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Effect of an amount of extra food on the dynamics of a food -web model with modified Holling type II functional response

Karzan Abdurahman Hamzah[1], Arkan Nawzad Mustafa[2]*

[1]Rangeen school, Sulaymaniyah, IRAQ

[2]Department of Mathematics, College of Education, University of Sulaimani, IRAQ

*Corresponding Author: Arkan Nawzad Mustafa

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ABSTRACT: This paper presents a food-web model including three species namely, prey, intermediate predators and top predators. The model incorporates the of effect of extra food supplies to intermediate predators, the top predator species predate both prey species and intermediate predator species according to extended Holling type II functional response for two prey species. Firstly, it is proved that the model solutions are bounded under a certain condition. All biological possible steady states of the model are explored and their local stability is analyzed based on the sample parameters. Critical values that make the occurrence of Hopf-bifurcation of the model near intermediate predators-free and top predator-free steady states are determined. Finally, with the help of MATLAB, it is performed numerical simulations to support the evidence of the analytical results regarding to stability and Hopf-bifurcation, further it is observed that when the amount of extra food increases, the dynamics of the model may induce a transition from a stability situation to the state where populations oscillate periodically.

Keywords: Food web, Extra food, Functional response, Stability analysis, Hopf-bifurcation

1. INTRODUCTION

To understand the dynamics of interaction between prey species and predator species, thousands of preys -predator model has been considered by mathematician author [1,2].
Food web models are important conceptual tool for illustrating the feeding relationships among species within a community, revealing species interactions and community structure, and understanding the dynamics of energy transfer in an ecosystem, therefore two-species model has been extended to the three-species model by many authors [3-7].
Functional response is defined as the number of consumed preys per predator per unit time [2], therefor it is the important element to represent the dynamics relationship between predator population and prey population, C.S. Holling identified three types of functional response Type I, Type II and Type- III. The most useful functional response is the Holling type II functional response, which is characterized by decelerating intake rate [8]. Those types of functional response for are used by many mathematicians to modeling the dynamics of interactions between predator and prey [9,10,17]. Jha and Ghorai [9] proposed a prey-predator model with selective harvesting between the species using a Holling-type functional response. Khan et al. [11] investigated bifurcation analysis of a three-species in discrete time. The authors [12, 13] have investigated the dynamic behavior of a three species system with a scavenger. Diana et al. [13] investigated the three species model's dynamic behavior with logistic growth in which disease was included. The behavior of a three-species model with time delay and noise was analyzed stochastically by Danane and Torres [15]. It has been explored the fear effect and stability analysis of the food chain model [16].

 *Corresponding author: arkan.mustafa@univsul.edu.iq

In nature, some of predator species consume more than type of prey For example, lions usually predate a number of large land-based animals, such as antelopes, buffaloes, crocodiles, giraffes, pigs, zebra, wild dogs and wildebeest. In 2022, the Holling type II functional response is extended to more than one prey species [2].
In this paper, a three species food-web incorporating the extended Holling type II functional response is considered. In the model consideration is to study the effect of We also consider the existence of an additional source of food for the intermediate predator. The amount of extra food assumes a linear increase in intake rate with food density and depends on the biomass of prey species [1].
This paper consists of six sections. In the next section, the model derivations and some of its solution property are given. In the third section, all feasible and possible steady stat points of model explored and their local stability are investigated. In the section four, the Hopf- bifurcation near to each steady state point is studied. In the section five, some numerical simulation is done, to observe the impact of parameters and confirm the analytical results in this work. Finally in section six. A brief conclusion on the total work is given.

2. THE MODEL DERIVATION

In the derivation of the proposed food web system, we assumed that $X(t)$, $Y(t)$, $Z(t)$ represent the individual numbers of the prey, intermediate predators and apex predators, respectively and the following assumptions are taken to consideration

.

