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# Approximate Solution of Fuzzy Caputo's-Katugampola differential equation with order $0 < \beta < 1$

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**ABSTRACT:** In this paper, some of the fuzzy second-order Caputo's- Katugampola fractionals which included also the first-order Caputo's- Katugmpola fractionals have been presented with analytic interesting results to explain the solution in fuzzy real numbers and distinguish space included the type of functions which suitable to the problem formulations which are under the studied. all the fuzzy results are supported by the numerical solutions used later on. The interesting illustrative examples for the application of some classes of fuzzy Caputo- Katugampla fractional order differential equations with $0 < \beta < 1$ and explained their systems (n, m) where n, m=1,2, moreover the tables of different parameters and fractional orders have been given in detail as different values of a fuzzy parameter. All tables are represented by figures that are given for the first time. Coupled figures for each table refer to the lower and upper of the fuzzy solution. The algorithm of reproducing kernel Hilbert space is used with their steps and Gram-Schmidt orthogonalization process to obtain the approximate solution

**Keywords:** Caputo- Katugampla, Riemann-Liouville, Gram-Schmidt, reproducing kernel, Riemann- Hadamard

## 1. INTRODUCTION

Fractional calculus is exactly the extension of classical ordinary differential and integral calculus, where integrals and derivatives have an arbitrary real order. Since the 17th century, after that they introduced several different derivatives such as Riemann-Liouville, Hadamard, Grunwald-Letnikov, Caputo, see [29,33,38], where its own advantages and disadvantages. The appropriate fractional derivative or integral have been chose it depends on the system was considered, with respect to that we find many researchers devoted to different fractional operator's equations. the interesting issue is how to generalized fractional operators. So U. Katugampola presented new interesting types of fractional operators, which generalized both fractional derivatives operators of the Riemann-Liouville and Hadamard fractional operators find recent references [25,26,27].

The Gram-Schmidt orthogonalization process have been used to implement RKHSM [7,41]. Since this process take a lot of time to run the algorithm, we make it unstable numerically for that here we act in a way and put in another way this process. Many approaches combine the methods introduced in [7,8]. More specifically, see [7,41]. On the other hand if not the orthogonalization process, the nonlinear problem used in RKSHM is applied successfully to solve their problems. Zaremba introduced the Reproducing Kernel Hilbert spaces (RKHS) in the early 20th century, where they mocked to study boundary value problems for biharmonic and harmonic functions as well as Aronzajn and Bergman in the mid-century studied a general theory or RKHS, first RKHS have seen increasing use for solving ordinary differential equation and partial differential, integral, and problems in optimal control , statistics and dynamical system moreover, RKHS have been studied of serval researchers (e.g., Akgul & Grow, Yao, and Yousefi) after Cui and Lin developed RKHS with piecewise polynomial kernels Hilbert space on $W_2^n[a,b]$ and $W_2^{(m,n)}[a,b]$. The important useful of a fuzzy simulation is the technological issues discussion. Many reseahers applied fuzzy set theory in various disciplines, including control systems, robotics, knowledge-based systems, image processing, industrial automation, power engineering,

*Corresponding author:* alihj97@ uomustansiriyah.edu.iq consumer electronics management, artificial intelligence/expert systems, and operation research. The fractional calculus has been assumed for solving challenging phenomena and sustainability due to its beneficial qualities such as non-locality, high dependability, inheritance, and analyticity [9,36].

The modification of fractional was considered to develop a solution of inhomogeneous equation. Various researchers have created the fractional derivative and fractional differential equation (FDEs), including Caputo, Liouville, Letnikov, Hadamard, Riez. Abel, Caputo Fabrizio, Grunwald, and Atangana-Baleanu [32,24] in their discussion. FDEs have been used in several interested interactions in acoustics, science electromagnetics, viscoelasticity, material and electrochemistry [35,14]. Minggen et al. in [30,31], introduced Reproducing Kernel Hilbert space (RKHS) and developed in approximate theory, learning theory, statistics, complex analysis, and machine group representation, the theory Reproducing Kernel Hilbert Space Method (RKHSM) is a component of a kernel-based approximation method that was applied for solving nonlinear boundary value problems [28-12], generalized singular nonlinear Lane-Emden type equation [12], integrodifferential equation [41-3], integrodifferential fractional equation [2], Bratus problem [20]. In [5], Ricardo Almeida and Agnieszka B. Malinowska studied Caputa-Katugampola differential equations.

The novel reproducing kernel algorithm was used for fined the approximate solutions introduced in [1], also numerical solution of coupled system of fractional order in [37]. The nonlinear integral equations classes and its applications as well as solving second order fuzzy integro- differential equations by using kernel Hilbert space method in [23],[13]. The new reproducing kernel Hilbert spaces method studied in the semi-infinite domain for approximate solution investigated in [16]. The dirichlet partial integrodifferentiall equations have been solved by using the computational algorithm in [6], Moreover, Hifer- katugampola fractional differential equations and their existence and uniqueness are presented in [10]. The reliable method for the fuzzy solution of problems included the system of fuzzy fractional equations, has been studied in [15]. The fuzzy fractional Kortewege-de varies equation with double parameters was studied in [41]. The main goal of their study is to study the interest issue approximate solution of $0 < \alpha < 1$ of Fuzzy Caputo's- Katugmpola fractional differential equations with some important detailed by necessary and sufficient conditions which explained all the results with their procedure of analytic for systems types such as different (n,m)-system and studied on some regular and suitable inner product spaces of analytic solution and using an efficient method such as reproducing kernel Hilbert space method and which reproducing depended on presented problem which is under studied and type of related spaces that used in the application problem formulation which modeled by fuzzy fractional order differential equations with different fractional orders can be extended later on .see also[22,40].

The reproducing kernel method has been used with fuzzy Caputo's-Katugmpola fractional order integral equations with different choosing of fractional order. The technical was a generalization of fuzzy Caputo's fractional order differential equations. The convergence of approximate solutions explained by a series of continuous functions approaches converge to an exact solution. The approximate solution was computed by applying the Gram-Schmidt orthogonal process. We introduced the effectiveness and efficiency of the method by some illustrative examples of presented fuzzy fractional differential equations with a maximum of order of 1 or 2. The orthogonal functions are computed by using the producing kernel method depending on the integral operator and reproducing kernel. The parameters of Caputo's- Katugmpola derivative and the parameter of fuzzy set made a good approximation with different values and clear the efficiently of the method by figures and some tables for all system types with different values of fractional orders.

## 2. BASIC DEFINITIONS AND CONCEPTS:

The following concepts and elementary results for $\mathrm{Caputo-Katugampola}$ fractional are very interesting for complet the new results and illustrative examples.

**Lemma (2.1),[11]:**

Let $\beta \in (0,1)$ and $\gamma > 0$, the to side of Caputo − Katugampola fractional derivatives of function $f \in C^1[a,b]$ given by

$$ {}^{ck}_{a}D^{\beta,\gamma}_{t} f(x) = \frac{\gamma^{\beta}}{\Gamma(1-\beta)} \int_a^t (t^{\gamma} - s^{\gamma})^{-\beta} f'(s)ds, \qquad (1)$$

$$ {}^{ck}_{t}D^{\beta,\gamma}_{b} f(t) = \frac{-\gamma^{\beta}}{\Gamma(1-\beta)} \int_b^t (s^{\gamma} - t^{\gamma})^{-\beta} f'(s)ds \qquad (2)$$

**Lemma (2.2),[11]:**

Let Re $(\alpha) \geq 0, n = [Re(\alpha)] + 1$ and $f \in C[a,b]$

1. If Re $(\alpha) \neq 0$ or $\propto \in N$, then
   ${}^{Ck}_{a}D^{\propto,\gamma}\, {}^{Ck}_{a}I^{\propto,\gamma} f(x) = f(x), \quad {}^{C}_{b}D^{\propto,\gamma}\, {}^{C}_{b}I^{\propto,\gamma} f(x) = f(x).$
2. If Re $(\alpha) \neq 0$ or Re$(\propto) \in N$, then
   ${}^{Ck}_{a}D^{\propto,\gamma}\, {}_{a}I^{\propto,\gamma} f(x) = f(x) - \frac{{}_{a}I^{\propto+1-n,\gamma} f(a)}{\Gamma(n-\alpha)} \left(\frac{x^{\gamma}-a^{\gamma}}{\gamma}\right)^{n-\alpha}, \quad {}^{Ck}D_b^{\propto,\gamma}\, I_b^{\propto,\gamma} f(x) = f(x) - \frac{I_b^{\propto+1-n,\gamma} f(b)}{\Gamma(n-\alpha)} \left(\frac{b^{\gamma}-x^{\gamma}}{\gamma}\right)^{n-\alpha}.$
3. If $0 < \propto \leq 1$ we have ${}^{Ck}_{a}I^{\propto,\gamma}\, {}^{Ck}_{a}D^{\propto,\gamma} f(x) = f(x) - f(a),$

$$ {}^{Ck}I_b^{\propto,\gamma}\, D_b^{\propto,\gamma} f(x) = f(x) - f(b)$$

4. Let Re $(\alpha) > 0, n = -[-Re(\propto)]$, ${}^{Ck}_{\ a}I^{\propto,\gamma} f \in AC_\delta^n[a,b]({I}_b^{\propto,\gamma} f \in AC_\delta^n[a,b])$. Then

$$({}^{Ck}_{\ a}D^{\propto,\gamma}\, {}^{Ck}_{\ a}I^{\propto,\gamma}) f(x) = f(x) - \sum_{j=1}^{n} \frac{{}^{Ck}_{\ a}D^{\propto-j,\gamma}}{\Gamma(\propto-j+1)} f(a) \left(\frac{x^\gamma - a^\gamma}{\gamma}\right)^{\alpha-j}$$

$$({}^{Ck}I_b^{\propto,\gamma} D_b^{\propto,\gamma}) f(x) = f(x) - \sum_{j=1}^{n} (-1)^j \frac{D_b^{\propto-j,\gamma}}{\Gamma(\propto-j+1)} f(b) \left(\frac{b^\gamma - x^\gamma}{\gamma}\right)^{\alpha-j}$$

**Definition (2.3),[43]**

Let $\mathbb{X}$ be a collection of objects denoted generically by $x$ then a fuzzy set $\widetilde{\mathbb{A}}$ in $\mathbb{X}$ is a set of ordered pairs: $\widetilde{\mathbb{A}} = \{(\mathbb{x}, \mu_{\widetilde{\mathbb{A}}}(\mathbb{x}))|\mathbb{x} \in \mathbb{X}\}$, where $\mu_{\widetilde{\mathbb{A}}}(\mathbb{x})$is called the membership function or grade of membership (also the degree of compatibility) of $\mathbb{x}$ in $\widetilde{\mathbb{A}}$ that maps $\mathbb{X}$ to the membership space $\mathbb{M}$ when 0,1 this only points belong to is $\mathbb{M}$ , $\widetilde{\mathbb{A}}$ nonfuzzy and $\mu_{\widetilde{\mathbb{A}}}$is identical to the characteristic of nonfuzzy set.

