

Hûta Method for Solving n^{th} Order State-Space Equations of Nonlinear Continuous-Time Control Systems

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Abstract

The paper presents a proposed method with new algorithms written in Matlab language to find the numerical solution of n^{th} order state-space equations (SSE's) of nonlinear continuous-time control system using Hûta method. The state-space equation is the modern representation to analysis linear and nonlinear continuous time invariant and time varying system. It was treated numerically to the single-input-single-output (SISO) time invariant and time varying systems as well as multiple-input-multiple-output (MIMO) time invariant and time varying systems by using sixth-order-eight-stage Hûta method. We show that it is possible to find the output values of nonlinear state-space model using Hûta method. Finally, comparison between the numerical and exact results has been given with illustrative figures for some test examples for solving different types of nonlinear state-space equations using Hûta method for conciliated the accuracy of the results of the proposed method.

Key words: State-space equation, Hûta method, Nonlinear control system and Algorithms.

طريقة هوتا لحل معادلات فضاء الحالة من الرتبة n لأنظمة السيطرة اللاخطية المستمرة الزمن

الخلاصة

يقدم البحث طريقة مقترحة مع خوارزميات جديدة لإيجاد الحل العددي لمعادلات فضاء الحالة من الرتبة n لأنظمة السيطرة اللاخطية المستمرة الزمن باستخدام طريقة هوتا. حيث تمت معالجة أنظمة فضاء الحالة عددياً للمنظومات اللاخطية المعتمدة و غير المعتمدة الزمن الفردية المدخل والمخرج مثلما للمنظومات اللاخطية المعتمدة و غير المعتمدة الزمن المتعددة المدخل والمخرج باستخدام طريقة هوتا من الرتبة السادسة. بالإضافة إلى ذلك تم إيجاد النتائج العددية لمعادلة الإخراج لتمثيل فضاء الحالة. استخدمت لغة (Matlab) لبرمجة خوارزميات هذه الطريقة. كما تمت مقارنة النتائج العددية و الحقيقية لأنواع مختلفة من معادلات فضاء الحالة من خلال بعض الأمثلة والرسوم التوضيحية وقد تم الحصول على نتائج دقيقة.

1. Introduction

Control systems are playing vital role in our life for instance: thermostat, automatic control of airplane, etc. The system is a

combination of component that act together and perform a certain objective [1,2].

In recent years, automatic control systems have assumed an

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increasingly important role in the development and advancement of modern civilization and technology. They are employed in numerous applications, such as quality control of manufactured products and machine tooling. The basic control system problem may be described by the simple block diagram shown in figure (1) [1,3].

Modern control theory adopts what known as state-space equations (SSE's) for mathematical representation of systems. Among its different advantages it makes possible to deal with [4,5]:

- Time variant systems.
- Linear and nonlinear systems.
- Multiple-input-multiple-output system.

The important control systems are nonlinear systems where in some systems there are no linear relationship between the input and output of the controller as in on-off control system.

The nonlinear *state-space equation* (SSE) is given by [1,4]:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t))\end{aligned}$$

where $x(t) \in IR^n$ is the n -dimensional state vector, IR^n is the set of real numbers with n -dimensional space, $u(t) \in IR^m$ is the control input or the forcing function of the system, $y(t) \in IR^p$ is the output of the system and n , m and p are positive integer numbers with $n \geq m$ and $n \geq p$.

State-space representation is helpful to represent a complex system by simple first-order differential equations [6]. In this work different types of nonlinear state-space

representation are solved numerically by using sixth order *Hûta method*. The nonlinear continuous-time invariant and varying systems with their modeling using state space method have been presented in the following sections.

2. Nonlinear Continuous-Time Control Systems (CTCS):

Most real-life systems, including physical and biological systems show nonlinear dynamical behavior. Their behavior in time depends not only on the inputs of the systems, but also on the state of the system [7].

A continuous system which can be described by differential equation is said to be nonlinear if its present output is not a linear combination of past input and output signal elements. A nonlinear continuous-time control system (CTCS) arises in many realistic models of problems in science, engineering and medicine. The general form of the n^{th} order nonlinear continuous-time *invariant* control system (i.e. systems with fixed constant coefficients), in which the forcing function involves derivative terms is given by [6,7]:

$$y^{(n)}(t) = f \left(\begin{matrix} a_1 y(t), a_2 y'(t), \dots, a_n \\ y^{(n-1)}(t), b_0 u(t), b_1 u'(t) \\ \dots, b_n u^{(n)}(t) \end{matrix} \right) \quad \dots (1)$$

where $u(t)$ is the forcing function of the system, $y(t)$ is the output of the system, $a_1, a_2, \dots, a_n, b_0, b_1, \dots, b_n$ are constants and n is a fixed nonnegative integer. Eq.(1) is called a single-input-single-output (SISO) system.

For multiple-input-multiple-output (MIMO) nonlinear continuous

time invariant systems of differential equations with m forcing function, the general form of this system is [1]:

$$\frac{dx_i(t)}{dt} = f_i \left(a_1 x_1(t), a_2 x_2(t), \dots, a_n x_n(t), b_1 u_1(t), b_2 u_2(t), \dots, b_m u_m(t) \right) \quad \dots (2)$$

where $i=1,2,\dots,n$ and the outputs ($y_q(t)$, $q=1,2,\dots,p$) of the system are related to the state variables and the input through the following expression:

$$y_q(t) = g_q \left(a_1 x_1(t), a_2 x_2(t), \dots, a_n x_n(t), b_1 u_1(t), b_2 u_2(t), \dots, b_m u_m(t) \right) \quad \dots (3)$$

where $x_1(t), x_2(t), \dots, x_n(t)$ are the state variables, $u_1(t), u_2(t), \dots, u_m(t)$ are the input variables of the system, $a_1, a_2, \mathbf{K}, a_n, b_1, b_2, \mathbf{K}, b_m$ are constants, f_i , $i=1,2,\dots,n$ denotes the i^{th} nonlinear functional relationship and g_q , $q=1,2,\dots,p$ denotes the q^{th} nonlinear functional relationship [5].