1. The prey specie grows logistically with Intrinsic growth rate > 0 and carrying capacity $K > 0$.
2. The top predators predate both prey and intermediate predators according to extended Holling type II functional response$\frac{\alpha_1 XZ}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y}$ and$\frac{\alpha_2 YZ}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y}$, respectively. Where α_1 and α_2 are predator's search efficiency of prey, intermediate predators, respectively.T_1 and T_2 are predator's average handling times of prey, intermediate predators, respectively. The extended Holling type II functional response functional response is presented in [2].
3. The prey population predated by intermediate predators according to Holling type II functional response $\frac{\beta XY}{1+\beta TX}$, where β is by intermediate predators' predation rate and T the predator's average handling time of intermediate predators.
4. An extra food quantity $aY\left(1-\frac{X}{K}\right)$ supplied to intermediate predators.
5. The top predator and intermediate predators numbers decreased by natural death with d_y and d_z, respectively.
6. The biomass of prey convers ate to biomass of with rate $0 < c < 1$. while the biomass of prey and intermediate predator convers ate to biomass of top predator with rate $c_1 < 1$ and$0 < c_2 < 1$, respectively.

The dynamics of such above interaction dynamics can be modeled as follows:

$$\begin{aligned}
\frac{dX}{dt} &= rX\left(1-\frac{X}{K}\right)-\frac{\beta XY}{1+\beta TX}-\frac{\alpha_1 XZ}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y} \\
\frac{dY}{dt} &= \frac{c\beta XY}{1+\beta TX}+aY\left(1-\frac{X}{K}\right)-\frac{\alpha_2 YZ}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y}-d_y Y \qquad (1) \\
\frac{dZ}{dt} &= \frac{(c_1\alpha_1 X+c_2\alpha_2 Y)Z}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y} - d_z Z
\end{aligned}$$

with initial conditions $X(0) \geq 0$, $Y(0) \geq 0$ and $Z(0) \geq 0$
System (1) satisfies the Lipschitzian condition, because the right side of it is continuous and has partial derivatives on the spaceR^3. Therefore, it has unique solution. Further, the time derivative of X, Y and Z are zero in $YZ-Plane, XZ-Plane$ and $XY-plane$, respectively. And this guarantees that the component $X, Y$and Z of the solution points of system (1), cannot cross any coordinates of the solution points. Hence components $X, Y$and Z of solution points are always non negative.

Theorem 1: System (1) is uniformly bounded, lf the following inequality holds.

$$a \leq d_y \qquad (2)$$

Proof. The first equation in system 1 gives that

$$\frac{dX}{dt} rX\left(1-\frac{X}{K}\right),$$

So, $\lim_{t\to\infty} Sup\big(X(t)\big) \le K$

Let $M = Min\{d_y - a, d_z\}$ and apply above inequality at system 1, it gets

$$\frac{d(X+Y+Z)}{dt} \le rX - (d_y - a)Y - d_z Z$$
$$\le rK + MK - M(X+Y+Z).$$

So, $\lim_{t\to\infty} Sup X + Y + Z \le \frac{rK+MK}{M}$ and this completes the proof.

3. STABLITY ANALYSIS

This section including two subsections. In the first subsection, all feasible and possible steady stat points of system (1) are determined and their local asymptotically stability LAS are investigated in the second

.

3.1 Existence of steady states

System (1) has at most the following six steady states:

i. The trivial steady state $S_0(0,0,0)$.

ii. The only prey existence steady state $S_1(K,0,0)$.

iii. The prey-free steady state $S_2(0,Y_2,Z_2)$ where,

$$Y_2 = \frac{d_z}{\alpha_2(c_2 - d_z T_2)} \text{ and } Z_2 = \frac{1}{\alpha_2}(a - d_y)(1 + \alpha_2 T_2 Y_2)$$

iv. The intermediate predators-free steady state $S_3(X_3, 0, Z_3)$ where,

$$X_3 = \frac{d_z}{\alpha_1(c_1 - d_z T_1)} \text{ and } Z_3 = \frac{r}{\alpha_1}\left(1 - \frac{X_3}{K}\right)(1 + \alpha_1 T_1 X_3)$$

v. The top predator-free steady state $S_4(X_4, Y_4, 0)$ where,

$Y_4 = \frac{r}{\beta}\left(1 - \frac{X_4}{K}\right)(1 + \beta Y X_4)$ and X_4 is a positive root for the following system

$$a\beta T X^2 + \big(a - c\beta K + (d_y - a)\beta T K\big)X + (d_y - a)K = 0$$

vi. coexistence steady state $S_5(X_5, Y_5, Z_5)$, where X_5, Y_5 and Z_5 are positive roots for the following system