**Definition (2.4),[21]:**

A fuzzy number is a fuzzy set $\mathcal{T}$: R → [0,1] which satisfies:

1. $\mathcal{T}$ is upper semi-continuous
2. $\mathcal{T}(x) = 0$ outside some interval [a, d].
3. There are real numbers b, c : $a \le b \le c \le d$ for which

**a.** $\mathcal{T}(\boldsymbol{x})$ is monotonic increasing on [a, b],

**b.** $\mathcal{T}(\boldsymbol{x})$ is monotonic decreasing on [c, d],

**c.** $\mathcal{T}(\boldsymbol{x}) = 1, b \le x \le c$ .

$$i.e.\mathcal{T}(x) = \begin{cases} 0,\ if & x \le a \\ f(x) & a \le x \le b \\ 1, & b \le x \le c \end{cases},$$

Where f is an increasing function and is called the left side

While if $\mathcal{T}(x) = \begin{cases} g(x), & c \le x \le d \\ 0 & x \ge d \end{cases}$

Where g is a decreasing function and is called right side.

- $\mathcal{T}$ is called symmetric fuzzy number if $\mathcal{T}(z+x) = \mathcal{T}(z-x)$for all x belong to R, where $z = \frac{b+c}{2}$.
- The set of all the fuzzy number is denoted by $\mathbb{E}^1$.
- If $\mathcal{T}(x)$ in the interval $[a,b]$ $and$ $[c,d]$ is linear then it is called a trapezoidal fuzzy number (which we will discuss later) and we write $\mathcal{T}(x) = (a,b,c,d)$.

**Definition (2.5),[22]:**

An arbitrary fuzzy number parametric form is represented by an ordered pair of Function $\left(\overline{\mathcal{T}(\mathrm{r})}, \underline{\mathcal{T}(r)}\right)$, r ∈ [0,1], which satisfy the following requirement:

1. $\underline{\mathcal{T}(r)}$is a bounded left-continuous non-decreasing function over [0,1].
2. $\overline{\mathcal{T}(r)}$is a bounded left-continuous non-increasing function over [0,1].
3. $\underline{\mathcal{T}(r)} \le \overline{\mathcal{T}(r)};\ 0 \le r \le 1$..

**Remark (2.6),[22]:**

A crisp number $\alpha$ is simply represented by

$\underline{\mathcal{T}(r)} = \overline{\mathcal{T}(r)} = \alpha, 0 \le r \le 1.$

Also $\mathcal{T} = (\underline{\mathcal{T}}, \overline{\mathcal{T}})$is called a symmetric fuzzy number in parametric from if

$\mathcal{T}^c(r) = \frac{\underline{\mathcal{T}} + \overline{\mathcal{T}}}{2}$ is a real constant for all $r \in [0,1]$.

**Theorem (2.7),[13]:**

Let $F: [a,b] \to R_F$ be continuous fuzzy function, where $[F(t)]^r = \quad [F_{1r}(t), F_{2r}(t)]$. If $F_{1r}(t) and\ F_{2r}(t)$ are integrable functions over$[a,b]$ ,then

$$\int_a^b F(t)dt \in R_F\ and\ , \quad \left[\int_a^b F(t)dt\right]^r = \left[\int_a^b F_{1r}(t)dt, \int_a^b F_{2r}(t)dt\right]$$

**Definition (2.8),[23]:**

A Hilbert space $W_2^m[a,b]$ is defined as $W_2^m[a,b] = \{u(t): u(t), u'(t),,, u^{(m-1)}(t)$ are absolutely continuous, $u^m(m) \in L^2[a,b]$ and $u(a) = u'(a) = \cdots = u^{(m-1)}(t) = 0$ whenever $m \neq 1$. Whilst the inner product and the norm in $W_2^m[a,b]$ are defined by

$$\langle u_1(t), u_2(t)\rangle_{W_2^m} = \sum_{i=0}^{m-1} u_1^{(i)}(a)u_2^{(i)}(a) + \int_a^b u_1^m(t)u_2^m(t)dt \qquad (3)$$

And $\|u_1(t)\|_{W_2^m} = \sqrt{\langle u_1(t), u_2(t)\rangle}_{W_2^m}\ where\ u_1, u_2 \in W_2^m[a,b]$.

**Theorem (2.9),[23]:**

Functional space $W_2^m[a,b]$ is Reproducing Kernel Hilbert space.

Now, it is taken away that expression from the reproducing kernel Hilbert space function $R_x(t) \in W_2^m[a,b]$.

Based on the essay, it is easy to prove that $R_x(t)$ id answer of the following generalized differential equation [30]:

$$\frac{\partial^{2m}R_x(t)}{\partial t^{2m}}(-1)^m = \delta(t-x),$$
$$\frac{\partial^i R_y(a)}{\partial t^i} - (-1)^{m-i-1}\frac{\partial^{2m-i-1}R_y(a)}{\partial t^{2m-i-1}} = 0 \qquad (4)$$
$$\frac{\partial^{2m-i-1}R_y(b)}{\partial t^{2m-i-1}} = 0,$$
$$i = 0,1, \dots, m-1,$$

Where $\delta$ is Dirac's delta function. While $x \neq t, R_x(t)$ is the answer of the following constant linear homogenous differential equation with 2m order:

$$\frac{\partial^{2m}R_x(t)}{\partial t^{2m}}(-1)^m = 0, \qquad (5)$$

With a boundary condition

$$\frac{\partial^i - y(a)}{\partial t^i} - (-1)^{m-i-1}\frac{\partial^{2m-i-1}R_y(a)}{\partial t^{2m-i-1}} = 0,$$
$$\frac{\partial^{2m-i-1}R_y(b)}{\partial^{2m-i-1}} = 0, \qquad i = 0,1, \dots, m-1 \qquad (6)$$

Equation (6) is characteristic where is $\varphi^{2m} = 0$ in equation (6). Then the general solution of Equation (6) is

$$R_x(t)\begin{cases}\sum_i^{2m} c_i(x)\, t^{i-1}, & t \leq x\\ \sum_i^{2m} d_i(x) t^{i-1}, & t > x\end{cases} \qquad (7)$$

Where coefficient $c_i(x) and\ d_i(x), i = 0,1,.., 2m$, could be calculated by solving

The following linear equations:

$$\frac{\partial^i R_x(x+0)}{\partial t^i} = \frac{\partial^i R_x(x-0)}{\partial t^i},\ \ i = 0,1, \dots, 2m-2,$$
$$\left(\frac{\partial^{2m-1}R_x(x+0)}{\partial t^{2m-1}} - \frac{\partial^{2m-1}R_x(x-0)}{\partial t^{2m-1}}\right) = (-1)^m, \qquad (8)$$
$$\frac{\partial^i R_y(a)}{\partial t^i} - (-1)^{m-i-1}\frac{\partial^{2m-i-1}R_y(a)}{\partial t^{2m-i-1}} = 0,$$
$$\frac{\partial^{2m-i-1}R_y(b)}{\partial t^{2m-i-1}} = 0,\ \ i = 0,1, \dots, m-1,$$
$$\frac{\partial^{2m-i-1}R_y(b)}{\partial t^{2m-i-1}} = 0,\ \ i = 0,1, \dots, m-1,$$

**Theorem (2.10),[24]:**

The space $W_2^1[0,1]$ is reproducing kernel space. That is for any fixed $x \in [0,1]$

There exists $R_1(x,y) \in W_2^1[0,1]$ such that $u(x) = (u(y), R_1(x,y))_1$ for any u(x) $\in W_2^1[0,1]$. The reproducing kernel $R_1(x,y)$ can be denoted by:

$$R_1(x) = \begin{cases}1+y, & y \leq x,\\ 1+x, & y < x.\end{cases}$$

The following algorithm to solve differential and fractional differential equations by using reproducing kernel method

**2.1. Algorithm:**

**Step1:** fix $a \leq \mathbb{x}\ and\ \mathbb{t} \leq b\ if\ \mathbb{t} \leq \mathbb{x}, set\ R_{\mathbb{x}}(\mathbb{t}) = 1 + \mathbb{t}$.

Else set $R_{\mathbb{x}}(\mathbb{t}) = 1 + \mathbb{x}$.

**Step2:** for $i = 1,2, \cdots, N\ set\ \mathbb{x}_i = \frac{(i-1)}{(N-1)}$.

Set $\psi_i(\mathbb{x}) = L_{\mathbb{t}}R_{\mathbb{x}}(\mathbb{t})|_{\mathbb{t}=\mathbb{x}_i}$.

**Step3:** for $i = j = 1,\ then\ set\ \beta_{11} = \frac{1}{\|\psi_1\|} = \frac{1}{\sqrt{\langle\psi_1, \psi_1\rangle}}$

Else $\beta_{ij} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} Z_{ik}^2}}$; $\quad Z_{ik} = \langle \psi_i, \overline{\psi_k} \rangle \;\; if\ i = j \neq 1$

Eles $\beta_{ij} = \frac{-\sum_{k=1}^{i-1} Z_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} Z_{ik}^2}}$ $\quad Z_{ik} = \langle \psi_i, \overline{\psi_k} \rangle \;\; if\ i \neq j$

**Step4:** for $i = 1,2,\cdots, N\ set\ \overline{\psi_i}(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$.

**Step5:** set $u_0(x_1) = u(x_1)$.

**Step6:** set $n = 1$

**Step7:** set $\mathbb{B}_n = \sum_{k=1}^{n} \beta_{nk}\psi_{k-1}(x_k)$.

**Step8:** $u_n(x) = \sum_{k=1}^{n} \mathbb{B}_i \overline{\psi_i}(x)$.

**Step9:** if $n < N$ then set $n = n + 1$ and go to step 7. Eles stop (7)

## 3. PROBLEM FORMULATION

Consider the following F- CKFDE

$$\left({}^{ck}D_{a+}^{\beta,\rho}\tilde{x}\right)(t) = \widetilde{K}(\tilde{x}(t),t)\widetilde{N}(\tilde{x},t), 1 < \beta < 2, t > a$$

$$\tilde{x}(t_0) = [\bar{x}_0, \underline{x}_0] \in \mathfrak{R}_F,\ \tilde{\acute{x}}(t_0) = [\bar{\acute{x}}_0, \underline{\acute{x}}_0] \qquad (9)$$

where K: $[\mathfrak{R} \times t_0, t_0 + a] \to \mathfrak{R}_F$ such that :

1) $[\widetilde{K}(t, x(t)]^r = [K_{1r}(x_{1r}(t), x_{2r}(t)t), K_{2r}(x_{1r}(t), X_{2r}(t)t)]$