Note that, when the system in equations (2) and (3):

- 1- Has one input ($m = 1$) and one output ($p = 1$), then the system is called system with single-input-single-output (SISO).
- 2- Has one input ($m = 1$) and (p) outputs, then the system is called system with single-input-multiple-output (SIMO).
- 3- Has (m) inputs and one output ($p = 1$), then the system is called system with multiple-input-single-output (MISO).
- 4- Has (m) inputs and (p) outputs, then the system is called system with multiple-input-multiple-output (MIMO) variables.

3. State - Space Representation of Nonlinear Continuous Time Systems:

State-space method has emerged in the last fifty years, where toward the end of 1950's, the concept of representing a continuous-time system by a set of first order differential equations has become a standard tool in control theory [2]. It has become popular in the early 1970's with high-speed digital computers since it permits a simple notation that is easily accepted and processed by digital computer. State space method is ideally suited for the analysis of MIMO systems as well as SISO systems where any state can be represented by a point in the state space [8].

The state of a dynamic system is the smallest set of variables, called state variables, such that the knowledge of these variables at time $t = t_0$, together with the input $u(t)$ for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$ where t_0 is the initial state [9].

A natural description for dynamical systems is the nonlinear state-space equation. The state completely describes the time history of the system while the output of the system at a certain time (t) is completely described by its state and inputs at the time (t) [9].

A mathematical model of a nonlinear CTCS is a state space model which can be computed as follows:

The SISO system in eq.(1) can be written as:

Let

$$\left. \begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \mathfrak{X}(t) \\ &\mathbf{M} \\ x_n(t) &= y^{(n-1)}(t) \end{aligned} \right\} \dots (4)$$

Then Eq.(4) can be written as:

$$\left. \begin{aligned} \mathfrak{X}_1(t) &= x_2(t) \\ \mathfrak{X}_2(t) &= x_3(t) \\ &\mathbf{M} \\ \mathfrak{X}_{n-1}(t) &= x_n(t) \\ \mathfrak{X}_n(t) &= f \left(\begin{aligned} &a_1 x_1(t), a_2 x_2(t), \dots, \\ &a_n x_n(t), b_0 u(t), b_1 u'(t), \\ &\dots, b_n u^{(n)}(t) \end{aligned} \right) \end{aligned} \right\} \dots (5)$$

or

$$\mathfrak{X}(t) = f(x(t), u(t)) \dots (6)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)] \in \mathbb{R}^n$ are the state variables and $u(t)$ is the control input which is the forcing function of the system. Eq.(6) is called the state equation.

The output equation of state space model becomes:

$$y(t) = g(x(t), u(t)) \dots (7)$$

where g denotes the nonlinear functional relationship related to the state variables and the input of the system.

For MIMO nonlinear continuous time invariant system in eq.(2) and eq.(3) the state space model of these equations can be computed as follows:

Eq.(2) can be written as:

$$\mathfrak{X}(t) = f_i(x(t), u(t)) \dots (8)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)] \in \mathbb{R}^n$ are the state variables, $u(t) = [u_1(t), u_2(t), \dots, u_m(t)] \in \mathbb{R}^m$

are the input variables of the system, $a_1, a_2, \mathbf{K}, a_n, b_1, b_2, \mathbf{K}, b_m$ are constants and $f_i, i=1,2,\dots,n$ denotes the i^{th} nonlinear functional relationship. The output variables in eq.(3) of the system can be written as:

$$y_q(t) = g_q(x(t), u(t)) \dots (9)$$

where g_q denotes the q^{th} nonlinear functional relationship and $q = 1, 2, \dots, p$.

Nonlinear state space equations (8) and (9) can represent complicated multivariable and it can be take several kinds according to the state and output equations form as illustrated in the following subsection.

3.1 Classification of Nonlinear State-Space Equation (SSE):

- (i) If the state and output equations are nonlinear then the general form of state space equation can be written as in eq.(8) and eq.(9).
- (ii) If the state equation is linear while the output equation is nonlinear then the general form of state space equation can be written as:

$$\mathfrak{X}(t) = Ax(t) + Bu(t) \dots (10)$$

$$y(t) = g(x(t), u(t)) \dots (11)$$

where A is the square $(n \times n)$ time-invariant system matrix and B is the $(n \times m)$ time-invariant input matrix of the system.

3.2 State-Space Equation of Nonlinear Time Varying System:

An advantage of state space approach to the control system analysis is that it can be extended to nonlinear time varying systems.

The system is called time-varying system if the coefficients $a_1, a_2, \mathbf{K}, a_n, b_0, b_1, \mathbf{K}, b_n$ of

equations (1), (2) and (3) are functions of continuous time t [10,11].

The state and output equations of state space representation of nonlinear time varying system can be computed as described in section (3).

4. Hûta Method :

One of the most important developments in numerical analysis is Hûta method. The concept of Hûta method is essentially an extrapolation procedure and has the advantage of being self-starting. It uses only the information from the last step computed; therefore, it is called single-step method. Hûta method has been widely used for solving linear and nonlinear problems arising in various fields of science and engineering.

In this paper Hûta method was employed for finding the numerical solution for different types of n^{th} order nonlinear SSE's of CTCS.

Consider the following first order nonlinear differential equation:

$$y' = f(t, y(t)) \quad \dots (12)$$

with initial condition $y(t_0) = y_0$.