$$r(K - X) - \frac{\beta K Y}{1+\beta T X} - \frac{\alpha_1 K Z}{1+\alpha_1 T_1 X + \alpha_2 T_2 Y} = 0$$
$$\frac{c\beta K X}{1+\beta T X} + a(K - X) - \frac{\alpha_2 K Z}{1+\alpha_1 T_1 X + \alpha_2 T_2 Y} - d_y K = 0$$
$$c_1\alpha_1 X + c_2 \alpha_2 Y - d_z(1 + \alpha_1 T_1 X + \alpha_2 T_2 Y) = 0$$

3.2 Local stability

Here, LAS for all the steady states of system (1) is studied. firstly, we Linearize system (1) near a point (X,Y,Z). using the perturbed variables $U(t) = X(t) - X$ and $V(t) = Y(t) - Y$ and $W(t) = Z(t) - Z$. system (1) can be linearized as follows:

$$\begin{pmatrix} \frac{dU(t)}{dt} \\ \frac{dV(t)}{dt} \\ \frac{dW(t)}{dt} \end{pmatrix} = A(X,Y,Z)\begin{pmatrix} U(t) \\ V(t) \\ W(t) \end{pmatrix} \quad \text{Where,} \quad A(X,Y,P) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{33} & A_{33} \end{pmatrix}$$

With

$$A_{11} = r - \frac{2rX}{K} - \frac{\beta Y}{(1+\beta T X)^2} - \frac{(1+\alpha_2 T_2 Y)\alpha_1 Z}{(1+\alpha_1 T_1 X + \alpha_2 T_2 Y)^2},$$
$$A_{12} = -\frac{\beta X}{1+\beta T X} + \frac{\alpha_1\alpha_2 T_2 X Z}{(1+\alpha_1 T_1 X + \alpha_2 T_2 Y)^2}, \quad A_{13} = -\frac{\alpha_1 X}{1+\alpha_1 T_1 X + \alpha_2 T_2 Y},$$
$$A_{21} = \frac{c\beta Y}{(1+\beta T X)^2} + \frac{\alpha_1\alpha_2 T_1 Y Z}{(1+\alpha_1 T_1 X + \alpha_2 T_2 Y)^2} - \frac{aY}{K},$$
$$A_{22} = \frac{c\beta X}{1+\beta T X} + a\left(1 - \frac{X}{K}\right) - \frac{(1+\alpha_1 T_1 X)\alpha_2 Z}{(1+\alpha_1 T_1 X + \alpha_2 T_2 Y)^2} - d_y,$$
$$A_{23} = \frac{-\alpha_2 Y}{1+\alpha_1 T_1 X + \alpha_2 T_2 Y}, A_{31} = \frac{(c_1 + \alpha_2(c_1 T_2 - c_2 T_1)Y)\alpha_1 Z}{(1+\alpha_1 T_1 X + \alpha_2 T_2 Y)^2},$$
$$A_{32} = \frac{(c_2 + \alpha_1(c_2 T_1 - c_1 T_2)X)\alpha_2 Z}{(1+\alpha_1 T_1 X + \alpha_2 T_2 Y)^2} \text{ and } A_{33} = \frac{c_1\alpha_1 X + c_2\alpha_2 Y}{1+\alpha_1 T_1 X + \alpha_2 T_2 Y} - d_z$$

Theorem 2. In system (1), $S_0(0,0,0)$ and $S_2(0,Y_2,Z_2)$ are always unstable.

Proof. The eigenvalues of $A(0,0,0)$, are $\lambda_{0X} = r > 0$, $\lambda_{0Y} = a - d_y$ and $\lambda_{0Z} = -d_z$.
Therefore, $S_0(0,0,0)$ is unstable.
The eigenvalues of $A(0, Y_2, Z_2)$, are $\lambda_{2X} = r - \beta Y_2 - \frac{\alpha_1 Z_2}{1+\alpha_2 T_2 Y_2}$ and λ_{2Y} and λ_{2Z} are roots of the equation

$$\lambda^2 + \left(\frac{(d_y - a)\alpha_2 T_2 Z_2}{1+\alpha_2 T_2 Y_2}\right)\lambda + \frac{\alpha_1^2 Y_2 (a - d_y)}{\alpha_2 (1+\alpha_2 T_2 Y_2)^2} = 0.$$

The existence condition of $S_2(0, Y_2, Z_2)$ is $a > d_y$. So $\lambda_{2Y} > 0$ or $\lambda_{2Z} > 0$
Therefore, $S_2(0, Y_2, Z_2)$ is unstable.

