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An Approximate Solutions of Fuzzy Linear Fredholm Integral Equations

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Abstract

The main aims of this paper are studying and modifying an approximate to solve fuzzy linear integral equations of Fredholm type.

Two different Kinds of fuzzy functions are used to transform the ordinary linear integral equations of Fredholm type to the fuzzy form.

Keywords: fuzzy, integral equation, fredholm

الحلول التقريبية للمعادلة فريد هولم الخطية الضبابية

الخلاصة

أن الهدف الرئيسي من هذا البحث هو دراسة و تطوير طريقة تقريبية لحل المعادلات التكاملية الضبابية الخطية من نوع فريدهولم, وقد تم استخدام نوعين من الدوال الضبابية لتحويل المعادلات التكاملية من هذا النوع من الحالات الاعتيادية إلى الحالة الضبابية.

1. Introduction

The concept of a fuzzy set was introduced by professor Lotfi A. Zadeh in 1965, he introduced a paper about fuzzy set, since that many papers had been introduced in different mathematical fields theoretical and applications[7]

Although the applications of fuzzy systems has developed significantly during the last 40 years, there was little research investigating the approximation of fuzzy systems until recently[2].

The concept of fuzzy integral introduced by Sugeno in 1974 to provide means by which information may be integrand from differing sources to derive a combined classification[4].

2. Fuzzy Sets

Definition2.1 :(Membership Function)[6] The expression of crisp set by using characteristic function $X_A: X \to \{0,1\}$,

Such that $X_A = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases} \quad \forall x \in X$

This characteristic function is represented as $X_A : x \to I = [0,1]$

Such that $0 \le X_A(x) \le 1$ for all $x \in X$, called fuzzy subset of X denoted by A as a membership function $A: X \to I$ such that $0 \le A(x) \le 1$ for all $x \in X$.

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Definition2.2 [1]

Let f: X$\rightarrow$Y be a function from the crisp set X to the crisp set Y , and let A and B be fuzzy sets on X and Y respectively, there exists a fuzzy set $\overset{-1}{f}(B) : X \rightarrow I$

Such that $\overset{-1}{f}(B)(x) = B(f(x))$ for all x $\in$ X, and there exists a fuzzy set $\tilde{f}(A) : Y \rightarrow I$ such that

$$\tilde{f}(A)(y) = \begin{cases} \sup\left\{A(x) : x \in \overset{-1}{f}(y)\right\} & \text{if } \overset{-1}{f}(y) \neq f \\ 0 & \text{if } \overset{-1}{f}(y) = f \end{cases}$$

Therefore

$$\tilde{f}(A)(y) = \sup\left\{A(x) : x \in \overset{-1}{f}(y)\right\} \; when$$

we denote sup$f$ by 0

3. Fuzzy Real Number [1]

Fuzzy numbers can be introduced in order to model imprecise situations involving real numbers, and one of the first problem one meets working with these is to decide what type of order use on the fuzzy number set.

Most of fuzzy numbers are defined by a function, which maps each fuzzy number into an ordered set and transfers the order of one set to the other.

Fuzzy numbers have been utilized on many fields to represent some linguistic terms.

4. Linear Fuzzy Equations [1]

A fuzzy equation is an equation whose coefficients and/or variables are fuzzy sets of R. The concept of equation can be extended to deal with fuzzy quantities in several ways.

Consider the simple equation ax+ b=x, where $a, b \in R^{+}, x$ is real variable and $a \neq 1$, so that the unique solution is $x = \dfrac{b}{1-a}$,

then the fuzzy equation $\tilde{a}\tilde{x} + \tilde{b} = \tilde{x}$, *where* $\tilde{a}, \tilde{b} \in F(R), x \in R$ means that the fuzzy set $\tilde{a}\tilde{x} + \tilde{b}$ is the same as $\tilde{x}$.

The fuzzy equation can be solved if the fuzzy equation in a - level set, for all $a \in [0,1]$ is treated in two different ways.