In order to solve equation (12) numerically, a sixth-order eight-stage method which is valid for a nonlinear differential equation is derived by Hûta [12, 13] as:

$$y_{n+1} = y_n + \frac{h}{840} \begin{pmatrix} 41H_1 + 216H_3 + \\ 27H_4 + 272H_5 + \\ 27H_6 + 216H_7 + \\ 41H_8 \end{pmatrix} \quad \dots (13)$$

where

$$\left. \begin{aligned} H_1 &= f(t_n, y(t_n)) \\ H_2 &= f\left(t_n + \frac{1}{9}h, y(t_n) + \frac{1}{9}hH_1\right) \\ H_3 &= f\left(t_n + \frac{1}{6}h, y(t_n) + \frac{h}{24}(H_1 + 3H_2)\right) \\ H_4 &= f\left(t_n + \frac{1}{3}h, y(t_n) + \frac{h}{6}\left(H_1 - 3H_2\right) + 4H_3\right) \\ H_5 &= f\left(t_n + \frac{1}{2}h, y(t_n) + \frac{h}{8}\begin{pmatrix} -5H_1 + \\ 27H_2 - \\ 24H_3 + \\ 6H_4 \end{pmatrix}\right) \\ H_6 &= f\left(t_n + \frac{2}{3}h, y(t_n) + \frac{h}{9}\begin{pmatrix} 221H_1 - \\ 981H_2 + \\ 867H_3 - \\ 102H_4 + H_5 \end{pmatrix}\right) \\ H_7 &= f\left(t_n + \frac{5}{6}h, y(t_n) + \frac{h}{48}\begin{pmatrix} -183H_1 + \\ 678H_2 - \\ 472H_3 - \\ 66H_4 + \\ 80H_5 + \\ 3H_6 \end{pmatrix}\right) \\ H_8 &= f\left(t_n + h, y(t_n) + \frac{h}{82}\begin{pmatrix} 716H_1 - \\ 2079H_2 + \\ 1002H_3 + \\ 834H_4 - \\ 454H_5 - \\ 9H_6 + 72H_7 \end{pmatrix}\right) \end{aligned} \right\} \quad \dots (14)$$

5. Hûta Method for Solving SSE of Nonlinear CTCS:

In this section different types of nonlinear SSE's have been solved using Hûta method.

5.1 The Solution of n^{th} Order SSE of Nonlinear CTCS In Which The Forcing Function Involves Derivative Terms :

In this subsection Hûta method including sixth order is candidate to find the numerical solution for the following nonlinear SSE:

Recall eq.(5), eq.(5) can be written as:

$$\frac{dx_i(t)}{dt} = f_i \left(\begin{matrix} a_1 x_1(t), a_2 x_2(t), \dots, \\ a_n x_n(t), b_0 u(t), b_1 u'(t) \\ \dots, b_n u^{(n)}(t) \end{matrix} \right) \quad \dots (15)$$

where $x_i(t)$ are the state variables and f_i , $i=1,2,\dots,n$ denotes the i^{th} nonlinear functional relationship while $u(t)$ is the forcing function of the system. The output of the system is obtained from eq.(7).

By applying Hûta method to find the numerical solution of eq.(15) using equations (13) and (14), one gets the following formula:

$$x_i(t_{j+1}) = x_i(t_j) + \frac{h}{840} \left(\begin{matrix} 41H_{1i} + \\ 216H_{3i} + \\ 27H_{4i} + \\ 272H_{5i} + \\ 27H_{6i} + \\ 216H_{7i} + \\ 41H_{8i} \end{matrix} \right) \quad \dots (16)$$

where $i=1,2,\dots,n$, $j=0,1,\dots,k$, $(k+1)$ is the number of points (t_0, t_1, \dots, t_k) and

$$\left. \begin{aligned} H_{1i} &= f_i \left(\begin{matrix} a_1 x_1(t_j), a_2 x_2(t_j), \dots, a_n x_n(t_j), \\ b_0 u(t_j), b_1 u'(t_j), \dots, b_n u^{(n)}(t_j) \end{matrix} \right) \\ H_{2i} &= f_i \left(\begin{matrix} a_1 x_1(t_j) + \frac{1}{9} h H_{11}, a_2 x_2(t_j) + \frac{1}{9} h H_{12}, \dots, a_n x_n(t_j) + \frac{1}{9} h H_{1n}, \\ b_0 u(t_j + \frac{1}{9} h), b_1 u'(t_j + \frac{1}{9} h), \dots, \\ b_n u^{(n)}(t_j + \frac{1}{9} h) \end{matrix} \right) \\ H_{3i} &= f_i \left(\begin{matrix} a_1 x_1(t_j) + \frac{h}{24} (H_{11} + 3H_{21}), \dots, \\ a_n x_n(t_j) + \frac{h}{24} (H_{1n} + 3H_{2n}), \\ b_0 u(t_j + \frac{1}{6} h), b_1 u'(t_j + \frac{1}{6} h), \dots, \\ b_n u^{(n)}(t_j + \frac{1}{6} h) \end{matrix} \right) \\ H_{4i} &= f_i \left(\begin{matrix} a_1 x_1(t_j) + \frac{h}{6} (H_{11} - 3H_{21} + 4H_{31}), \dots, a_n x_n(t_j) + \frac{h}{6} (H_{1n} - 3H_{2n} + 4H_{3n}), \\ b_0 u(t_j + \frac{1}{3} h), b_1 u'(t_j + \frac{1}{3} h), \dots, b_n u^{(n)}(t_j + \frac{1}{3} h) \end{matrix} \right) \\ H_{5i} &= f_i \left(\begin{matrix} a_1 x_1(t_j) + \frac{h}{8} (-5H_{11} + 27H_{21} - 24H_{31} + 6H_{41}), \dots, a_n x_n(t_j) + \frac{h}{8} (-5H_{1n} + 27H_{2n} - 24H_{3n} + 6H_{4n}), \\ b_0 u(t_j + \frac{1}{2} h), b_1 u'(t_j + \frac{1}{2} h), \dots, b_n u^{(n)}(t_j + \frac{1}{2} h) \end{matrix} \right) \end{aligned} \right\}$$

