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The Collocation Method for Solving the Linear Fredholm Integral Equation of the Second Kind Using Bernstein Polynomials

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Abstract

Integral equation of the second kind which has, extensively, been solved by different ways, but not this one that deals with the collocation method using Bernstein polynomials together with some useful examples to declare the method.

Keywords: Collocation method Fredholm integral equation of the second kind, Bernstien polynomials, least square error.

استخدام متعددات الحدود برتشتاين كل المعادلات فردهولم التكاملية من النوع الثاني

الخلاصة:

بدلالة طريقة التوضع(collocation method) وباستخدام متعددات الحدود برنشـتاين وجدنا حلاً لمعادلة فردهوم التكاملية من النوع الثاني واعطيت بعض الامثلة المفيـدة فـي توضـيح الطريقة المذكورة.

Introduction

The well-known Fredholm integral equation of the second kind has been discussed thoroughly in the recent time using various kinds of approximation methods, in this paper; we submit another known method that is the collocation method to solve this integral equation of Fredholm. Also has been given some illustrative examples which has evaluated by a computer program.

Recall Fredholm of the second kind in the form:

$$U(x) = f(x) + l\int_a^b K(x,y)U(y)dy \quad (1)$$

where the integral limits *a* and *b* are constants and *K(x,y)* is the kernel function and *f(x)* is any continuous function.

Without loss of generality, let *a=0* and *b=1*.

Numerical Solution: The discrete form for the exact solution *U(x)* for equation (1) can be written in the form: $U(x) \approx U_N(x)$,

where *N* is a positive integer. This paper pivoted to implement Bernstein polynomials (B-spline) as a discrete function (polynomial) of $U_N(x)$ i.e.

$$(2)\, U_N(x) = \sum_{i=0}^{N} a_i B_{i,N}(x) \quad x \in R$$

$$B_{i,N}(x) = \binom{N}{i} x^i (1-x)^{N-i} \text{ where}$$

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for $i=0,1,2,\ldots,n$, where the combination

$$\binom{N}{i}=\frac{N!}{i!(N-i)!}$$

There are $N+1$ Nth degree Bernstein polynomials [1], [2].
The Bernstein polynomials of degree 1 are:

$$B_{0,1}(x)=1-x$$
$$B_{1,1}(x)=x$$

The Bernstein polynomials of deg.2 are

$$B_{0,2}(x)=(1-x)^2$$
$$B_{1,2}(x)=2x(1-x)$$
$$B_{2,2}(x)=x^2$$

Some necessary characters of Bernstein polynomials: (for the proof see [5, 6])

i) A recursive definition of the Bernstein polynomials of degree N can be written as:

$$B_{i,N}(x)=(1-x)B_{i,N-1}(x)+xB_{i-1,N-1}(x)$$

$i=1,2,\ldots,N$

ii) The Bernstein Polynomials are all non-negative for $0\leq x\leq 1$ (see[5]).

iii) The Bernstein polynomials form a partition of unity.

iv) Converting from Bernstein basis to power basis (proof see 5) as:

$$B_{i,N}(x)=\sum_{k=i}^{N}(-1)^{k-i}\binom{N}{k}\binom{k}{i}x^k$$

$i=1,2,\ldots,N.$

Therefore we could write the function

$$U(x)\approx U_N(x)=a_0B_{0,N}(x)+a_1B_{1,N}(x)+\ldots+a_NB_{N,N}(x)$$
$$=a_0\sum_{k=0}^{N}(-1)^{k}\binom{N}{k}\binom{k}{0}x^k+\ldots+$$
$$+a_N\sum_{k=N}^{N}(-1)^{k-N}\binom{N}{k}\binom{k}{N}x^k$$
$$=a_0+a_1\left[\sum_{k=1}^{N}(-1)^{k-1}\binom{N}{1}\binom{1}{1}\right]x^1+\ldots+$$
$$+a_N\left[\sum_{k=0}^{N}(-1)^{k-N}\binom{N}{N}\binom{N}{N}\right]x^N$$

This system can be written in a matrix form as follows

$$U_N(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix}\begin{bmatrix}a_{0,0} & 0 & \cdots & 0\\ a_{1,0} & a_{1,1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{N,0} & a_{N,1} & \cdots & a_{N,N}\end{bmatrix}=\begin{bmatrix}b_0\\ b_1\\ \vdots\\ b_N\end{bmatrix} \quad (3)$$

Using operator forms, equation (3) can be written as $L[U]=f(x)$ (4)
where the operator L is defined as

$$L[U]=U(x)-\int_a^b k(x,y)U(y)dy$$

The unknown function $U(x)$ is approximated in the form

$$(5)\quad U_N(x)=\sum_{i=0}^{N}a_iB_i(x)$$

Substituting equation (5) in (4) to obtain

$$L[U_N]=f(x)+E_N(x)$$

where

$$L[U_N(x)]=U_N(x)-\int_a^b k(x,y)U_N(y)dy$$

For which we have the residue equation

$$E_N(x) = L[U_N(x)] - f(x) \quad (6)$$

Substituting Eq.(5) in Eq.(6) to get

(7)

$$E_N(x) = L\left(\sum_{i=0}^{N} a_i B_i(x)\right) - f(x)$$

$$= \sum_{i=1}^{N} a_i L(B_i(x) - f(x))$$

Obviously, the weighted function sets its weighted integral equals to zero i.e.