$[\widetilde{N}(x(t))] = [N_{1r}(x_{1r}(t), x_{2r}(t), t), N_{2r}(x_{1r}(t), x_{2r}(t), t)]$

2) for any $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$|K_{1r}(x,y,t)N_{1r}(x,y,t) - K_{1r}(x_1,y_1.t_1)N_{1r}(x_1,y_1,t_1)| < \varepsilon$ and

$|K_{2r}(x,y,t)N_{2r}(x,y,t) - K_{2r}(x_1,y_1,t_1)N_{2r}(x_1,y_1,t_1)| < \varepsilon$ for

all $r \in [0,1]$, whenever $(x, y, t), (x_1, y_1, t_1) \in R_f \times [t_0, t_0 + a]$

$, \left\|(x,y,t) - (x_1,y_1,t_1)\right\|_{\mathfrak{R}^3} < \delta$ and $K_{1r}$ and $K_{2r}$ are uniformly bounded on any bounded set

3) There is an $L > 0$ such that $|K_{1r}(x_2,y_2,t_2)N_{1r}(x_2,y_2,t_2) - K_{1r}(x_1,y_1,t_1)N_{1r}(x_1,y_1,t_1)| = |K_{1r}(x_2,y_2,t_2)N_{1r}(x_2,y_2) - K_{1r}(x_2,y_2,t_2)N_{1r}(x_1,y_1) + K_{1r}(x_2,y_2,t_2)N_{1r}(x_1,y_1) - k_{1r}(x_1,y_1,t_1)|$

$\leq |K_{1r}(x_2,y_2,t_2)||N_{1r}(x_2,y_2) - N_{1r}(x_1,y_1)| + |K_{1r}(x_2,y_2,t_2) - K_{1r}(x_1,y_1,t_1||N_{1r}(x_1,y_1)| \leq L_1 Max\{|x_2 - x_1|, |y_2 - y_1|\} +$

$\mathrm{Max}\{|x_2 - x_1|, |y_2 - y_1|L_2$

$\in [0,1]$and $|K_{1r}(x_2,y_2,t_2)N_{1r}(x_2,y_2,t_2) - K_{1r}(x_1,y_1,t_1)N_{1r}(x_1,y_1,t_1) \leq L\ \ Max\{|x_2 - x_1|, |z_2 - z_1|\}$for all $r \in [0,1]$.

Then the F-CKFDE (9) is equivalent to the system of ordinary fractional differential equations (OFDEs):

$$({}^{ck}D_{t_0^+}^{\beta,\rho} x_{1r})(t) = K_{1r}(x_{1r}(t), x_{2r}(t), t)N_{1r}(x_{1r}(t), x_{2r}(t))$$

$$({}^{ck}D_{t_0^+}^{\beta,\rho} x_{2r})(t) = K_{2r}(x_{1r}(t), x_{2r}(t), t)N_{2r}(x_{1r}(t), x_{2r}(t))$$

$$x_{1r}(t_0) = x_{01r}, x_2(t_0) = x_{02r}$$

if x(t) is ${}^{c}[(1) - \beta] -$ differentiable. if $x(t)^{c}[(2) - \beta] -$ sifferentiable ,

Then (2.1) is equivalent to the following system of OFDEs:

$$({}^{ck}D_{t_0^+}^{\beta,\rho} x_{1r})(t) = K_{2r}(x_{1r}(t), x_{2r}(t), t)N_{2r}(x_{1r}(t), x_{2r}(t))$$

$$({}^{ck}D_{t_0^+}^{\beta,\rho} x_{2r})(t) = K_{1r}(x_{1r}(t), x_{2r}(t), t)N_{1r}(x_{1r}(\mathrm{t}), x_{2r}(\mathrm{t}))$$

Using this theorem. a F-CKFDE can be converted to a system of ODEs of fractional order. Then a numerical method can be applied to solve the resulting system.

Now we define a fuzzy Caputo-Katugampola fractional derivative of order $\beta \in (0,1)$for a fuzzy function $F: [a,b] \to \mathfrak{R}_F$,moreover, we give some

properties of the mentioned fractional H-derivative.

**Definition (3.2),[4]:**

Let $\mathcal{V}, \mathcal{U} \in \mathfrak{R}_F$ . If there exsist an element $d \in \mathfrak{R}_F$ such that $\mathcal{V} = \mathcal{U} + d$,then

we say that $d$ is the Hukuhara difference (H-difference) of $\mathcal{U}$and $\mathcal{V}$ denoted by

$\mathcal{V} \ominus \mathcal{U}$

**Definition (3.3),[4]:**
Let K be a fuzzy function on [a, b], then the r-cut function on [a, b] is an interval-valued function $k_r: [a, b] \to \mathfrak{R}_{\mathbb{C}}$ defined by $K_r(x) = [K(x)]^r, \forall r \in [0,1]$. Hence
$K_r(x) = [K_{1r}(x), K_{2r}(x)]$ where $k_{1r}$ and $k_{2r}$ are real valued function on $[a, b]$
given by $K_{1r}(x) = \min\{K_r(x)\}$ and $K_{2r}(x) = \max\{K_r(x)\}, \forall r \in [0,1]$L

**Definition (3.4),[4]:**
A mapping $K: [a, b] \to \mathfrak{R}_F$ is said to be Hukuhara differentiable, or simply H -differentiable, at $x_\circ \in [a, b]$ if there is a fuzzy number $K'(x_\circ)$ such that
$K'(x_\circ) = Lim_{h\to 0^+} \frac{K(x_\circ+h)\ominus K(x_\circ)}{h}$ and $Lim_{h\to 0^+} \frac{K(x_\circ)\ominus K(x_\circ-h)}{h}$ exist and are equal to $K'(x_\circ)$ which is called the H − derivative

**Definition (3.5),[4]:**
Let $K : (a, b) \to \mathfrak{R}_K$ and $x_0 \in (a, b)$. We say that K is strongly generalized differentiable at $x_0$ if there exists an element $K'(x) \in \mathfrak{R}_K$, such that

1. For all $h > 0$ sufficiently small, there exists $K(x_0 + h) \ominus K(x_0)$, there exists $K(x_0) \ominus K(x_0 - h)$ and the limits
$$\lim_{h\to 0^+} \frac{K(x_0+h)\ominus K(x_0)}{h} = \lim_{h\to 0^+} \frac{K(x_0)\ominus K(x_0-h)}{h} = K'(x_0),$$
2. For all $h > 0$ sufficiently small, there exists $K(x_0) \ominus K(x_0 + h)$, there exists $K(x_0 - h) \ominus K(x_0)$ and the limits
$$\lim_{h\to 0^+} \frac{K(x_0)\ominus K(x_0+h)}{-h} = \lim_{h\to 0^+} \frac{K(x_0-h)\ominus K(x_0)}{-h} = k'(x_0),$$
3. For all $h > 0$ sufficiently small, there exists $K(x_0 + h) \ominus K(x_0)$, there exists $K(x_0 - h) \ominus K(x_0)$ and the limits
$$\lim_{h\to 0^+} \frac{K(x_0+h)\ominus K(x_0)}{h} = \lim_{h\to 0^+} \frac{K(x_0-h)\ominus K(x_0)}{-h} = K(x_0),$$
4. For all $h > 0$ sufficiently small, there exists $K(x_0) \ominus K(x_0 + h)$, there exists $K(x_0) \ominus K(x_0 - h)$ and the limits
$$\lim_{h\to 0^+} \frac{K(x_0)\ominus K(x_0+h)}{-h} = \lim_{h\to 0^+} \frac{K(x_0)\ominus K(x_0-h)}{h} = k'(x_0)$$

**Definition (3.6),[4]:**
let $\beta \in (0,1]$ and $F: [a, b] \to \mathfrak{R}_F$ be such that $F \in C^F[a, b] \cap L^F[a, b]$, the first order Caputo's – Katugmpola H-derivative of F at $t \in (a, b)$ is defined as
$$({}^{ck}D^{\beta,\rho}_{a^+}F)(t) = \frac{\rho^\beta}{\sqrt{1-\beta}} \int_a^t (t^\rho - \tau^\rho)^{-\beta} F'(\tau)d\tau, t > a \qquad (10)$$
we say that F is ${}^c[(m, n) - \beta] -$ differentiable for $m, n \in \{0,1\}$ if (2.2) exists and F is (m, n)-differentiable.

**Definition (3.7):**
Let $f; [a, b] \to R$ and $f \in AC^F[a, b] \cap L^F[a, b]$ be a fuzzy set-value function and $\varphi(x) = \frac{\rho^\beta}{\Gamma(1-\beta)} \int_a^t (t^\rho - s^\rho)^{-\beta} F'(s)ds, t > a$. And then $f$ is side to be H-Caputo- Katugampola fuzzy fractional differentiable at x, when
$$({}^{ck}_aD^{\beta,\rho}_t F)(x) = \lim_{h\to 0} \frac{\varphi(x+h)\ominus_H \varphi(x)}{h}, \qquad (11)$$
Were
$$({}^{ck}_aD^{\beta,\rho}_t F_{1r})(x) = \left[\frac{\rho^\beta}{\Gamma 1-\beta} \int_a^t (t^\rho - s^\rho)^{-\beta} F'_{1r}(s)ds\right].$$
$$({}^{ck}_aD^{\beta,\rho}_t F_{2r})(x) = \left[\frac{\rho^\beta}{\Gamma(1-\beta)} \int_a^t (t^\rho - s^\rho)^{-\beta} F'_{2r}(s)ds\right]. \qquad (12)$$

**Lemma (3.8),[15];**
Let $u: [a, b] \to R_F$ be a fuzzy-valued function, where $[u(t)]_\alpha = [u_{1,\alpha}(t), u_{2,\alpha}(t)]$ for each $\alpha \in [0,1]$.

1- If u is (1)-differentiable, then $u_{1,\alpha}$ and $u_{2,\alpha}$ are differentiable function and
$$[D_1^1 u(t)]_\alpha = [u'_{1,\alpha}(t), u'_{2,\alpha}(t)]$$
2- If u is (3)-differentiable, then $u_{1,\alpha}$ and $u_{2,\alpha}$ are differentiable function and

$[D_2^1 u(t)]_\alpha = [u'_{2,\alpha}(t), u'_{1,\alpha}(t)]$.

**Lemma (3.9),[15];**

Let $D_1^1 u$ or $D_2^1 u: [a,b] \to R_F$ be a fuzzy -valued function where $[u(t)]_\alpha = [u_{1,\alpha}(t), u_{2,\alpha}(t)]$ for each $\alpha \in [0,1]$.