$$\begin{aligned}
 H_{6i} &= f_i \left(\begin{aligned} &a_1 x_1(t_j) + \frac{h}{9} (221H_{11} - 981H_{21} + \\ &867H_{31} - 102H_{41} + H_{51}), \dots, \\ &a_n x_n(t_j) + \frac{h}{9} (221H_{1n} - 981H_{2n} + \\ &867H_{3n} - 102H_{4n} + H_{5n}), \\ &b_0 u(t_j + \frac{2}{3}h), b_1 u'(t_j + \frac{2}{3}h), \dots, \\ &b_n u^{(n)}(t_j + \frac{2}{3}h) \end{aligned} \right) \\
 H_{7i} &= f_i \left(\begin{aligned} &a_1 x_1(t_j) + \frac{h}{48} (-183H_{11} + 678H_{21} \\ &- 472H_{31} - 66H_{41} + 80H_{51} + \\ &3H_{61}), \dots, a_n x_n(t_j) + \frac{h}{48} (-183H_{1n} \\ &+ 678H_{2n} - 472H_{3n} - 66H_{4n} + \\ &80H_{5n} + 3H_{6n}), b_0 u(t_j + \frac{5}{6}h), \\ &b_1 u'(t_j + \frac{5}{6}h), \dots, b_n u^{(n)}(t_j + \frac{5}{6}h) \end{aligned} \right) \\
 H_{8i} &= f_i \left(\begin{aligned} &a_1 x_1(t_j) + \frac{h}{82} (716H_{11} - 2079H_{21} \\ &+ 1002H_{31} + 834H_{41} - 454H_{51} - \\ &9H_{61} + 72H_{71}), \dots, a_n x_n(t_j) + \frac{h}{82} \\ &(716H_{1n} - 2079H_{2n} + 1002H_{3n} + \\ &834H_{4n} - 454H_{5n} - 9H_{6n} + \\ &72H_{7n}), b_0 u(t_j + h), b_1 u'(t_j + h) \\ &\dots, b_n u^{(n)}(t_j + h) \end{aligned} \right) \\
 &\dots (17)
 \end{aligned}$$

for each $i=1,2,\dots,n$ and $j=0,1,\dots,k$.

The output values of nonlinear SSE in eq.(7) can be computed using Hûta method as :

$$y(t_j) = g(x(t_j), u(t_j))$$

where $j=0,1,\dots,k$.

The following (SSSE-HM) algorithm summarizes the steps for finding the numerical solution for the SISO SSE (5) in which the forcing function involves derivative terms using Hûta method.

SSSE-HM Algorithm :

Input

- t_0 (the initial state).
- k ($(k+1)$ is the number of points (t_0, t_1, \dots, t_k)).
- n (the order of the SSE).
- The functions f_i , $i=1,2,\dots,n$ of state equation (15).
- The function g of the output equation (7) of state space model.

Output

- The state variables $x_i(t_j)$ where $i=1,2,\dots,n$ and $j=0,1,\dots,k$.
- The output values of SSE $y(t_j)$, $j=0,1,\dots,k$.

Step 1: Set $h = \frac{t_k - t_0}{k}$.

Step 2: Set $j=0$

Step 3: For $i=1:n$ compute H_{1i} in eq.(17).

Step 4: For $i=1:n$ compute H_{2i} in eq.(17).

Step 5: For $i=1:n$ compute H_{3i} in eq.(17).

Step 6: For $i=1:n$ compute H_{4i} in eq.(17).

Step 7: For $i=1:n$ compute H_{5i} in eq.(17).

Step 8: For $i=1:n$ compute H_{6i} in eq.(17).

Step 9: For $i=1:n$ compute H_{7i} in eq.(17).

Step 10: For $i=1:n$ compute H_{8i} in eq.(17).

Step 11: For $i=1:n$ compute:

$$t_{j+1} = t_j + h$$

$$x_i(t_{j+1}) = x_i(t_j) + \frac{h}{840} \begin{pmatrix} 41H_{1i} + \\ 216H_{3i} + \\ 27H_{4i} + \\ 272H_{5i} + \\ 27H_{6i} + \\ 216H_{7i} + \\ 41H_{8i} \end{pmatrix}$$

Step 12: Put $j = j+1$

Step 13: If $j = k$ then
go to (step 14).

Step 14: Else go to (step 3)
For $j=0,1,\dots, k$ compute
the output values of SSE :
 $y(t_j) = g(x(t_j), u(t_j))$.

5.2 The Solution of n^{th} Order SSE of Nonlinear CTCS With (m) Forcing Functions :

The sixth order Huta method is applied to find the numerical solution for the MIMO nonlinear SSE's (8) and (9) as follows:

Consider the state equation in eq.(8), by applying Hûta method for it using equations (13) and (14), one gets the following formula :

$$x_i(t_{j+1}) = x_i(t_j) + \frac{h}{840} \begin{pmatrix} 41H_{1i} + \\ 216H_{3i} + \\ 27H_{4i} + \\ 272H_{5i} + \\ 27H_{6i} + \\ 216H_{7i} + \\ 41H_{8i} \end{pmatrix} \quad \dots (18)$$

where $i=1,2,\dots,n$, $j=0,1,\dots, k$,
($k + 1$) is the number of points
(t_0, t_1, \dots, t_k) and

$$H_{1i} = f_i \begin{pmatrix} a_1 x_1(t_j), a_2 x_2(t_j), \dots, a_n x_n(t_j), \\ b_1 u_1(t_j), b_2 u_2(t_j), \dots, b_m u_m(t_j) \end{pmatrix} \quad \dots (19)$$

$$H_{2i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{1}{9} h H_{11}, a_2 x_2(t_j) + \frac{1}{9} h H_{12}, \dots, a_n x_n(t_j) + \frac{1}{9} h H_{1n}, \\ b_1 u_1(t_j + \frac{1}{9} h), b_2 u_2(t_j + \frac{1}{9} h), \\ \dots, b_m u_m(t_j + \frac{1}{9} h) \end{pmatrix} \quad \dots (20)$$

$$H_{3i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{h}{24} (H_{11} + 3H_{21}), \dots, \\ a_n x_n(t_j) + \frac{h}{24} (H_{1n} + 3H_{2n}), \\ b_1 u_1(t_j + \frac{1}{6} h), b_2 u_2(t_j + \frac{1}{6} h) \\ \dots, b_m u_m(t_j + \frac{1}{6} h) \end{pmatrix} \quad \dots (21)$$

$$H_{4i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{h}{6} (H_{11} - 3H_{21} + \\ 4H_{31}), \dots, a_n x_n(t_j) + \frac{h}{6} (H_{1n} - \\ 3H_{2n} + 4H_{3n}), b_1 u_1(t_j + \frac{1}{3} h), \\ b_2 u_2(t_j + \frac{1}{3} h), \dots, b_m u_m(t_j + \frac{1}{3} h) \end{pmatrix} \quad \dots (22)$$

