)!}\right|.$$
$$\left|f(s,x(s),x'(s),x''(s),...,x^{(n-1)}(s))\,\mathrm{ds}-\right.$$
$$\left.f(s,y(s),y'(s),y''(s),...,y^{(n-1)}(s))\,\mathrm{ds}\right|$$

$$\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j_!}\right|\left|x^{(j)}(0)-y^{(j)}(0)\right|+\int_0^t\left|\frac{(t-s)^{(n-1)}}{(n-1)_!}\right|.$$

$$\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j_!}\right|\left|x^{(j)}(0)-y^{(j)}(0)\right|+\int_0^t\left|\frac{(t-s)^{(n-1)}}{(n-1)_!}\right|$$
$$L\left[\sum_{j=0}^{n-1}\left|(x(s)-y(s))^{(j)}\right|\right]\mathrm{ds}$$
$$\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j_!}\right|\left|(x(0)-y(0))^{(j)}\right|+\int_0^t\left|t^{(n-1)}\right|$$
$$L\left[\sum_{j=0}^{n-1}\left|(x(s)-y(s))^{(j)}\right|\right]\mathrm{ds}$$

Let

$$v(t)=\sum_{j=0}^{n-1}\left|(x(t)-y(t))^{(j)}\right|\left|t^{(n-1)}\right|L$$

Then

$$|x(t)-y(t)|\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j_!}\right|\left|(x(0)-y(0))^{(j)}\right|+$$

$$\int_0^t v(s)\,\mathrm{ds}$$

Let r(t) equal to the right hand side of the above inequality ,then
$v(s)\le r(s)$

$$r(t)\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j!}\right|\left|(x(0)-y(0))^{(j)}\right|+\int_0^t r(s)\,ds$$

By the above inequality (Gronwall's inequality)

$$r(t)\le\sum_{j=0}^{n-1}\left|\frac{t^j}{j!}\right|\left|(x(0)-y(0))^{(j)}\right|e^{\int_0^t ds}$$

Since $x^{(j)}(0)=y^{(j)}(0)\quad,\forall j$ then
$r(t)\le 0$

since $|x(t)-y(t)|\le r(t)\le 0$

Then $|x(t)-y(t)|\le 0$ and since the absolute value larger than or equal to zero then

$x(t)=y(t)$ $t\in$ J.

## 4- Conclusions

It is easy to note that the uniqueness of a special cases solution (n=1 or n=2) can be obtain by using Bihari's and Gronwall's inequality which is give the work more accuracy and easer .

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