1- If $D_1^1 u$ is (1)-differentiable, then $u'_{1,\alpha}$ and $u'_{2,\alpha}$ are differentiable function and $[u''(t)]_\alpha = [u''_{1,\alpha}(t), u''_{2,\alpha}(t)]$
2- If $D_1^1 u$ is (2)-differentiable, then $u'_{1,\alpha}$ and $u'_{2,\alpha}$ are differentiable function and $[u''(t)]_\alpha = [u''_{2,\alpha}(t), u''_{1,\alpha}(t)]$
3- If $D_2^1 u$ is (1)-differentiable, then $u'_{1,\alpha}$ and $u'_{2,\alpha}$ are differentiable functions and $[u''(t)]_\alpha = [u''_{2,\alpha}(t), u''_{1,\alpha}(t)]$
4- If $D_2^1 u$ is (2)-differentiable, then $u'_{1,\alpha}$ and $u'_{2,\alpha}$ are differentiable functions and $[u''(t)]_\alpha = [u''_{1,\alpha}(t), u''_{2,\alpha}(t)]$

**Theorem (3.10);**

Let $\beta \in (0,1) and\ F \in AC^F[a,b] be\ such\ that\ [K(x)]^r = [K_{1r}(t), K_{2r}(t)], r \in [0,1]$. Then the first order Caputo's-Katugmpola H-differentiable exists almost everywhere on (a, b) and

1) if K is (1,1)- differentiable, then $[({}^{ck}D_{a^+}^{\beta,\rho})(t)]^r =$
$\frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau = [({}^{ck}D_{a^+}^{\beta,\rho}K_{1r})(t), ({}^{ck}D_{a^+}^{\beta,\rho}K_{2r}(t))]$

2) if F is (1,2)-differentiable, then $[({}^{ck}D_{a^+}^{\beta,\rho}K(t))]^r =$
$\frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau = [({}^{ck}D_{a^+}^{\beta,\rho}K_{2r}(t)), ({}^{cr}D_{a^+}^{\beta,\rho}K_{1r}(t))]$.

3) if K is (2,1)-differentiable, then $[({}^{ck}D_{a^+}^{\beta,\rho}k(t)]^r =$
$\frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}}{(t^\rho-\tau^\rho)^\beta}d\tau = [({}^{ck}D_{a^+}^{\beta,\rho}K_{2r})(t), ({}^{ck}D_{a^+}^{\beta,\rho}K_{1r})(t)]$

4) if k is (2,2)- differentiable, then $[({}^{ck}D_{a^+}^{\beta,\rho}K(t)] =$
$\frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}}{(t^\rho-\tau^\rho)^\beta}d\tau = [({}^{ck}D_{a^+}^{\beta,\rho}K_{1r})(t), ({}^{ck}D_{a^+}^{\beta,\rho}K_{2r})(t)]$

**proof:**

1) Since the other cases are analogous if h > 0, K∈ [0,1]

Since k is (1,1)- differentiable

$[K(t+h)\Theta k(t)]^r = [K_{1r}(t+h) - K_{1r}(t), K_{2r}(t+h) - K_{2r}(t)]$

and multiplying by 1/h, we have

$\frac{[K(t+h)\Theta K(t)]^r}{h} = [\frac{K_{1r}(t+h)-K_{1r}(t)}{h}, \frac{K_{2r}(t+h)-K_{2r}(t)}{h}]$

similarly, we obtain

$\frac{[K(t)\Theta K(t-h)]^r}{h} = [\frac{K_{1r}(t)-K_{1r}(t-h)}{h}, \frac{K_{2r}(t)-K_{2r}(t-h)}{h}]$

$[(D_{1,1}^1 K(t)]^r = [\frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau]$

passing to the limit, we have

$[(D_{1,1}^1 K(t)]^r = [{}^{ck}D_{a+}^{\beta,\rho}K_{1r}(t), {}^{ck}D_{a+}^{\beta,\rho}K_{2r}(t)]$

2) if K is (2,1)- differentiable. Then

$[K(t)\Theta K(t+h)]^r = [K'_{2r}(t+h) - K_{2r}(t), K'_{1r}(t+h) - K_{1r}(t)]$

and multiplying by 1/h, we have

$\frac{[K(t)\Theta K(t+h)]^r}{h} = [\frac{K_{2r}(t+h)-K_{2r}(t)}{h}, \frac{K_{1r}(t+h)-K_{1r}(t)}{h}]$

similarly, we obtain

$\frac{[K(t)\Theta K(t-h)]^r}{h} = [\frac{K_{2r}(t)-K_{2r}(t-h)}{h}, \frac{K_{1r}(t)-K_{1r}(t-h)}{h}]$

$[D_{1,2}^1 F(t)]^r = \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau$

passing to the limit, we have

$[D^1_{1,2}F(t)]^r = [{}^{ck}D^{\beta,\rho}_{a+}K'_{2r}(t), {}^{ck}D^{\beta,\rho}_{a+}K'_{1r}(t)]$

3) if k is (1,2)- differentiable

$[K(t+h)\theta K(t)]^r]^r = [K'_{2r}(t+h) - K_{2r}(t), K'_{1r}(t+h) - K_{1r}(t)]$

and multipliying by 1/h, we have

$\frac{[\mathrm{K(t+h)\Theta\,K(t)}]^r}{h} = [\frac{K'_{2r}(t+h)-K'_{2r}(t)}{h}, \frac{K'_{1r}(t+h)-K'_{1r}(t)}{h}]$

similarly, we obtain

$\frac{[K(t)\Theta\,K(t-h)]^r}{h} = [\frac{K'_{2r}(t)-K'_{2r}(t-h)}{h}, \frac{K'_{1r}(t)-K'_{1r}(t-h)}{h}]$

$[D^1_{2,1}F(t)]^r = \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau$

passing to the limit, we have

$[(D^1_{2,2}F(t)]^r = [{}^{ck}D^{\beta,\rho}_{a+}K'_{2r}(t), {}^{ck}D^{\beta,\rho}_{a+}K'_{1r}(t)]$

4) if k is (2,2)- differentiable. Then

$[K(t+h)\theta K(t)]^r = [K_{1r}(t+h) - K_{1r}(t), K_{2r}(t+h) - K_{2r}(t)]$

and multipliying by 1/h, we have

$\frac{[K(t+h)\Theta\,K(t)]^r}{h} = [\frac{K'_{1r}(t+h)-K'_{1r}(t)}{h}, \frac{K'_{2r}(t+h)-K'_{2r}(t)}{h}]$

similarly, we obtain

$\frac{[K(t)\Theta\,K(t-h)]^r}{h} = [\frac{K'_{1r}(t)-K'_{1r}(t-h)}{h}, \frac{K'_{2r}(t)-Kk_{2r}(t-h)}{h}]$

$[(D^1_{2,2}F(t)]^r = \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{1r}(t)}{(t^\rho-\tau^\rho)^\beta}d\tau, \frac{\rho^\beta}{\sqrt{1-\beta}}\int_a^t \frac{K'_{2r}}{(t^\rho-\tau^\rho)^\beta}d\tau$

passing to the limit, we have

$[(D^1_{2,2}F(t)]^r = [{}^{ck}D^{\beta,\rho}_{a+}K'_{1r}(t), {}^{ck}D^{\beta,\rho}_{a+}K'_{2r}(t)]$

this completes the proof theorem

**Theorem (3.11):**

let $\beta \in (0,1]$ and $K \in AC^F[a,b]$.

1) if $K$ is (1,1)-differentiable then,

$(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x) = K(x) - K((a))$

2) if K is (1,2)-differentiable then,

3) $(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K) = (-K(x) - K(a))$

4) if k is (2,1)-differentiable then,

$(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x) = (-K(x) - k(a))$

5) if K is (2,2)-differentiable then,

$(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x) = K(x) - k(a)$

**proof:**

Let $(K(x)]^r = [K_{1r}(x), K_{2r}(x)]$ for all $r \in [0,1]$ then we have the real valued function $k_{1r}$ and $k_{2r}$,

$(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K_{1r})(x) = K_{1r}(x)$ and $(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K_{2r})(x) = K_{2r}(x)$.

Now if $K$ is (1,1)-differentiable or (2,2)-differentiable, then by theorem (3.10), we can write $[({}^{ck}D^\beta_{a+}K)(x)]^r = [({}^{ck}D^\beta_{a+}K_{1r})(x), ({}^{ck}D^\beta_{a+}K_{2r})(x)]$. Hence

$[(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x)]^r = [(I^\beta_{a+}\,{}^{ck}D^\beta_{a+}K_{1r})(x), (I^\beta_{a+}\,{}^{ck}D^\beta_{a+}K_{2r})(x)] =$

$[K_{1r}(x), K_{2r}]$. So $(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x) = K(x) - K(a)$ if $K$ is (1,1)- differentiable and $(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x) = K(x) - K(a)$ if k is (2,2)- differentiable

Now, if $K$ is (1,2)-differentiable or (2,1)- differentiable then from theorem(3,1) We have, $[({}^{ck}D^\beta_{a+}k)(x)]^r = [({}^{ck}D^\beta_{a+}K_{1r})(x), ({}^{ck}D^\beta_{a+}K_{2r})(x)]$. So

$[(I^\beta_{a+}\,{}^{ck}D^{\beta,\rho}_{a+}K)(x)]^r = [(I^\beta_{a+}\,{}^{ck}D^\beta_{a+}K_{1r})(x), (I^\beta_{a+}\,{}^{ck}D^\beta_{a+}K_{2r})(x)] =$

$[K_{1r}(x), K_{2r}(x)]$. Hence $(I_{a+}^{\beta}\ {}^{ck}D_{a+}^{\beta,\rho}K)(x) = -K(x) - K(a)$ if $K$ is (1,2)- differentiable and $(I_{a+}^{\beta}\ {}^{ck}D_{a+}^{\beta,\rho}K)(x) = -K(x) - K(a)$ if $K$ is (2,1)- differentiable.