$$H_{5i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{h}{8} (-5H_{11} + 27H_{21} - \\ 24H_{31} + 6H_{41}), \dots, a_n x_n(t_j) + \\ \frac{h}{8} (-5H_{1n} + 27H_{2n} - 24H_{3n} + \\ 6H_{4n}), b_1 u_1(t_j + \frac{1}{2} h), \\ b_2 u_2(t_j + \frac{1}{2} h), \dots, b_m u_m(t_j + \frac{1}{2} h) \end{pmatrix} \quad \dots (23)$$

$$H_{6i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{h}{9}(221H_{11} - 981H_{21} + 867H_{31} - 102H_{41} + H_{51}), \dots, \\ a_n x_n(t_j) + \frac{h}{9}(221H_{1n} - 981H_{2n} + 867H_{3n} - 102H_{4n} + H_{5n}), \\ b_1 u_1(t_j + \frac{2}{3}h), \dots, b_m u_m(t_j + \frac{2}{3}h) \end{pmatrix} \dots (24)$$

$$H_{7i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{h}{48}(-183H_{11} + 678H_{21} - 472H_{31} - 66H_{41} + 80H_{51} + 3H_{61}) \\ \dots, a_n x_n(t_j) + \frac{h}{48}(-183H_{1n} + 678H_{2n} - 472H_{3n} - 66H_{4n} + 80H_{5n} + 3H_{6n}), b_1 u_1(t_j + \frac{5}{6}h) \\ \dots, b_m u_m(t_j + \frac{5}{6}h) \end{pmatrix} \dots (25)$$

$$H_{8i} = f_i \begin{pmatrix} a_1 x_1(t_j) + \frac{h}{82}(716H_{11} - 2079H_{21} + 1002H_{31} + 834H_{41} - 454H_{51} - 9H_{61} + 72H_{71}), \dots, a_n x_n(t_j) + \frac{h}{82} \\ (716H_{1n} - 2079H_{2n} + 1002H_{3n} + 834H_{4n} - 454H_{5n} - 9H_{6n} + 72H_{7n}) \\ b_1 u_1(t_j + h), \dots, b_m u_m(t_j + h) \end{pmatrix} \dots (26)$$

for each $i=1,2,\dots,n$ and $j=0,1,\dots,k$.

The outputs of nonlinear SSE in eq.(9) can be computed using Hûta method as :

$$y_q(t_j) = g_q \begin{pmatrix} a_1 x_1(t_j), a_2 x_2(t_j), \\ \dots, a_n x_n(t_j), b_1 u_1(t_j), \\ b_2 u_2(t_j), \dots, b_m u_m(t_j) \end{pmatrix} \dots (27)$$

where $q=1,2,\dots,p$ and $j=0,1,\dots,k$.

The following (MSSE-HM) algorithm summarizes the steps for finding the numerical solution for the MIMO nonlinear SSE in eq.(8) and eq.(9) using Hûta method.

MSSE-HM Algorithm :

Input

- t_0 (the initial state).
- k ($(k + 1)$ is the number of points (t_0, t_1, \dots, t_k)).
- n (the order of the SSE).
- The functions f_i , $i=1,2,\dots,n$ of state equation (8).
- The functions g_q , $q=1,2,\dots,p$ of the output equation (9) of state space model.

Output

- The state variables $x_i(t_j)$ where $i=1,2,\dots,n$ and $j=0,1,\dots,k$.
- The output values of SSE $y_q(t_j)$ where $q=1,2,\dots,p$ and $j=0,1,\dots,k$.

Step 1: Set $h = \frac{t_k - t_0}{k}$.

Step 2: Set $j=0$

Step 3: For $i = 1 : n$ compute H_{1i} in eq.(19).

Step 4: For $i = 1 : n$ compute H_{2i} in eq.(20).

Step 5: For $i = 1 : n$ compute H_{3i} in eq.(21).

Step 6: For $i = 1 : n$ compute H_{4i} in eq.(22).

Step 7: For $i = 1 : n$ compute H_{5i} in eq.(23).

Step 8: For $i = 1 : n$ compute H_{6i} in eq.(24).

Step 9: For $i = 1 : n$ compute H_{7i} in eq.(25).

Step 10: For $i = 1 : n$ compute H_{8i} in eq.(26).

Step 11: For $i = 1 : n$ compute:

$$t_{j+1} = t_j + h$$

$$x_i(t_{j+1}) = x_i(t_j) + \frac{h}{840} \begin{pmatrix} 41H_{1i} + \\ 216H_{3i} + \\ 27H_{4i} + \\ 272H_{5i} + \\ 27H_{6i} + \\ 216H_{7i} + \\ 41H_{8i} \end{pmatrix}$$

Step 12: For $q = 1 : p$ compute the output values of MIMO-SSE :

$$y_q(t_j) = g_q \begin{pmatrix} a_1 x_1(t_j), a_2 x_2(t_j), \\ \dots, a_n x_n(t_j), b_1 u_1(t_j), \\ b_2 u_2(t_j), \dots, b_m u_m(t_j) \end{pmatrix}$$

Step 13: Put $j = j+1$

Step 14: If $j = k$ then stop.
Else go to (step 3)

6. Test Examples:

The previous methods in section (5) are illustrated in the following examples:-

Example (1) :

In the SISO sort occurring in robotic time invariant system problem shown in the figure (2) [7],

The state-space model of figure (2) was derived as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \frac{1}{2}x_2(t)x_3(t) - \frac{1}{2}x_2^2(t) + 2x_1(t) \\ + 2u(t) - 6u(t) + 2u(t)u(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where the initial state is: $x(0) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

and the forcing function: $u(t) = t, t \geq 0$.

The exact solution of the above SISO state-space model is:

$$x(t) = \begin{bmatrix} Exact_1 \\ Exact_2 \\ Exact_3 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e' + 2t \\ e' + 2 \\ e' \end{bmatrix}$$

When the algorithm (SSSE-HM) is applied, table (1) presents the comparison between the exact and numerical solution of state-space model using Hûta method for $k=10$, $h=0.1$ and $t_i = ih, i = 0,1,\dots,k$ depending on least square error (L.S.E.). The output variables $y(t)$ of state space model by applying (SSSE-HM) algorithm is also tabulated.