Theorem 3. In system (1),

1. $S_1(K, 0,0)$ is LAS if and only if,

$$\begin{aligned} \frac{\beta K}{1+\beta T K} &< \frac{d_y}{c} \\ \frac{\alpha_1 K}{1+\alpha_1 T_1 K} &< \frac{d_z}{c_1} \end{aligned} \qquad (3)$$

2. If $S_3(X_3, 0, Z_3)$ exists, then it is LAS if and only if,

$$\begin{aligned} a &< \frac{r\alpha_2}{\alpha_1} + \frac{d_y}{(K - X_3)} - \frac{c\beta K X_3}{(K - X_3)(1+\beta T X_3)} \\ 1 + 2\alpha_1 T_1 X_3 &> K\alpha_1 T_1 \end{aligned} \qquad (4)$$

3. If $S_4(X_4, Y_4, 0)$ exists, then it is LAS if and only if

$$\begin{aligned} c_1\alpha_1 X_4 + c_2\alpha_2 Y_4 &< d_z\,(1 + \alpha_1 T_1 X_4 + \alpha_2 T_2 Y_4) \\ 1 + 2\beta T X_4 > \beta T K &> \frac{aT}{c}(1 + \beta T X_4)^2 \end{aligned} \qquad (5)$$

4. If $S_5(X_5, Y_5, Z_5)$ exists, then it is LAS if and only if, all the following criteria hold:

$$\begin{aligned} A &> 0 \\ C &> 0 \\ AB &> C \end{aligned} \qquad (6)$$

A, B and C to be determined in the proof.

Proof 1. The eigenvalues of $A(K, 0,0)$ are $\lambda_{1X} = -r$, $\lambda_{1Y} = \frac{c\beta K}{1+\beta T K} - d_y$ and $\lambda_{1Z} = \frac{c_1\alpha_1 K}{1+\alpha_1 T_1 K} - d_z$.
Therefore, eigenvalues are negative, if and only if condition (3) holds. This completes the proof.

Proof 2. The eigenvalues of $A(X_3, 0, Z_3)$ are $\lambda_{3Y} = \frac{c\beta X_3}{1+\beta T X_3} + a\left(1 - \frac{X_3}{K}\right) - \frac{\alpha_2 Z_3}{1+\alpha_1 T_1 X_3} - d_y$ and λ_{3X} and λ_{3Z} are roots of the equation

$$\lambda^2 + \frac{r X_3 (1 + 2\alpha_1 T_1 X_3 - K\alpha_1 T_1)}{K(1+\alpha_1 T_1 X_3)}\lambda + \frac{c_1 \alpha_1^2 X_3 Z_3}{(1+\alpha_1 T_1 X_3)^2} = 0$$

Therefore, all the eigenvalues are negative, if and only if condition (4) holds. This completes the proof.

Proof 3. The eigenvalues of $A(X_4, Y_4, 0)$ are $\lambda_{4z} = \frac{c_1\alpha_1 X_4 + c_2\alpha_2 Y_4}{1+\alpha_1 T_1 X_4 + \alpha_2 T_2 Y_4} - d_z$ and λ_{4X} and λ_{4y} are roots of the equation

$$\lambda^2 + \frac{r X_4 (1 + 2\beta T X_4 - \beta T K)}{k(1+\beta T X_4)}\lambda + \frac{\beta X_4 y_4}{1+\beta T X_4}\left[\frac{c\beta}{(1+\beta T X_4)^2} - \frac{a}{K}\right] = 0$$

Therefore, all the eigenvalues are negative, if and only if condition (5) holds. This completes the proof.