5. Fuzzy Function

Fuzzy systems are widely used for linguistic modeling of functional relationships. The fuzzy system generates a fuzzy function $\tilde{F} : X \longrightarrow F(y)$, from the input space X to the space F(Y) of all subsets of the set Y, where X$\subset R^m$ and Y$\subset R^n$.

There are many types of definitions for fuzzy functions in the literature. In this section more attention is paid to a new form of definition of fuzzy function, which is a form of fuzzy bunch of functions $\tilde{F}$.

Definition (3.4)[1]

Let $\tilde{F}$:X$\longrightarrow$ F(R) for all $a \in I$, the functions f_a^- and f_a^+ in R$^{\times}$ satisfies:

1- $F(f_a^-) = F(f_a^+) = a$

2- $\tilde{F}(x) = \bigcup_{a \in I} [f_a^-(x), f_a^+(x)]$

3- $\tilde{F} = \bigcup_{a \in I} [f_a^-, f_a^+]$.

Definition (5.1)[4]

Let $\tilde{F} : X \to F(R)$, if $\tilde{F}$ written as $\tilde{F} = \{(f_i, a_i)\}_{i=1}^n$,then $\tilde{F}$ is finite, and it also written in the form $\tilde{F}(x) = \{(f_i(x), a_i)\}_{i=1}^n$ for all $x \in X$.

6.Fuzzy Functional[1]

In this section the concept of functional will be extended to a real fuzzy function ($\tilde{F}$ (x)).

Differential of fuzzy function at fuzzy point is discussed.

Integral of fuzzy function over fuzzy interval also discussed.

Given a fuzzy function $\tilde{F}$(x) with $\tilde{F} : x \to F(R)$ and a functional $J : R^x \to R$, then a fuzzy functional $\tilde{J} : \tilde{R}^x \to \tilde{R}$, such that $\tilde{J}(F)(y) = \sup\{(F(f))(y) : y = J(f), f \in R^x\}$ for all y∈R

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Fuzzy Integral Operator [1]

Given a fuzzy function $\tilde{F} : X \to F(R)$, let $J : R^x \to R^x$ be an ordinary integral operator, then fuzzy integral operator defined as:

$$(J(\tilde{F}))(g) = (\tilde{J}(\tilde{F})(x))(g)$$
$$= \sup\left\{\tilde{F}(f) : g = \int_a^x f\right\}$$

for all a∈ R and for all x∈ X

7.Wavelet solution

Now we modify the method of wavelet solution used in solving ordinary linear Fredholm integral equations for solving fuzzy linear Fredholm integral equations of the first kind and the second kind, the discussion of this method restrict here for the equation defined on the positive real axis.

We write equation (1.1) and equation (1.2)

$$\tilde{U}(x) = \int_0^a \tilde{K}(x,t)\tilde{U}(t)\,dt \quad 1.1$$

$$\tilde{U}(x) = \tilde{F}(x) + \int_0^a \tilde{K}(x,t)\tilde{U}(t)\,dt \quad 1.2$$

7.1 If each function in equation (1.1) or in equation (1.2) has the form of the function in the definition(5.1) , then for all $a \in I$, the method of wavelet solution discussed as follow:

If we consider a linear Fredholm integral equations of the second kind which are of the form

$$u_a^-(x) = f_a^-(x) + \int_0^a K_a^-(x,t)u_a^-(t)dt,$$

$$u_a^+(x) = f_a^+(x) + \int_0^a K_a^+(x,t)u_a^+(t)dt,$$

The symbols K_a^- *and* K_a^+ are used to denote the integral operator of the above equations, which are given by the formula

$$((Ku)(x))_a^- = \int_0^a K_a^-(x,t)u_a^-(t)dt$$

$$((Ku)(x))_a^+ = \int_0^a K_a^+(x,t)u_a^+(t)dt$$

Then the original integral equations written in operators form as

$$((I-K)u)_a^- = f_a^-$$

$$((I-K)u)_a^+ = f_a^+$$

The wavelet approximation to the solutions $u_a^-(x)$ *and* $u_a^+(x)$ at the scale m are:

$$u_a^-(x) = \sum_K (\tilde{c}_K)_a^- 2^{m/2} f_a^-(2^m x - k)$$

$$u_a^+(x) = \sum_K (\tilde{c}_K)_a^+ 2^{m/2} f_a^+(2^m x - k)$$

Where $(\tilde{c}_K)_a^-$ *and* $(\tilde{c}_K)_a^+$ are the wavelet coefficients of u_a^- *and* u_a^+ respectively.