Figure (3) shows the solution of nonlinear state-space model, which was given in example (1) by using Hûta method with the exact solutions.

Example (2) :

Consider the MIMO time-invariant control system shown in figure (4):

The MIMO state-space model was derived from figure (4) as follows:

$$\begin{aligned}\dot{x}_1(t) &= x_1^2(t) - x_1(t)x_2(t) - u_1(t) \\ \dot{x}_2(t) &= x_1(t) + 2x_2(t)x_3(t) - u_1(t) + u_2(t) - u_3(t) \\ \dot{x}_3(t) &= 2x_1(t)x_3(t) - x_3(t) + u_2(t) - u_3(t) \\ y_1(t) &= x_1(t) \\ y_2(t) &= x_2(t)x_3(t) + x_2(t) \\ y_3(t) &= x_3(t) + 2u_3(t)\end{aligned}$$

The initial state of the MIMO state-space model is:

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ and}$$

the forcing function is:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} t-1 \\ 2e^t \\ 2te^t \end{bmatrix}, t \geq 0.$$

The exact solution of the above nonlinear MIMO state-space model is:

$$x(t) = \begin{bmatrix} Exact_1 \\ Exact_2 \\ Exact_3 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} t \\ t-1 \\ e^t \end{bmatrix}$$

When the algorithm (MSSE-HM) is applied, table (2) presents the comparison between the exact and numerical solution using Hûta method for $k=10$, $h=0.1$ and $t_i = ih$, $i = 0,1,...,k$ depending on least square error (L.S.E.). The output

variables $y(t)$ of state space model by applying (MSSE-HM) algorithm are also tabulated in table (3).

Example (3):

Consider the following SIMO time varying gyro controller system [10]:

$$\begin{aligned}\dot{x}(t) + 2t\dot{x}(t) - x(t) &= 2t\dot{x}(t) + 2tu(t) \\ y_1(t) &= 4x^2(t) + u(t) \\ y_2(t) &= x(t)\dot{x}(t)\end{aligned}$$

with initial conditions: $x(0) = 0$, $\dot{x}(0) = 1$ and the forcing function $u(t) = \cos(t)$, $t \geq 0$.

The state-space equation was derived using eq.(10) and eq.(11) as follows :

$$\begin{aligned}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & -2t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2tu(t) + 2t\dot{x}(t) \end{bmatrix} \\ y_1(t) &= 4x_1^2(t) + u(t) \\ y_2(t) &= x_1(t)x_2(t)\end{aligned}$$

where the initial state of the state-space model is:

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The exact solution of the above SIMO state-space model is:

$$x(t) = \begin{bmatrix} Exact_1 \\ Exact_2 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

When the algorithm (MSSE-HM) is applied, table (4) presents the comparison between the exact and numerical solution using Hûta method for $k=10$, $h=0.01$ and $t_i = ih$, $i = 0,1,...,k$ depending on

least square error (L.S.E.). The output variables $y(t)$ of state space model by applying (MSSE-HM) algorithm is also tabulated.

Figure (5) shows the solution of nonlinear state-space model, which was given in example (3) by using Hûta method with the exact solutions.

Conclusions

Hûta method has been presented to find the numerical solution for different types of n^{th} order SSE's of nonlinear continuous-time control system. The results show a marked improvement in the least square errors (L.S.E.). From solving some examples the following points are listed:

- 1- Hûta method solves the nonlinear SSE of the SISO time invariant and time varying system as well as MIMO time invariant and time varying system.
- 2- Hûta method gives a better accuracy and consistent to the solution of different types of n^{th} order nonlinear SSE's of CTCS where it solves the nonlinear SSE of n^{th} order by reducing it to a system of first order equations.
- 3- The good approximation depends on the size of h , if h is decreased then the number of points (knots) increases and the L.S.E. approaches to zero where this gives the advantage of numerical computation.

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Table (1) The solution $x(t)$ and the output variables $y(t)$ of state space model for Ex.(1).

t	$Exact_1$	$\hat{H}u_{x_1(t)}$	$Exact_2$	$\hat{H}u_{x_2(t)}$	$Exact_3$	$\hat{H}u_{x_3(t)}$	Output $y(t)$	$\hat{H}u_{y(t)}$
0	1.0000	1.0000	3.0000	3.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.3052	1.3052	3.1052	3.1052	1.1052	1.1052	1.3052	1.3052
0.2	1.6214	1.6214	3.2214	3.2214	1.2214	1.2214	1.6214	1.6214
0.3	1.9499	1.9499	3.3499	3.3499	1.3499	1.3499	1.9499	1.9499
0.4	2.2918	2.2918	3.4918	3.4918	1.4918	1.4918	2.2918	2.2918
0.5	2.6487	2.6487	3.6487	3.6487	1.6487	1.6487	2.6487	2.6487
0.6	3.0221	3.0221	3.8221	3.8221	1.8221	1.8221	3.0221	3.0221
0.7	3.4138	3.4138	4.0138	4.0138	2.0138	2.0138	3.4138	3.4138
0.8	3.8255	3.8255	4.2255	4.2255	2.2255	2.2255	3.8255	3.8255
0.9	4.2596	4.2596	4.4596	4.4596	2.4596	2.4596	4.2596	4.2596
1	4.7183	4.7183	4.7183	4.7183	2.7183	2.7183	4.7183	4.7183
L.S.E.		0.825e-20	L.S.E.	0.825e-20	L.S.E.	0.825e-20	L.S.E.	0.825e-20

Table (2) The solution $x(t)$ of state space model for Ex.(2).