Proof 4. The eigenvalues of $A(X_4, Y_4, 0)$ are λ_{5X}, λ_{5Y} and λ_{5Z} are roots of the equation

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0,$$

Where, $A = -(R_1 + R_5)$,
$B = R_1 R_5 - R_2 R_4 - r_3(R_1 + R_3 + R_5)$,
and $C = R_1 R_5 - R_2 R_4 - R_3 R_4 + R_3 R_5$

With $R_1 = r - \frac{2rX_5}{K} - \frac{\beta Y_5}{(1+\beta T X_5)^2} - \frac{(1+\alpha_2 T_2 Y_5)\alpha_1 Z_5}{(1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5)^2}$,
$R_2 = -\frac{\beta X_5}{1+\beta T X_5} + \frac{\alpha_1\alpha_2 T_2 X_5 Z_5}{(1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5)^2}$, $R_3 = -\frac{\alpha_1 X_5}{1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5}$,
$R_4 = \frac{c\beta Y_5}{(1+\beta T X_5)^2} + \frac{\alpha_1\alpha_2 T_1 Y_5 Z_5}{(1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5)^2} - \frac{aY_5}{K}$,
$R_5 = \frac{\alpha_2^2 T_2 Y_5 Z_5}{(1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5)^2}$, $R_6 = -\frac{\alpha_2 Y_5}{1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5}$,
$R_7 = \frac{(c_1\alpha_1+\alpha_1\alpha_2(c_1T_2-c_2T_1)Y_5)Z_5}{(1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5)^2}$ and $R_8 = \frac{(c_2\alpha_2+\alpha_1\alpha_2(c_2T_1-c_1T_2)X_5)Z_5}{(1+\alpha_1 T_1 X_5+\alpha_2 T_2 Y_5)^2}$

Condition (6) is the Routh-Hurwize criteria; therefore, all the eigenvalues are negative. This completes the proof.

4. HOPF-BIFURCATION

Here, the occurrence of Hopf- bifurcation in system1 near all steady states, are discussed as follows:

From Theorem (2) it is observed that $S_0(0,0,0)$ and $S_2(0, Y_2, Z_2)$ is always unstable. Therefor there is no possibility to have a Hopf bifurcation near S_0 and S_2 . From Theorem 3(1), it is observed that the eigenvalues of $A(K, 0,0)$ are $\lambda_{1X} = -r$, $\lambda_{1Y} = \frac{c\beta K}{1+\beta T K} - d_y$ and $\lambda_{1Z} = \frac{c_1\alpha_1 K}{1+\alpha_1 T_1 K} - d_z$. this guarantee that the eigenvalues cannot be imaginary complex number and hence there is no possibility to have a Hopf bifurcation near$S_1(K, 0,0)$.

The conditions that guarantee the occurring of Hopf- bifurcation near, $S_3(X_3, 0, Z_3)$ and If $S_4(X_4, Y_4, 0)$ $S_4 = (X^*, Y^*, Z^*)$ are established in Theorem 4 and Theorem (5), respectively. Regarding to$S_5(X_5, Y_5, Z_5)$ Form theorem 3(4) we can note that it is difficult to determine where the eigenvalue become imaginary complex, so it is complex analytically study Hopf-bifurcation near$S_5(X_5, Y_5, Z_5)$. However, we observed the occurring of Hopf bifurcation near$S_5(X_5, Y_5, Z_5)$ numerically in the next section.

Theorem 4. System (1) has a Hopf bifurcation near $S_3(X_3, 0, Z_3)$ as the parameter value K passes through the value $K_1 = \frac{1}{\alpha_1 T_1} + 2X_3$, if the following condition holds.

$$\frac{c\beta K X_3}{1+\beta T X_3} + a(K - X_3) > \frac{r\alpha_2}{\alpha_1}(K - X_3) \quad (7)$$

Proof. According to $A(X_3, 0, Z_3)$, $\lambda_{3Y} = \frac{c\beta X_3}{1+\beta T X_3} + a\left(1 - \frac{X_3}{K}\right) - \frac{\alpha_2 Z_3}{1+\alpha_1 T_1 X_3}$ and

$$\lambda_{3X}, \lambda_{3Z} = \frac{1}{2}\left[-A_1 \pm \sqrt{A_1^2 - 4B_1}\right]$$