## 4. EXTENSION PROBLEM FORMULATION

Consider the following F-CKFDEs of the from

$$\left({}^{ck}D_{a+}^{\beta,\rho}x\right)(t) = \tilde{f}(t)x(t) + \tilde{K}(x(t),t)\tilde{N}(x,t), 1 < \beta < 2, t > a \qquad (13)$$

$$\tilde{x}(a) = [\bar{x}_0, \underline{x}_0] \in \mathfrak{R}_F,\ \tilde{\acute{x}}(a) = [\acute{\bar{x}}_0, \acute{\underline{x}}_0],$$

where $\tilde{f}(t)$ is a continuous fuzzy function with nonnegative values on [0,1],
$k: R_F \times [a,b] \to R_F$ is a linear or nonlinear continuous fuzzy function, and $\alpha \in R_F$. An $(m,n) -$ solution of (2.5)is an ${}^{c}[(m,n) - \beta] -$ differentiable
function x: [a, b] $\to R_F$ that satisfies (2.5). To solve this problem, we convert it to a based on the selection of the derivative type. This system will be called (m, n)-system. Let
$\tilde{K}(x(t),t)\tilde{N}(x,t)[K_{1r}N_{1r}(x_{1r}, x_{2r}, t), k_{2r}N_{2r}(x_{1r}, x_{2r}, t)], [x(t)]^r$
$[x_{1r}(t), x_{2r}(t)], [x(a)]^r = [x_{1r}(a), x_{2r}(a)] = [\alpha_{1r}, \alpha_{2r}]$, be the $r -$ cut
Representations of $\tilde{K}(x(t),t)\tilde{N}(x,t)$ and $x(t)$.This (13) can be translated to one of the following systems:

**(1,1)-system**

$$({}^{ck}D_{a+}^{\beta,\rho}x_{1r})(t) = f_{1r}(t)x_{1r}(t) + k_{1r}N_{1r}(x_{1r}(t), x_{2r}(t), t),$$
$$({}^{ck}D_{a+}^{\beta,\rho}x_{2r})(t) = f_{2r}(t)x_{2r}(t) + k_{2r}N_{2r}(x_{1r}(t), x_{2r}(t), t),$$
$$x_{1r}(a) = \alpha_{1r}, \qquad x_{2r}(a) = \alpha_{2r}$$
$$\acute{x}_{1r}(a) = \acute{\alpha}_{1r}, \qquad \acute{x}_{2r}(a) = \acute{\alpha}_{2r}$$

**(1,2)-system**

$$({}^{ck}D_{a+}^{\beta,\rho}x_{2r})(t) = f_{1r}(t)x_{1r}(t) + k_{1r}N_{1r}(x_{1r}(t), x_{2r}(t), t),$$
$$({}^{ck}D_{a+}^{\beta,\rho}x_{1r})(t) = f_{2r}(t)x_{2r}(t) + k_{2r}N_{2r}(x_{1r}(t), x_{2r}(t), t),$$
$$x_{1r}(a) = \alpha_{1r}, \qquad x_{2r}(a) = \alpha_{2r}$$
$$\acute{x}_{1r}(a) = \acute{\alpha}_{1r}, \qquad \acute{x}_{2r}(a) = \acute{\alpha}_{2r}$$

**(2,1)-system**

$$({}^{ck}D_{a+}^{\beta,\rho}x_{2r})(t) = f_{2r}(t)x_{2r}(t) + k_{1r}N_{1r}(x_{1r}(t), x_{2r}(t), t),$$
$$({}^{ck}D_{a+}^{\beta,\rho}x_{1r})(t) = f_{1r}(t)x_{1r}(t) + k_{2r}N_{2r}(x_{1r}(t), x_{2r}(t), t),$$
$$x_{1r}(a) = \alpha_{1r}, \qquad x_{2r}(a) = \alpha_{2r}$$
$$\acute{x}_{1r}(a) = \acute{\alpha}_{2r}, \qquad \acute{x}_{2r}(a) = \acute{\alpha}_{1r}$$

**(2,2)-system**

$$({}^{ck}D_{a+}^{\beta,\rho}x_{1r})(t) = f_{2r}(t)x_{2r}(t) + k_{1r}N_{1r}(x_{1r}(t), x_{2r}(t), t),$$
$$({}^{ck}D_{a+}^{\beta,\rho}x_{2r})(t) = f_{1r}(t)x_{1r}(t) + k_{2r}N_{2r}(x_{1r}(t), x_{2r}(t), t),$$
$$x_{1r}(a) = \alpha_{1r}, \qquad x_{2r}(a) = \alpha_{2r}$$
$$\acute{x}_{1r}(a) = \acute{\alpha}_{2r}, \qquad \acute{x}_{2r}(a) = \acute{\alpha}_{1r}$$

**Definition (4.1),[15]:**

Let $k: [a,b] \to \mathfrak{R}_F$ be a fuzzy function and n, m $\in$ {1,2}. One says k is an (n, m)-solution for problem (13) on [a, b], if $D_n^1 k D_{n,m}^2 k$ exist on [a, b] and$D_{n,m}^2 k(t) + a \cdot D_n^1 k + b \cdot k(t) = \sigma(t), k(0) = \alpha_0, D_n^1 k(0) = \alpha_1$

**Theorem (4.2),[17]:**

Let $[x(t)]^r = [x_{1r}, x_{2r}]$be an $(m,n) -$ solution of (13) Then $x_{1r}(t)$ *and* $x_{2r}(t)$ solve (m, n)-system for n, m$\in$ {1,2}. Moreover, if $x_{1r}(t)$ and $x_{2r}(t)$ solve the $(m,n) -$ system for each $r \in [0,1]$, $[x_{1r}(t), x_{2r}(t)]$ has valid level sets , and x(t) is ${}^{c}[(m,n) - \beta] -$ diff, then x(t) is an (m, n) solution of (13).

**proof:**

Suppose that $[x(t)]^r$is the $(m,n) -$ solution of equation (13), according to the lemma (3.9) then $D_n^1$and $D_{n,m}^2 x(t)$exist and satisfy problem (13) and by Lemma 3.8and 3.9 and substituting $x_{1r}, x_{2r}$and their derivatives in problem (13), we get the (m, n)-system corresponding to the (m, n)- solution, now since $[x(t)]^r = [x_{1r}, x_{2r}]$is$(m, n) -$ differetiable fuzzy function , by theorem 3.8 and 3.9 we can compute $D_n^1 k$ and $D_{n,m}^2 k(t)$according $x'_{1r}, x'_{2r}$. Due to the fact that $x_{1r}, x_{2r}$ solve $(m, n) -$ system from defintion (4.1),it comes that x(t) is an (m, n)- solution for (13).

## 5. Illustrative Numerical Fuzzy Fractional Order$0 < \beta < 1$ Examples:

In this Section the interesting examples are explained the efficiently of the reproducing kernel Hilbert space algorithm for different fractional orders such as $0 < \beta < 1$ and different systems (n, m) where n, m=1,2.

**Example (5.1):**

Consider the following

$$({}^{ck}D_{0^+}^{\beta,\rho} y)(t) = \delta + y(t), \qquad 0 \le \beta \le 1, \rho > 0, t \in [0,1] \qquad (14)$$

$\delta = \mathrm{y}(0) = [0.2\alpha + 0.8, 1.2 - 0.2\alpha]$ $\alpha \in (0,1)$ where $\delta$ are the fuzzy number whose $\alpha$ -cut representation is $[0.2\alpha + 0.8, 1.2 - 0.2\alpha]$ Depending on the type of differentiability, we have the following system

$$\text{system(1,1)} \begin{cases} ({}^{ck}D_{0^+}^{\beta,\rho} y_{1\alpha})(t) = 0.2\alpha + 0.8 + y_{1\alpha}(t), & , \\ ({}^{ck}D_{0^+}^{\beta,\rho} y_{2\alpha})(t) = 1.2 - 0.2\alpha + y_{2\alpha}(t) \\ y_{1\alpha}(0) = 0.2\alpha + 0.8, \;\; y_{2\alpha} = 1.2 - 0.2\cdot\alpha \end{cases}$$

$$\text{system(1,2)} \begin{cases} ({}^{ck}D_{0^+}^{\beta} y_{1\alpha})(t) = 1.2 - 0.2\alpha + y_{2\alpha}(t), \\ ({}^{ck}D_{0^+}^{\beta} y_{2\alpha})(t) = 0.2\alpha + 0.8 + y_{1\alpha}(t) \\ y_{1\alpha}(0) = 0.2\alpha + 0.8 \,, \;\; y_{2\alpha} = 0.2\alpha + 0.8 \end{cases}$$

**Solution:**

By using the reproducing kernel Hilbert space method for N=25.

Assume$(0 \le t \le 1)$. $(0 \le x \le 1$ and $x_i = \left(\frac{i-1}{N-1}\right), i = 1 \ldots N$.

$$x_i = \left[0, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{1}{2}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}, 1\right]$$

$$R(x) \begin{cases} t + 1, \;\; t \le x, \\ x + 1, x \le t. \end{cases}$$

For $f = (0.25 - 1) + (0.25 - 1)\cdot x$, and $K = (x - t)^{\beta - 1}, N1 = (0.25 - 1)$,

Where $\beta = 1.8$

$$\mathrm{G} = \mathrm{U} - \frac{1}{\Gamma(1.9)} - \int_0^x \mathrm{K}(\mathrm{t}, \mathrm{x}) \cdot \mathrm{N1}(\mathrm{U})$$

$$\psi_i(x) = L_t R_x(t)|_{t=x_i}$$

$$\psi_1(x) = \begin{cases} 1 & , 0 \le x \\ x + 1, x \le 0 \end{cases} \quad \psi_2(x) = \begin{cases} \frac{25}{24} & , \frac{1}{24} \le x \\ x + 1, x \le \frac{1}{24} \end{cases} \quad \psi_3(x) = \begin{cases} \frac{13}{12} & , \frac{1}{12} \le x \\ x + 1, x \le \frac{1}{12} \end{cases}$$

$$\psi_4(x) = \begin{cases} \frac{9}{8} & , \frac{1}{8} \le x \\ x + 1, x \le \frac{1}{8} \end{cases} \quad \psi_5(x) = \begin{cases} \frac{7}{6} & , \frac{1}{6} \le x \\ x + 1, x \le \frac{1}{6} \end{cases} \quad \psi_6(x) = \begin{cases} \frac{29}{24} & , \frac{5}{24} \le x \\ x + 1, x \le \frac{5}{24} \end{cases}$$

$$\psi_7(x) = \begin{cases} \frac{5}{4} & , \frac{1}{4} \le x \\ x + 1, x \le \frac{1}{4} \end{cases} \quad \psi_8(x) = \begin{cases} \frac{31}{24} & , \frac{7}{24} \le x \\ x + 1, x \le \frac{7}{24} \end{cases} \quad \psi_9(x) = \begin{cases} \frac{4}{3} & , \frac{1}{3} \le x \\ x + 1, x \le \frac{1}{3} \end{cases}$$

$$\psi_{10}(x) = \begin{cases} \frac{11}{8} & , \frac{3}{8} \le x \\ x + 1, x \le \frac{3}{8} \end{cases} \quad \psi_{11}(x) = \begin{cases} \frac{17}{12} & , \frac{5}{12} \le x \\ x + 1, x \le \frac{5}{12} \end{cases} \quad \psi_{12}(x) \begin{cases} \frac{35}{24} & , \frac{11}{24} \le x \\ x + 1, x \le \frac{11}{24} \end{cases}$$