t	$Exact_1$	$\hat{H}u_{x_1(t)}$	$Exact_2$	$\hat{H}u_{x_2(t)}$	$Exact_3$	$\hat{H}u_{x_3(t)}$
0	0.0000	0.0000	-1.0000	-1.0000	1.0000	1.0000
0.1	0.1000	0.1000	-0.9000	-0.9000	1.1052	1.1052
0.2	0.2000	0.2000	-0.8000	-0.8000	1.2214	1.2214
0.3	0.3000	0.3000	-0.7000	-0.7000	1.3499	1.3499
0.4	0.4000	0.4000	-0.6000	-0.6000	1.4918	1.4918
0.5	0.5000	0.5000	-0.5000	-0.5000	1.6487	1.6487
0.6	0.6000	0.6000	-0.4000	-0.4000	1.8221	1.8221
0.7	0.7000	0.7000	-0.3000	-0.3000	2.0138	2.0138
0.8	0.8000	0.8000	-0.2000	-0.2000	2.2255	2.2255
0.9	0.9000	0.9000	-0.1000	-0.1000	2.4596	2.4596
1	1.0000	1.0000	0.0000	0.0000	2.7183	2.7183
L.S.E.		0.852e-14	L.S.E.	0.159e-13	L.S.E.	0.486e-14

Table (3) The output variables $y(t)$ of state space model for Ex.(2).

t	Output $y_1(t)$	Hûta $y_1(t)$	Output $y_2(t)$	Hûta $y_2(t)$	Output $y_3(t)$	Hûta $y_3(t)$
0	0.0000	0.0000	-2.0000	-2.0000	1.0000	1.0000
0.1	0.1000	0.1000	-1.8947	-1.8947	1.5472	1.5472
0.2	0.2000	0.2000	-1.7771	-1.7771	2.1985	2.1985
0.3	0.3000	0.3000	-1.6449	-1.6449	2.9697	2.9697
0.4	0.4000	0.4000	-1.4951	-1.4951	3.8787	3.8787
0.5	0.5000	0.5000	-1.3244	-1.3244	4.9462	4.9462
0.6	0.6000	0.6000	-1.1288	-1.1288	6.1952	6.1952
0.7	0.7000	0.7000	-0.9041	-0.9041	7.6523	7.6523
0.8	0.8000	0.8000	-0.6451	-0.6451	9.3473	9.3473
0.9	0.9000	0.9000	-0.3460	-0.3460	11.3142	11.3142
1	1.0000	1.0000	0.0000	0.0000	13.5914	13.5914
L.S.E.		0.852e-14	L.S.E.		0.199e-12	0.486e-14

Table (4) The solution $x(t)$ and the output variables $y(t)$ of state space model for Ex.(3).

t	$Exact_1$	$H\hat{u}_t$ $x_1(t)$	$Exact_2$	$H\hat{u}_t$ $x_2(t)$	$Output$ $y_1(t)$	$H\hat{u}_t$ $y_1(t)$	$Output$ $y_2(t)$	$H\hat{u}_t$ $y_2(t)$
0	0.0000	0.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000
0.1	0.0998	0.0998	0.9950	0.9950	1.0349	1.0349	0.0993	0.0993
0.2	0.1987	0.1987	0.9801	0.9801	1.1379	1.1379	0.1947	0.1947
0.3	0.2955	0.2955	0.9553	0.9553	1.3047	1.3047	0.2823	0.2823
0.4	0.3894	0.3894	0.9211	0.9211	1.5276	1.5276	0.3587	0.3587
0.5	0.4794	0.4794	0.8776	0.8776	1.7970	1.7970	0.4207	0.4207
0.6	0.5646	0.5646	0.8253	0.8253	2.1006	2.1006	0.4660	0.4660
0.7	0.6442	0.6442	0.7648	0.7648	2.4249	2.4249	0.4927	0.4927
0.8	0.7174	0.7174	0.6967	0.6967	2.7551	2.7551	0.4998	0.4998
0.9	0.7833	0.7833	0.6216	0.6216	3.0760	3.0760	0.4869	0.4869
1	0.8415	0.8415	0.5403	0.5403	3.3726	3.3726	0.4546	0.4546
$L.S.E.$		0.194e-18	$L.S.E.$	0.151e-18	$L.S.E.$	0.638e-17	$L.S.E.$	0.333e-18

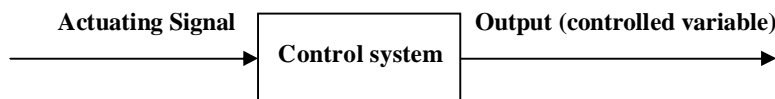


Figure (1) The basic control system.

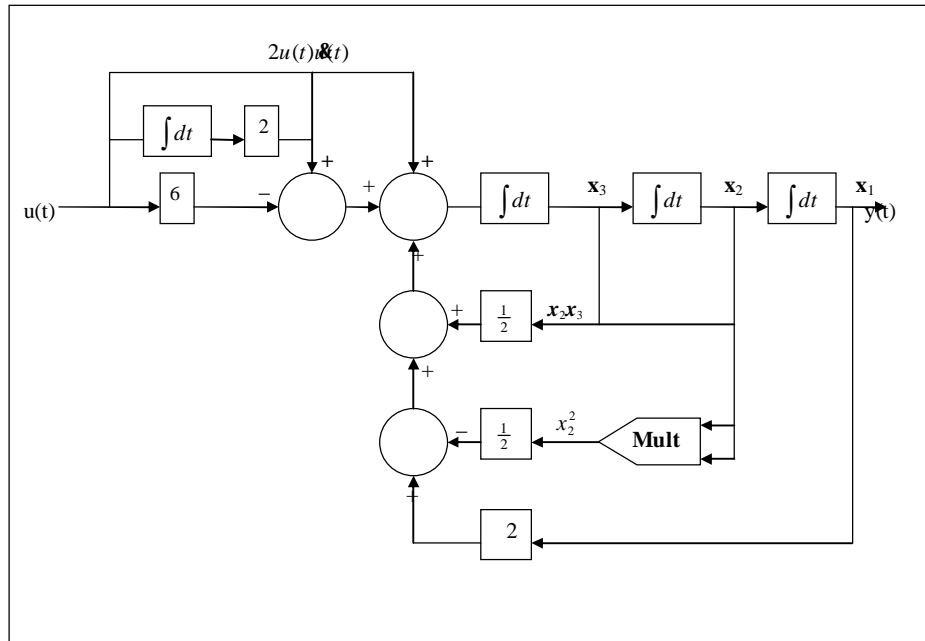


Figure (2) Simulation diagram for sort occurring in robotic system.