Where $A_1 = \frac{rX_3(2\alpha_1 T_1 X_3 - K\alpha_1 T_1 + 1)}{K(1+\alpha_1 T_1 X_3)}$ and $B_1 = \frac{c_1\alpha_1^2 X_3 Z_3}{(1+\alpha_1 T_1 X_3)^3}$

Clearly, as shown above, λ_{3Y} is negative if and only if condition (7) holds. However,

$\lambda_{3X}, \lambda_{3Z} = \pm i\sqrt{B_1}$ at $K = K_1$, so there is neighborhood around $K = K_1$ such that

$\lambda_{4Y}, \lambda_{4z} = \omega(K) \pm i\varpi(K)$, where $\omega(K) = -\frac{A_1}{2}$, and

$$\left[\frac{d\omega(K)}{dK}\right]_{K=K_1} = -\frac{rX_3(1+2\alpha_1 T_1 X_3)}{(1+\alpha_1 T_1 X_3)\left(\frac{1}{\alpha_1 T_1} + 2X_3\right)^2} \neq 0$$

Therefore, system (**1**) has a Hopf- bifurcation near $S_3(X_3, 0, Z_3)$at $K = K_1$, and hence the proof is

Theorem 5 System (1) has a Hopf bifurcation near $S_4(X_4, Y_4, 0)$ as the parameter value K passes through the value $K_2 = \frac{1}{\beta T} + 2X_4$, if

$$c_1\alpha_1 X_4 + c_2\alpha_2 Y_4 < d_z\,(1 + \alpha_1 T_1 X_4 + \alpha_2 T_2 Y_4) \tag{8}$$

Proof According to $A(X_4, Y_4, 0)$, $\lambda_{4Y} = \frac{c_1\alpha_1 X_4 + c_2\alpha_2 Y_4}{1+\alpha_1 T_1 X_4 + \alpha_2 T_2 Y_4} - d_z$ and

$$\lambda_{4X},\ \lambda_{4Z} = \frac{1}{2}\left[-A_2 \pm \sqrt{A_2^2 - 4B_2}\right]$$

Where $A_2 = \frac{r}{c\beta K^2}\left(d_z K - a(K - X_4)\right)(1 + 2\beta T X_4 - \beta T K)$ and $B_2 = \frac{\beta X_4 y_4}{1+\beta T X_4}\left[\frac{c\beta}{(1+\beta T X_4)^2} - \frac{a}{K}\right]$

Clearly, as shown above, λ_{4Y} is negative if and only if condition (8) holds. However,

$\lambda_{3X},\ \lambda_{3Z} = \pm i\sqrt{B_1}$ at $K = K_2$, so there is neighborhood around $K = K_2$ sch that

$$\lambda_{4Y},\ \lambda_{4z} = \omega(K) \pm i\varpi(K)$$

where$\omega(K) = -\frac{A_1}{2}$, and

$$\left[\frac{d\omega(K)}{dK}\right]_{K=K_2} = \frac{d_z K}{(K_2 - X_4)^2}\left[\frac{dX_4}{da}\right]_{K=K_2} \neq 0$$

Therefore, system (**1**) has a Hopf- bifurcation near $S_4(X_4, Y_4, 0)$at $K = K_2$, and hence the proof is

5. NUMERICAL SIMULATION

In order to support the analytical finding in this paper, some numerical simulations are performed; all the simulations are carried out through Runga -Kutta method of order six method, using MATLAB. First, let choose the set of parameter values as given in (9).

$$r = 1.1;\ K = 40;\ \beta = 0.01; T = T_1 = T_2 = 5; a = 0.008;$$
$$\alpha_1 = \alpha_2 = 0.001; c = c_1 = c_2 = 0.5; d_y = d_z = 0.03. \tag{9}$$

The parameter values in (9), satisfy the condition for LAS of the coexistence steady state. The numerical solution with parameter values in (1) illustrated in Fig.1.