We make substitute on y=2mx, to write

$$u_a^-(x) = \sum_K (c_K)_a^- \; f_a^-(y-k)$$

$$u_a^+(x) = \sum_K (c_K)_a^+ \; f_a^+(y-k)$$

where

$$(c_k)_a^- = (\tilde{c}_k)_a^- 2^{m/2}$$

$$and \quad (c_k)_a^+ = (\tilde{c}_k)_a^+ 2^{m/2}.$$

Similarly, the wavelet expansion for $f_a^-(x), f_a^+(x), K_a^-(x,t)$ and $K_a^+(x,t)$ are

$$K_a^-(x,t) = \sum_K \sum_L (c_{k,l})_a^- f_a^-(y-k) f_a^-(s-l)$$

$$K_a^+(x,t) = \sum_K \sum_L (c_{k,l})_a^+ f_a^+(y-k) f_a^+(s-l)$$

where

$(c_{k,l})_a^- = 2^m(\tilde{c}_{k,l})_a^-, (c_{k,l})_a^+ = 2^m(\tilde{c}_{k,l})_a^+ \; and \; s = 2^{m/2}t$.

And

$$f_a^-(x) = \sum_K (g_K)_a^- \; f_a^-(y-k),$$

$$f_a^+(x) = \sum_K (g_K)_a^+ \; f_a^+(y-k),$$

Where $(g_K)_a^- = 2^{m/2}(\tilde{g}_K)_a^-$ $and \; (g_K)_a^+ = 2^{m/2}(\tilde{g}_K)_a^+$.

Now, substitute the expansion of $u_a^-(x), u_a^+(x), f_a^-(x)$, $f_a^+(x), K_a^-(x,t) \; and \; K_a^+(x,t)$ in the original equations, to obtain

$$\sum_K (c_K)_a^- f_a^-(y-k) - 2^{-m}\sum_K f_a^-(y-k) \sum_L (c_{K,L})_a^-(c_L)_a^- = \sum_K (g_K)_a^- f_a^-(y-k)$$

$$\sum_K (c_K)_a^+ f_a^+(y-k) - 2^{-m}\sum_K f_a^+(y-k) \sum_L (c_{K,L})_a^+(c_L)_a^+ = \sum_K (g_K)_a^+ f_a^+(y-k)$$

Taking the inner product of both sides of the above equations with $f_a^-(y-j)$ and $f_a^+(y-j)$ respectively, to obtain

$$\sum_K (c_K)_a^- \int_0^a f_a^-(y-k) f_a^-(y-j)dy - 2^{-m}\sum_K \int_0^a f_a^-(y-k) f_a^-(y-j)dy \sum_L (c_{K,L})_a^-(c_L)_a^- = \sum_K (g_K)_a^- \int_0^a f_a^-(y-k) f_a^-(y-j)dy,$$

$$\sum_K (c_K)_a^+ \int_0^a f_a^+(y-k) f_a^+(y-j)dy - 2^{-m}\sum_K \int_0^a f_a^+(y-k) f_a^+(y-j)dy \sum_L (c_{K,L})_a^+(c_L)_a^+ = \sum_K (g_K)_a^+ \int_0^a f_a^+(y-k) f_a^+(y-j)dy,$$

then

$$\int_0^a f_a^-(y-k) f_a^-(y-j)dy = (d_{j,k})_a^-,$$

$$\int_0^a f_a^+(y-k) f_a^+(y-j)dy = (d_{j,k})_a^+,$$

and

$$(c_j)_a^- - 2^{-m}\sum_L (c_{j,L})_a^- \; (c_L)_a^- = (g_j)_a^-,$$

$$(c_j)_a^+ - 2^{-m}\sum_L (c_{j,L})_a^+ \; (c_L)_a^+ = (g_j)_a^+.$$

This completes the solution of $u_a^- \; and \; u_a^+$.