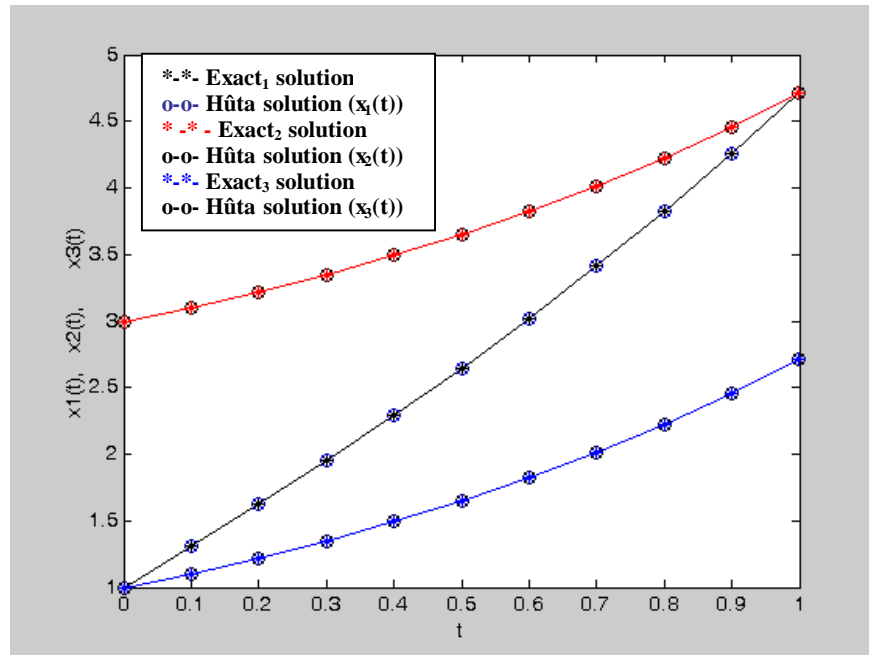


Figure (3) The comparison between the exact and Hûta solution for nonlinear state-space model in Ex.(1).

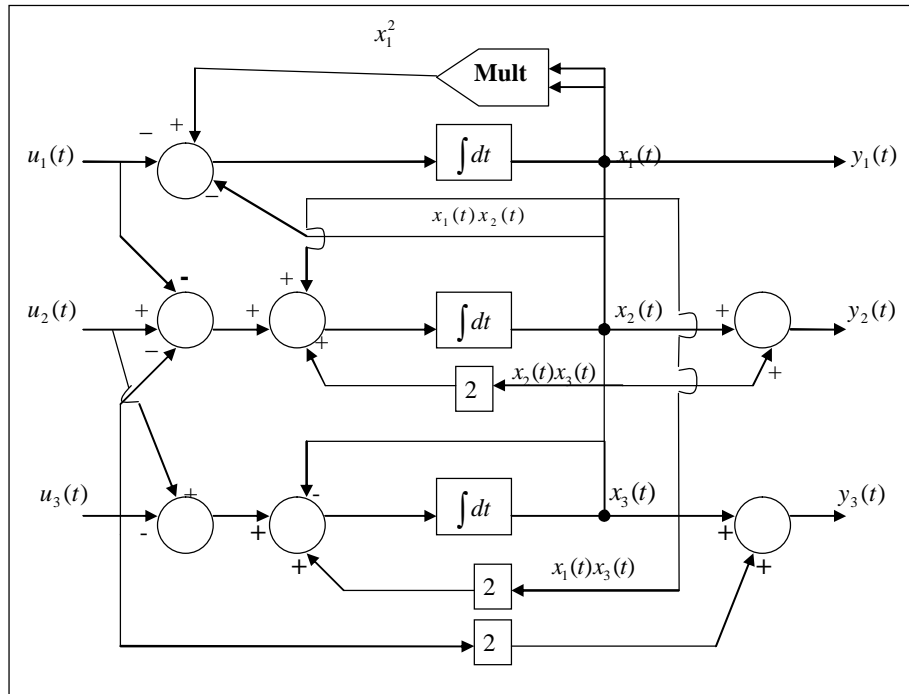


Figure (4) Simulation diagram for nonlinear MIMO system.

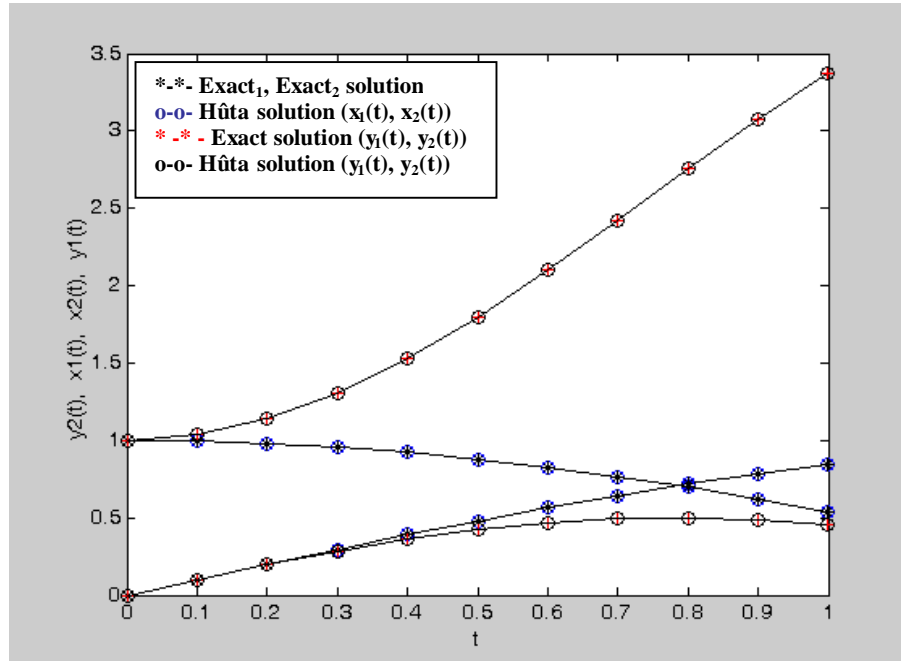


Figure (5) The comparison between the exact and Hûta solution for nonlinear state-space model of SIMO gyro system in Ex.(3).