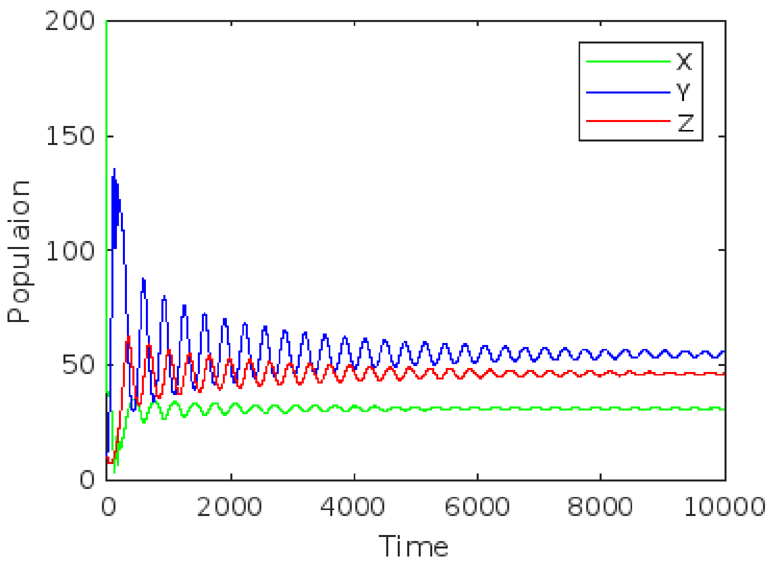

Figure 1: With parameter values as given in (9), system (1) approaches coexistence steady state.

In Fig.2, the value of $\boldsymbol{a}$ increased to $\mathbf{0.011}$ and other parameter values are fixed as given in (9). It is observed that and system trajectories show periodic oscillations around coexistence, and this indicates the emergence of Hopf bifurcations as parameter $\boldsymbol{a}$ increases.

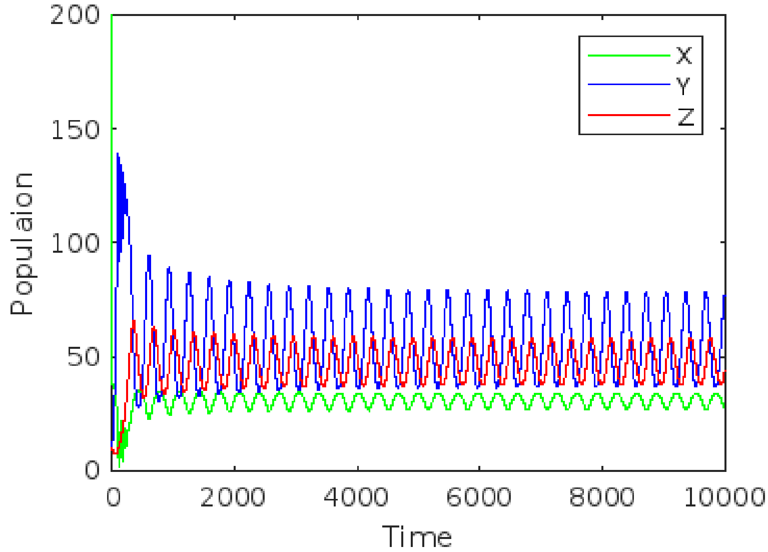

Figure 2: The time series shows periodic oscillations around coexistence steady state where, a= 0.011 and other parameter values are fixed as given in (9).

In Fig.3, values $K, a, \alpha_1, c, c_1, c_2, d_1$ and d_2 changed to 83, 0.025, 0.005, 0.9, 0.7, 0.7, 0.07 and 0.05, respectively and fixed others as given in (9). Those parameter values satisfy the conditions for LAS of the intermediate predator free Steady state

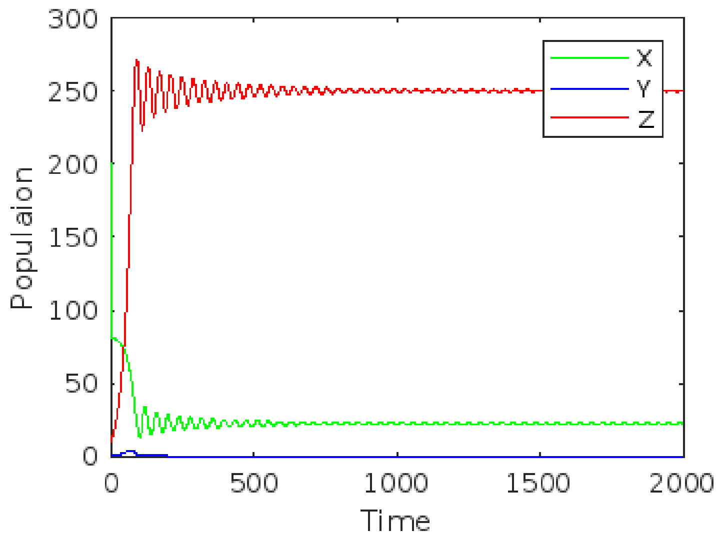

Figure 3: The time series shows that system (1) approaches intermediate predator steady state.