7.2 If each function in equation (1.1) or in equation (1.2) has the form of the function in definition (5.3), then for all $a_i \in [0,1]$, the wavelet solution discussed as follow:

Consider a linear Fredholm integral equation of the second Kind which is of the form

$$(u_i(x),a_i)=(f_i(x),a_i)+\int_0^a (k_i(x,t),a_i)\ (u_i(t),a_i)dt$$

where i=1,2,……,m

(K_i,a_i) used to denote the integral operator of the above equation, which given by the formula

$$(((Ku)(x))i,a_i)=\int_0^a (k_i(x,t),a_i)\ (u_i(t),a_i)dt$$

Then the original integral equation written in operator form as $(((I-K)u)i,a_i)=(f_i,a_i)$

The wavelet approximation to the solution ui(x) at scale m is

$$(u_i(x),a_i)=\sum_K((\tilde{c}_K)_i,a_i)\ 2^{m/2}\left(f_i\left(\overset{m}{2}x-k\right),a_i\right)$$

where $(\tilde{c}_K)_i$ is the wavelet coefficients of ui.

We make the substitute on $y=\overset{m}{2}t$, then we may write

$$(u_i(x),a_i)=\sum_K((c_K)_i,a_i)(f_i(y-k),a_i)$$

where

$$((c_k)_i,a_i)=((\tilde{c}_k)_i,a_i)\ 2^{m/2}.$$

Similarly, the wavelet expansion for fi(x) and ki(x,t) are

$$(k_i(x,t),a_i)=\sum_K\sum_L((c_{K,L})_i,a_i)$$

$$(f_i(y-k),a_i)\ (f_i(s-l),a_i),$$

where

$$((c_{k,l})_i,a_i)=\overset{m}{2}((\tilde{c}_{k,l})_i,a_i)\ \ and\ \ s=\overset{m/2}{2}t,$$

and

$$(f_i(x),a_i)=\sum_K((g_K)_i,a_i)(f_i(y-k),a_i)$$

Where

$$((g_k)_i,a_i)=((\tilde{g}_k)_i,a_i)2^{m/2}$$

Now, if we substitute the expansion of $u_i(x)$, $f_i(x)$ and $k_i(x,t)$ into our original integral equation, we have

$$\sum_K((c_K)_i,a_i)\ (f_i(y-k),a_i)-2^{-m}\sum_K(f_i(y-k),a_i)\sum_L((c_{K,L})_i,a_i)$$

$$=\sum_K((g_K)_i,a_i)\ (f_i(y-k),a_i).$$

Taking the inner product of both sides with $(f_i(y-j),a_i)$ to obtain

$$\sum_{K}((c_K)_i, a_i)\int_0^a (f_i(y-k), a_i)$$

$$(f_i(y-j), a_i)dy - 2^{-m}$$

$$\sum_{K}\int_0^a (f_i(y-k), a_i)\ (f_i(y-j), a_i)dy$$

$$\sum_{L}((c_{K,L})_i, a_i) = \sum_{K}((g_K)_i, a_i)$$

$$\int_0^a (f_i(y-k), a_i)\ (f_i(y-j), a_i)dy,$$

and

$$\int_0^a (f_i(y-k), a_i)\ (f_i(y-j), a_i)dy = ((d_{j,k})_i, a_i)$$

so

$$((c_j)_i, a_i) - 2^{-m}\sum_{L}((c_{j,L}c_L)_i, a_i) = ((g_j)_i, a_i)$$

This completes the solution of u_i

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