$$\psi_{13}(x)=\begin{cases}\frac{3}{2} & ,\frac{1}{2}\le x\\ x+1, x\le\frac{1}{2}\end{cases}\quad \psi_{14}(x)=\begin{cases}\frac{37}{24} & ,\frac{13}{24}\le x\\ x+1, x\le\frac{13}{24}\end{cases}\quad \psi_{15}(x)\begin{cases}\frac{19}{12} & ,\frac{7}{12}\le x\\ x+1, x\le\frac{7}{12}\end{cases}\quad \psi_{16}(x)=\begin{cases}\frac{13}{8} & ,\frac{5}{8}\le x\\ x+1, x\le\frac{5}{8}\end{cases}\quad \psi_{17}(x)=\begin{cases}\frac{5}{3} & ,\frac{2}{3}\le x\\ x+1, x\le\frac{2}{3}\end{cases}\quad \psi_{18}(x)=\begin{cases}\frac{41}{24} & ,\frac{17}{24}\le x\\ x+1, x\le\frac{17}{24}\end{cases}$$

$$\psi_{19}(x)=\begin{cases}\frac{7}{4} & ,\frac{3}{4}\le x\\ x+1, x\le\frac{3}{4}\end{cases}\quad \psi_{20}(x)=\begin{cases}\frac{43}{24} & ,\frac{19}{24}\le x\\ x+1, x\le\frac{19}{24}\end{cases}\quad \psi_{21}(x)=\begin{cases}\frac{11}{6} & ,\frac{5}{6}\le x\\ x+1, x\le\frac{5}{6}\end{cases}\quad \psi_{22}(x)=\begin{cases}\frac{15}{8} & ,\frac{7}{8}\le x\\ x+1, x\le\frac{7}{8}\end{cases}\quad \psi_{23}(x)=\begin{cases}\frac{23}{12} & ,\frac{11}{12}\le x\\ x+1, x\le\frac{11}{12}\end{cases}\quad \psi_{24}(x)=\begin{cases}\frac{47}{24} & ,\frac{23}{24}\le x\\ x+1, x\le\frac{23}{24}\end{cases}$$

$$\psi_{25}(x)=\begin{cases}2 & ,1\le x\\ x+1, x\le 1\end{cases}$$

$$\psi=\begin{bmatrix}
1.039738e+00 & 1.118872e+00 & 1.187884e+00 & 1.255231e+00 & 1.320637e+00 & 1.383822e+00 & 1.444574e+00, & 1.502731e+00 & 1.558169e+00\\
1.039738e+00 & 1.246534e+00 & 1.317523e+00 & 1.386737e+00 & 1.453877e+00, & 1.518646e+00, & 1.580817e+00, & 1.640219e+00 & 1.696718e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.518242e+00 & 1.587116e+00 & 1.653470e+00 & 1.717061e+00 & 1.777707e+00 & 1.835267e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.649747e+00 & 1.720356e+00 & 1.788294e+00 & 1.853305e+00 & 1.915196e+00 & 1.973816e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.649747e+00 & 1.853596e+00 & 1.923118e+00 & 1.989549e+00 & 2.052684e+00 & 2.112366e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.649747e+00 & 1.853596e+00 & 2.057942e+00 & 2.125792e+00 & 2.190172e+00 & 2.250915e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.649747e+00 & 1.853596e+00 & 2.057942e+00 & 12.262036e+00 & 2.327660e+00 & 2.389464e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.649747e+00 & 1.853596e+00 & 2.057942e+00 & 2.262036e+00 & 2.465148e+00 & 2.528013e+00\\
1.039738e+00 & 1.246534e+00 & 1.447163e+00 & 1.649747e+00 & 1.853596e+00 & 2.057942e+00 & 2.262036e+00 & 2.465148e+00 & 2.666562e+00
\end{bmatrix}$$

$$\beta_{11}=\frac{1}{\|\psi_1\|}=\frac{1}{\sqrt{\langle\psi_1,\psi_1\rangle}}$$

$$\beta_{ii}=\frac{1}{\sqrt{\|\psi_i\|^2-\sum_{k=1}^{i-1}Z_{ik}^2}};\qquad Z_{ik}=\langle\psi_i,\overline{\psi_k}\rangle$$

$$\beta_{ij}=\frac{-\sum_{k=1}^{i-1}Z_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2-\sum_{k=1}^{i-1}Z_{ik}^2}}$$

$$\beta_{ik}=\begin{bmatrix}
1.0706 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.3260 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 2.0811 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 2.0811 & 2.3402 & 0 & 0 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 2.0811 & 2.3402 & 2.6008 & 0 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 2.0811 & 2.3402 & 2.6008 & 2.8615 & 0 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 2.0811 & 2.3402 & 2.6008 & 2.8615 & 3.1211 & 0\\
1.0706 & 1.3260 & 1.5728 & 1.8248 & 2.0811 & 2.3402 & 2.6008 & 2.8615 & 3.1211 & 3.3781
\end{bmatrix}$$

The following tables and figures explain the approximant solution of FFC-KDe (14) with different parameters obtained in the tables, which some time decrease or increase over time also the reproducing kernel Hilbert space method is more efficiently more clearing and convergent to exact solution with suitable value of N.

**<u>Example (5.2):</u>**

Consider the following

$$({}^{ck}D^{\beta,\rho}_{0^+}y)(t)=\delta+t^2,\qquad 0\le\beta\le 1,\rho>0,t\in[0,1]\qquad (15)$$

$$y(0)=[\alpha,2-\alpha]\qquad \alpha\in(0,1)$$

where $\delta$ are the fuzzy number whose r-cut representation is $[\alpha,2-\alpha]$.

Depending on the type of differentiability, we have the following systems

$$\text{(1,1)-system}\begin{cases}({}^{ck}D^{\beta}_{0^+}y_{1\alpha})(t)=\alpha+t^2,\\ ({}^{ck}D^{\beta}_{0^+}y_{1\alpha})(t)=2-\alpha+t^2\\ y_{1\alpha}(0)=\alpha\ ,\ \ y_{2\alpha}=2-\alpha\end{cases}$$

$$\text{(1,2)-system}\begin{cases}({}^{ck}D^{\beta}_{0^+}y_{1\alpha})(t) = 2-\alpha+t^2,\\ ({}^{ck}D^{\beta}_{0^+}y_{1\alpha})(t) = \alpha+t^2\\ y_{1\alpha}(0)=\alpha\,,\;\; y_{2\alpha} = 2-\alpha\end{cases}$$

**Solution:**

By using the reproducing kernel Hilbert space method for N=25

Assume$(0 \le t \le 1)$. $(0 \le x \le 1$ and $x_i = \left(\frac{i-1}{N-1}\right), i = 1 \dots N.$

$$x_i = \left[0, \tfrac{1}{24}, \tfrac{1}{12}, \tfrac{1}{8}, \tfrac{1}{6}, \tfrac{5}{24}, \tfrac{1}{4}, \tfrac{7}{24}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{5}{12}, \tfrac{11}{24}, \tfrac{1}{2}, \tfrac{13}{24}, \tfrac{7}{12}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{17}{24}, \tfrac{3}{4}, \tfrac{19}{24}, \tfrac{5}{6}, \tfrac{7}{8}, \tfrac{11}{12}, \tfrac{23}{24}, 1\right]$$

$$R(x)\begin{cases}t+1,\; t \le x,\\ x+1, x \le t.\end{cases}$$

For $f = 0.25$, and $K = (x^{0.6} - t^{0.6})^{\beta-1}, N1 = (0.25 + t^2)\cdot t^{0.6-1}$,

Where $\beta = 0.5$

$$\mathrm{G} = \mathrm{U} - \frac{0.6^{1-0.5}}{\mathrm{gamma}(0.5)} - \int_0^{\mathrm{x}} \mathrm{K(t,x)} \cdot \mathrm{N1(U)}$$

$$\psi_i(x) = L_t R_x(t)|_{t=x_i}$$

$$\psi_1(x) = \begin{cases}1 & ,0 \le x\\ x+1, x \le 0\end{cases} \quad \psi_2(x) = \begin{cases}\frac{25}{24} & ,\frac{1}{24} \le x\\ x+1, x \le \frac{1}{24}\end{cases} \quad \psi_3(x) = \begin{cases}\frac{13}{12} & ,\frac{1}{12} \le x\\ x+1, x \le \frac{1}{12}\end{cases} \quad \psi_4(x) = \begin{cases}\frac{9}{8} & ,\frac{1}{8} \le x\\ x+1, x \le \frac{1}{8}\end{cases} \quad \psi_5(x) = \begin{cases}\frac{7}{6} & ,\frac{1}{6} \le x\\ x+1, x \le \frac{1}{6}\end{cases}$$

$$\psi_6(x) = \begin{cases}\frac{29}{24} & ,\frac{5}{24} \le x\\ x+1, x \le \frac{5}{24}\end{cases} \quad \psi_7(x) = \begin{cases}\frac{5}{4} & ,\frac{1}{4} \le x\\ x+1, x \le \frac{1}{4}\end{cases} \quad \psi_8(x) = \begin{cases}\frac{31}{24} & ,\frac{7}{24} \le x\\ x+1, x \le \frac{7}{24}\end{cases} \quad \psi_9(x) = \begin{cases}\frac{4}{3} & ,\frac{1}{3} \le x\\ x+1, x \le \frac{1}{3}\end{cases}$$

$$\psi_{10}(x) = \begin{cases}\frac{11}{8} & ,\frac{3}{8} \le x\\ x+1, x \le \frac{3}{8}\end{cases} \quad \psi_{11}(x) = \begin{cases}\frac{17}{12} & ,\frac{5}{12} \le x\\ x+1, x \le \frac{5}{12}\end{cases} \quad \psi_{12}(x)\begin{cases}\frac{35}{24} & ,\frac{11}{24} \le x\\ x+1, x \le \frac{11}{24}\end{cases} \quad \psi_{13}(x) = \begin{cases}\frac{3}{2} & ,\frac{1}{2} \le x\\ x+1, x \le \frac{1}{2}\end{cases}$$

$$\psi_{14}(x) = \begin{cases}\frac{37}{24} & ,\frac{13}{24} \le x\\ x+1, x \le \frac{13}{24}\end{cases} \quad \psi_{15}(x) = \begin{cases}\frac{19}{12} & ,\frac{7}{12} \le x\\ x+1, x \le \frac{7}{12}\end{cases} \quad \psi_{16}(x) = \begin{cases}\frac{13}{8} & ,\frac{5}{8} \le x\\ x+1, x \le \frac{5}{8}\end{cases} \quad \psi_{17}(x) = \begin{cases}\frac{5}{3} & ,\frac{2}{3} \le x\\ x+1, x \le \frac{2}{3}\end{cases}$$