$$(8) \int w_j E_N(x) d(x) = 0$$

Inserting Eq.(7) into Eq.(8) to obtain

$$\int w_j \left[\sum_{i=0}^{N} a_i L(B_i(x)) - f(x)\right] = 0$$

Whence

$$\sum_{i=0}^{N} a_i \int w_j L(B_i(x)) = \int w_j f(x)$$

where

$$L(B_j(x)) = B_j(x) - \int_a^b k(x,y) B_j(y) dy \quad (9)$$

Collocation method

Collocation method is a simple way to obtain a linear approximation U_N, this method uses the weighted function to be $w_j = \delta(x - x_j)$

where the points $x_j \in R$ and $j=0,1,\ldots,N$ are called collocation points, and Dirac delta, as it is known, is the function

$$\delta(x - x_j) = \begin{cases} 1 & if \quad x = x_j \\ 0 & elsewere \end{cases}$$

$$\int_D w_j E(x_j) dx = \int_D \delta(x - x_j) dx = \int_{x_j^-}^{x_j^+} \delta(x - x_j) E(x_j) dx$$

$$= E(x_j) \int_{x_j^-}^{x_j^+} \delta(x - x_j) dx = 0, \quad j = 0,1,\ldots,N$$

Therefore,

$E_N(x_j) = 0$,

where $x_j = \dfrac{1}{i+1}$

This implies that

$$\sum_{j=0}^{N} L(B_j(x_i)) a_j = f(x_i) \quad i = 0,1,\ldots,N \quad (10)$$

Equation (10) is merely a system of $N+1$ equations with $N+1$ unknowns a_i, $i=0,1,\ldots,N$ as follows:

$$\begin{bmatrix} L(B_0(x_0)) & L(B_1(x_1)) & \cdots & L(B_N(x_0)) \\ L(B_0(x_1)) & L(B_1(x_1)) & \cdots & L(B_N(x_1)) \\ \vdots & \vdots & \ddots & \vdots \\ L(B_0(x_N)) & L(B_1(x_N)) & \cdots & L(B_N(x_N)) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

Put

$K=$

$$\begin{bmatrix} L(B_0(x_0)) & L(B_1(x_1)) & \cdots & L(B_N(x_0)) \\ L(B_0(x_1)) & L(B_1(x_1)) & \cdots & L(B_N(x_1)) \\ \vdots & \vdots & \ddots & \vdots \\ L(B_0(x_N)) & L(B_1(x_N)) & \cdots & L(B_N(x_N)) \end{bmatrix}$$

,

$$\boldsymbol{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \text{ and } H = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

Then using Gauss elimination method to solve the obtained linear system

KA=H, for the coefficients a_j, $j=0,1,\ldots,N$ which satisfies eq.(6) . Finally, we obtain the approximate solution of equation (1).

Numerical Examples

Example 1: Consider the Fredholm integral equation of the second kind:

$$U(x)= f(x)+\int_0^1 k(x,y)U(y)$$

where

$$f(x)=\frac{3x}{2}-\frac{1}{3}$$

and let the linear kernel be

$$k(x,y)=y-x$$

the exact solution is taken to be $u(x)=x$.

For $h=0.1$ and $x=x_1=a+ih$, $i=0,1,\ldots,10$,

the table(1) result is obtained by applying the method involved in this paper i.e. the implementation of Bernstein polynomial; these numerical results are compared with the exact one in the same table below:

Example 2: Consider the Fredholm integral equation of the second kind

$$u(x)= f(x)+\int_{-1}^1 k(x,y)u(y)dy$$

where $f(x)=x$ and $k(x,y)=xt$, the exact solution is taken to be $u(x)=3x$; and the step size $h=0.1$. The application of Bernstein polynomial yields the results shown in the table(2) below together with the exact solution at each point x. The least square error is given as well:

Conclusions

Bernstein Polynomial is introduced to find the approximate solution of Fredholm integral equation of the second kind. Two numerical examples were submitted to illustrate the given idea with good approximate results were achieved. We conclude that:

1. The use of Bernstein polynomials give, as it is expected, like the other polynomials, an accurate numerical solution for the simple continuous functions.

2. An advantage of using the Bernstein polynomials lies in their dependence upon a free parameter n; this dependence gives the smallest least square error.

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Table (1)
Presents acomparison the exact solution and approximate solution (Bernstein polynomial)

x	Exact sol.	Bernstein Polynomial
0.0	*0.0000*	*0.0000*
0.1	*0.1000*	*0.1000*
0,2	*0.2000*	*0.2000*
0.3	*0.3000*	*0.3000*
0.4	*0.4000*	*0.4000*
0.5	*0.5000*	*0.5000*
0.6	*0.6000*	*0.6000*
0.7	*0.7000*	*0.7000*
0.8	*0.8000*	*0.8000*
0.9	*0.9000*	*0.9000*
1.0	*1.0000*	*1.0000*
L.Sq.E		*0.0000*

Table (2)
Presents acomparison the exact solution and approximate solution (Bernstein polynomial)

x	Exact solution	Bernstein Polynomial
-1.000	*-3.000*	*-3.000*
-0.900	*-2.700*	*-2.700*
-0.800	*-2.400*	*-2.400*
-0.700	*-2.100*	*-2.100*
-0.600	*-1.800*	*-1.800*
-0.500	*-1.500*	*-1.500*
-0.400	*-1.200*	*-1.200*
-0.300	*-0.900*	*-0.900*
-0.200	*-0.600*	*-0.600*
-0.100	*-0.300*	*-0.300*
0.000	*0.000*	*0.000*
0.100	*0.300*	*0.300*
0.200	*0.600*	*0.600*
0.300	*0.900*	*0.900*
0.400	*1.200*	*1.200*
0.500	*1.500*	*1.500*
0.600	*1.800*	*1.800*
0.700	*2.100*	*2.100*
0.800	*2.400*	*2.400*
0.900	*2.700*	*2.700*
1.000	*3.000*	*3.000*
L.Sq.Error		*0.000*