$$\psi_{18}(x) = \begin{cases}\frac{41}{24} & ,\frac{17}{24} \le x\\ x+1, x \le \frac{17}{24}\end{cases} \quad \psi_{19}(x) = \begin{cases}\frac{7}{4} & ,\frac{3}{4} \le x\\ x+1, x \le \frac{3}{4}\end{cases} \quad \psi_{20}(x) = \begin{cases}\frac{43}{24} & ,\frac{19}{24} \le x\\ x+1, x \le \frac{19}{24}\end{cases} \quad \psi_{21}(x) = \begin{cases}\frac{11}{6} & ,\frac{5}{6} \le x\\ x+1, x \le \frac{5}{6}\end{cases}$$

$$\psi_{22}(x) = \begin{cases}\frac{15}{8} & ,\frac{7}{8} \le x\\ x+1, x \le \frac{7}{8}\end{cases} \quad \psi_{23}(x) = \begin{cases}\frac{23}{12} & ,\frac{11}{12} \le x\\ x+1, x \le \frac{11}{12}\end{cases} \quad \psi_{24}(x) = \begin{cases}\frac{47}{24} & ,\frac{23}{24} \le x\\ x+1, x \le \frac{23}{24}\end{cases} \quad \psi_{25}(x) = \begin{cases}2 & ,1 \le x\\ x+1, x \le 1\end{cases}$$

$$\psi = \begin{bmatrix}
1.039738e+00 & 1.084800e+00 & 1.129857e+00 & 1.175739e+00 & 1.222133e+00 & 1.268702e+00 & 1.315097e+00, & 1.360968e+00 & 1.405961e+00\\
1.039738e+00 & 1.209800e+00 & 1.254857e+00 & 1.300739e+00 & 1.347133e+00 & 1.393702e+00 & 1.440097e+00 & 1.485968e+00 & 1.530961e+00\\
1.039738e+00 & 1.209800e+00 & 1.254857e+00 & 1.425739e+00 & 1.472133e+00 & 1.518702e+00 & 1.565097e+00 & 1.610968e+00 & 1.655961e+00\\
1.039738e+00 & 1.209800e+00 & 1.379857e+00 & 1.550739e+00 & 1.597133e+00 & 1.643702e+00 & 1.690097e+00 & 1.735968e+00 & 1.780961e+00\\
1.039738e+00 & 1.209800e+00 & 1.379857e+00 & 1.550739e+00 & 1.722133e+00 & 1.768702e+00 & 1.815097e+00 & 1.860968e+00 & 1.905961e+00\\
1.039738e+00 & 1.209800e+00 & 1.379857e+00 & 1.550739e+00 & 1.722133e+00 & 1.893702e+00 & 1.940097e+00 & 1.985968e+00 & 2.030961e+00\\
1.039738e+00 & 1.209800e+00 & 1.379857e+00 & 1.550739e+00 & 9.491245e-01 & 1.893702e+00 & 2.065097e+00 & 2.110968e+00 & 2.155961e+00\\
1.039738e+00 & 1.209800e+00 & 1.379857e+00 & 1.550739e+00 & 9.491245e-01 & 1.893702e+00 & 2.065097e+00 & 2.235968e+00 & 2.280961e+00\\
1.039738e+00 & 1.209800e+00 & 1.379857e+00 & 1.550739e+00 & 9.491245e-01 & 1.893702e+00 & 2.065097e+00 & 2.235968e+00 & 2.405961e+00,
\end{bmatrix}$$

$$\beta_{11} = \frac{1}{\|\psi_1\|} = \frac{1}{\sqrt{\langle\psi_1,\psi_1\rangle}}$$

$$\beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} Z_{ik}^2}}; \qquad Z_{ik} = \langle\psi_i, \overline{\psi_k}\rangle$$

$$\beta_{ij} = \frac{-\sum_{k=1}^{i-1} Z_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} Z_{ik}^2}}$$

$$\beta_{ik} = \begin{bmatrix}
1.0706 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.2615 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 0 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 1.8291 & 0 & 0 & 0 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 1.8291 & 2.0190 & 0 & 0 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 1.8291 & 2.0190 & 2.2090 & 0 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 1.8291 & 2.0190 & 2.2090 & 2.3991 & 0 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 1.8291 & 2.0190 & 2.2090 & 2.3991 & 2.5887 & 0\\
1.0706 & 1.2615 & 1.4504 & 1.6396 & 1.8291 & 2.0190 & 2.2090 & 2.3991 & 2.5887 & 2.7773
\end{bmatrix}$$

The following tables and figures explain the approximant solution of FFC-KDe (15) with different parameters obtain in the tables with some time decreasing or increasing also the reproducing kernel Hilbert space method is more efficiently and more clearing and convergent to exact solution with suitable value of N.

**Table 5.13:** Solution of $y_{1\alpha}$ in system (1,1) for $\beta = 0.5$ $and$ $r =$ 0.25

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 1.692264e+00, | 1.659690e+00, | 1.629613e+00, |
| 0.1 | 1.582263e+00, | 1.503754e+00, | 1.439001e+00, |
| 0.2 | 1.484762e+00, | 1.345392e+00, | 1.226662e+00, |
| 0.3 | 1.388507e+00, | 1.183446e+00, | 1.001755e+00 |
| 0.4 | 1.293755e+00, | 1.020377e+00, | 7.695187e-01, |
| 0.5 | 1.200849e+00, | 8.579925e-01, | 5.338181e-01, |
| 0.6 | 1.110148e+00, | 6.978403e-01, | 2.980299e-01, |
| 0.7 | 1.022049e+00, | 5.414314e-01, | 6.548341e-02, |
| 0.8 | 9.370121e-01, | 3.903677e-01, | -1.602718e-01, |
| 0.9 | 8.555785e-01, | 2.464242e-01 | -3.752812e-01, |

Figure of table 5.13

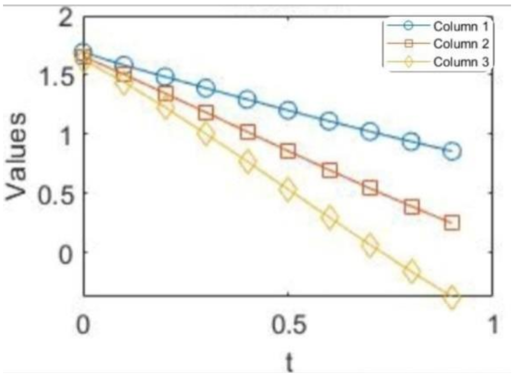



**Table 5.14:** Solution of $y_{2\alpha}$ in system (1,1) for $\beta = 0.5$ $and$ $r = 0.25$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 2.417520e-01, | 2.370986e-01, | 2.328019e-01, |
| 0.1 | 2.324373e-01, | 2.228578e-01, | 2.141706e-01, |
| 0.2 | 2.233540e-01, | 2.085272e-01, | 1.950420e-01, |
| 0.3 | 2.142424e-01, | 1.940189e-01, | 1.754997e-01, |
| 0.4 | 2.050893e-01, | 1.793352e-01, | 1.555597e-01, |
| 0.5 | 1.958962e-01, | 1.644896e-01, | 1.352438e-01, |
| 0.6 | 1.866780e-01, | 1.495202e-01, | 1.146141e-01, |
| 0.7 | 1.774643e-01, | 1.344984e-01, | 9.379492e-02, |
| 0.8 | 1.682998e-01, | 1.195346e-01, | 7.298883e-02, |
| 0.9 | 1.592448e-01 | 1.047825e-01, | 5.249079e-02, |

Figure of table 5.14

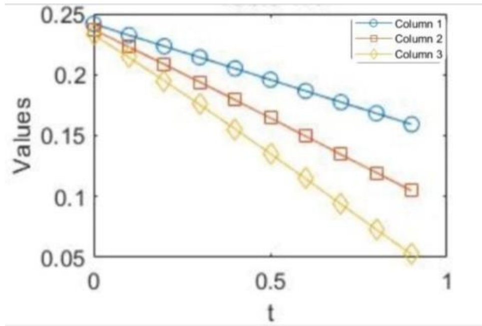



**Table 5.15:** Solution of $y_{1\alpha}$ in system (1,1) for $\beta = 0.7$ $and$ $r = 0.25$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 4.966709e-01, | 4.863082e-01, | 4.767811e-01, |
| 0.1 | 4.926804e-01, | 4.706163e-01, | 4.510684e-01, |
| 0.2 | 4.888155e-01, | 4.546007e-01, | 4.240208e-01, |
| 0.3 | 4.849271e-01, | 4.382187e-01, | 3.959228e-01, |
| 0.4 | 4.810189e-01, | 4.215433e-01, | 3.669297e-01, |
| 0.5 | 4.770971e-01, | 4.046343e-01, | 3.371670e-01, |
| 0.6 | 4.731698e-01, | 3.875570e-01, | 3.067789e-01, |
| 0.7 | 4.692483e-01, | 3.703951e-01, | 2.759613e-01, |
| 0.8 | 4.653476e-01, | 3.532592e-01, | 2.449886e-01, |
| 0.9 | 4.614872e-01 | 3.362941e-01, | 2.142356e-01, |

Figure of table 5.15

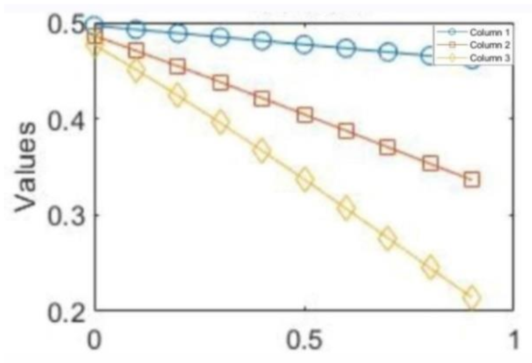


**Table 5.16:** Solution of $y_{2\alpha}$ in system(1.1) for $\beta = 0.7$ $and$ $r = 0.25$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 1.490013e+00, | 1.458925e+00, | 1.430343e+00, |
| 0.1 | 1.474072e+00, | 1.399848e+00, | 1.338243e+00, |
| 0.2 | 1.459308e+00, | 1.338058e+00, | 1.234457e+00, |
| 0.3 | 1.444436e+00, | 1.273507e+00, | 1.122242e+00, |
| 0.4 | 1.429534e+00, | 1.207158e+00, | 1.003728e+00, |
| 0.5 | 1.414679e+00, | 1.139744e+00, | 8.806167e-01, |
| 0.6 | 1.399939e+00, | 1.071900e+00, | 7.544741e-01, |
| 0.7 | 1.385385e+00, | 1.004243e+00, | 6.268948e-01, |
| 0.8 | 1.371089e+00, | 9.374134e-01, | 4.996172e-01, |
| 0.9 | 1.357134e+00, | 8.721176e-01 | 3.746131e-01 |

Figure of table 5.16

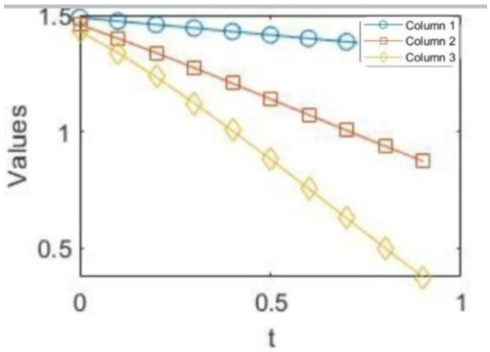



**Table 5.17:** Solution of system (1,1) for $\beta = 0.9$ $and$ $r = 0.75$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 1.285553e+00, | 1.255917e+00, | 1.228826e+00, |
| 0.1 | 1.334672e+00, | 1.263354e+00, | 1.203988e+00, |
| 0.2 | 1.381913e+00, | 1.271225e+00, | 1.176353e+00, |
| 0.3 | 1.430014e+00, | 1.279517e+00, | 1.146358e+00, |
| 0.4 | 1.478719e+00, | 1.288137e+00, | 1.114347e+00, |
| 0.5 | 1.527787e+00, | 1.297005e+00, | 1.080629e+00, |
| 0.6 | 1.577001e+00, | 1.306052e+00, | 1.045517e+00, |
| 0.7 | 1.626148e+00, | 1.315209e+00, | 1.009355e+00, |
| 0.8 | 1.675009e+00, | 1.324401e+00, | 9.725403e-01, |
| 0.9 | 1.723343e+00, | 1.333546e+00 | 9.355457e-01, |

Figure of table 5.17

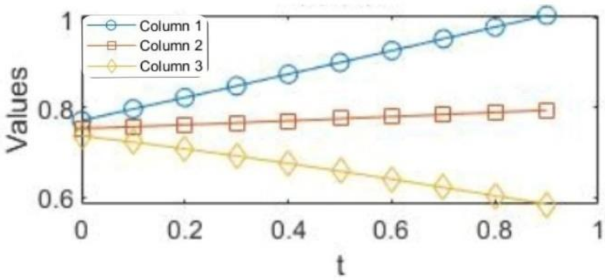



**Table 5.18:** Solution of system (1,1) for $\beta = 0.9$ $and$ $r = 0.75$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 7.713315e-01, | 7.535500e-01, | 7.372956e-01, |
| 0.1 | 7.975472e-01, | 7.576474e-01, | 7.232723e-01, |
| 0.2 | 8.230996e-01, | 7.619032e-01, | 7.082340e-01, |
| 0.3 | 8.489923e-01, | 7.663174e-01, | 6.923204e-01, |
| 0.4 | 8.751491e-01, | 7.708600e-01, | 6.756327e-01, |
| 0.5 | 9.014929e-01, | 7.755065e-01, | 6.582618e-01, |
| 0.6 | 9.279478e-01, | 7.802336e-01, | 6.403063e-01, |
| 0.7 | 9.544320e-01, | 7.850158e-01, | 6.218868e-01, |
| 0.8 | 9.808503e-01, | 7.898231e-01, | 6.031597e-01, |
| 0.9 | 1.007090e+00, | 7.946188e-01, | 5.843287e-01 |

Figure of table 5.18

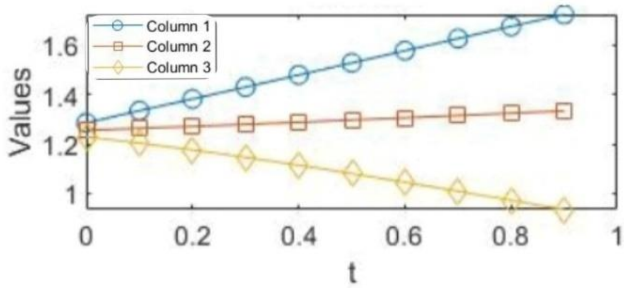



**Table 5.19:** Solution of system (2,1) for $\beta = 0.5$ $and$ $r = 0.25$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 2.417520e-01, | 2.370986e-01, | 2.328019e-01, |
| 0.1 | 2.260375e-01, | 2.148220e-01, | 2.055715e-01, |
| 0.2 | 2.121088e-01, | 1.921988e-01, | 1.752375e-01, |
| 0.3 | 1.983581e-01, | 1.690637e-01, | 1.431079e-01, |
| 0.4 | 1.848221e-01, | 1.457682e-01, | 1.099312e-01, |
| 0.5 | 1.715499e-01, | 1.225704e-01, | 7.625973e-02, |
| 0.6 | 1.585925e-01, | 9.969147e-02, | 4.257570e-02, |
| 0.7 | 1.460069e-01, | 7.734735e-02, | 9.354772e-03, |
| 0.8 | 1.338589e-01, | 5.576682e-02, | -2.289597e-02, |
| 0.9 | 1.222255e-01 | 3.520346e-02 | -5.361160e-02, |

Figure of table 5.19

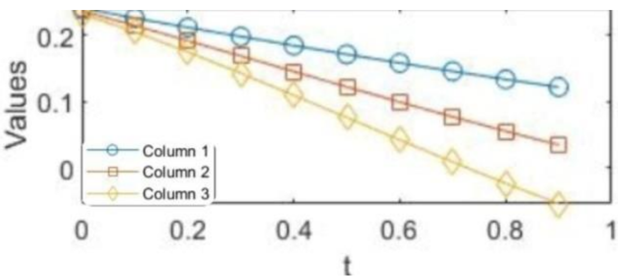


**Table 5.20:** Solution of system (2,1) for $\beta = 0.5$ and $r = 0.25$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 1.692264e+00, | 1.659690e+00, | 1.629613e+00, |
| 0.1 | 1.627061e+00, | 1.560005e+00, | 1.499194e+00, |
| 0.2 | 1.563478e+00, | 1.459690e+00, | 1.365294e+00, |
| 0.3 | 1.499697e+00, | 1.358132e+00, | 1.228498e+00, |
| 0.4 | 1.435625e+00, | 1.255347e+00, | 1.088918e+00, |
| 0.5 | 1.371273e+00, | 1.151427e+00, | 9.467064e-01, |
| 0.6 | 1.306746e+00, | 1.046642e+00, | 8.022990e-01, |
| 0.7 | 1.242250e+00, | 9.414889e-01, | 6.565644e-01, |
| 0.8 | 1.178098e+00, | 8.367423e-01, | 5.109218e-01, |
| 0.9 | 1.114713e+00 | 7.334777e-01 | 3.674355e-01, |

Figure of table 5.20

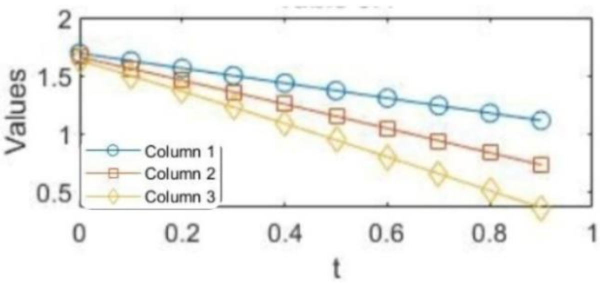



**Table5.21:** Solution of system (2,1) for $\beta = 0.7$ and $r = 0.5$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 4.966709e-01, | 4.863082e-01, | 4.767811e-01, |
| 0.1 | 4.913575e-01, | 4.666161e-01, | 4.460810e-01, |
| 0.2 | 4.864360e-01, | 4.460192e-01, | 4.114856e-01, |
| 0.3 | 4.814787e-01, | 4.245025e-01, | 3.740808e-01, |
| 0.4 | 4.765114e-01, | 4.023860e-01, | 3.345760e-01, |
| 0.5 | 4.715595e-01, | 3.799145e-01, | 2.935389e-01, |
| 0.6 | 4.666464e-01, | 3.573001e-01, | 2.514914e-01, |
| 0.7 | 4.617949e-01, | 3.347475e-01, | 2.089649e-01, |
| 0.8 | 4.570297e-01, | 3.124711e-01, | 1.665391e-01, |
| 0.9 | 4.523781e-01 | 2.907059e-01 | 1.248710e-01 |

Figure of table 5.21

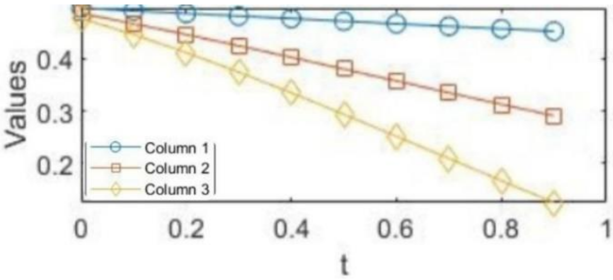
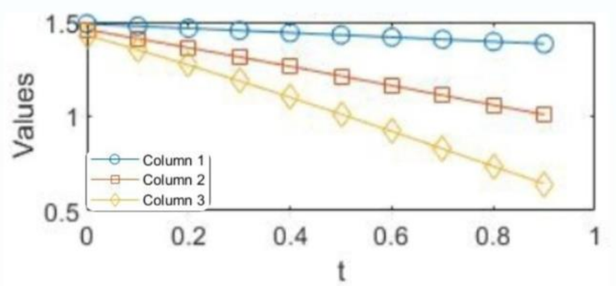



**Table 5.22:** Solution of system (2,1) for $\beta = 0.7$ and $r = 0.5$

| t | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1$ |
|---|---|---|---|
| 0 | 1.490013e+00, | 1.458925e+00, | 1.430343e+00, |
| 0.1 | 1.478041e+00, | 1.411849e+00, | 1.353205e+00, |
| 0.2 | 1.466447e+00, | 1.363802e+00, | 1.272062e+00, |
| 0.3 | 1.454781e+00, | 1.314656e+00, | 1.187768e+00, |
| 0.4 | 1.443057e+00, | 1.264630e+00, | 1.100789e+00, |
| 0.5 | 1.431291e+00, | 1.213903e+00, | 1.011501e+00, |
| 0.6 | 1.419509e+00, | 1.162671e+00, | 9.203367e-01, |
| 0.7 | 1.407745e+00, | 1.111185e+00, | 8.278840e-01, |
| 0.8 | 1.396043e+00, | 1.059778e+00, | 7.349657e-01, |
| 0.9 | 1.384461e+00, | 1.008882e+00, | 6.427067e-01, |

Figure of table 5.22




## CONCLUSION

1. The reproducing function depended on order of derivatives and drive it by some method.
2. The spaces of continuous functions of solutions depended on order of derivatives and drive it by some method.
3. The tables showed the efficiently of method since the values in columns are decreasing or increasing converge to exact solutions.
4. The method gives a permutation to use it on another fractional order type of fractional differential equations.
5. The parameters of Caputo- Katugmpola made a complexity for computed the approximate solution of integral equation
6. The reproducing kernel Hilbert space method cannot use it for fractional differential equation directly such ordinary differential equations but we can use it with fractional integral equation.

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