Note that $\boldsymbol{K_1} \approx \mathbf{84}$ for the parameters used Fig.3. Therefore, if we increased the value $\boldsymbol{K} = \mathbf{86}$ in Fig.4 and fixed others as used in Fig.3. Then it is observed that and system trajectories show periodic oscillations around intermediate predator free Steady state and this indicates the emergence of Hopf bifurcations as parameter $\boldsymbol{K}$ increases.

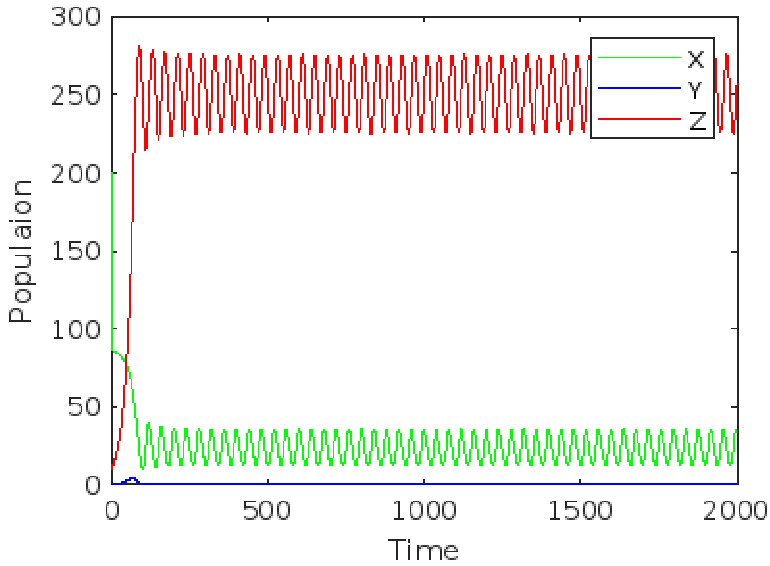

Figure 4: The time series shows periodic oscillations around intermediate predator steady state.

In Fig.5, values $K, a, c,\ d_1$ and d_2 changed to 50, 0.025, 0.7, 0.9, 0.08 and 0.08, respectively and fixed others as given in (9). Those parameter values satisfy the conditions for LAS of the top predator free Steady state.

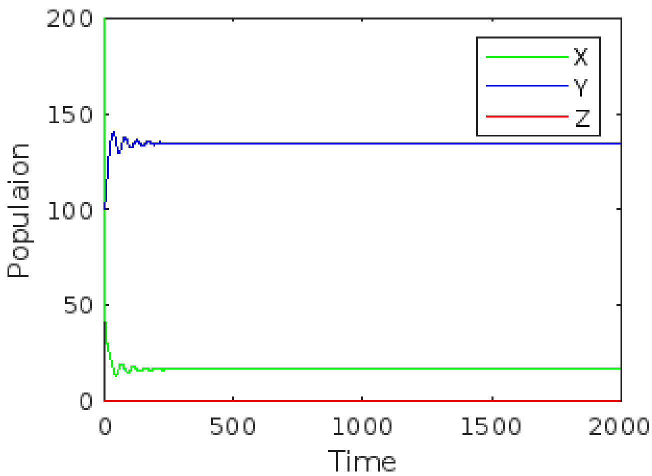

Figure 5: The time series shows that system (1) approaches top predator steady state.

Note that $\boldsymbol{K_2 \approx 52}$ for the parameters used Fig.5, therefore if we increased the value $\boldsymbol{K = 53}$ in Fig.6 and fixed others as used in Fig.5. Then it is observed that and system trajectories show periodic oscillations around top predator free Steady state and this indicates the emergence of Hopf- bifurcations as parameter $\boldsymbol{K}$ increases.

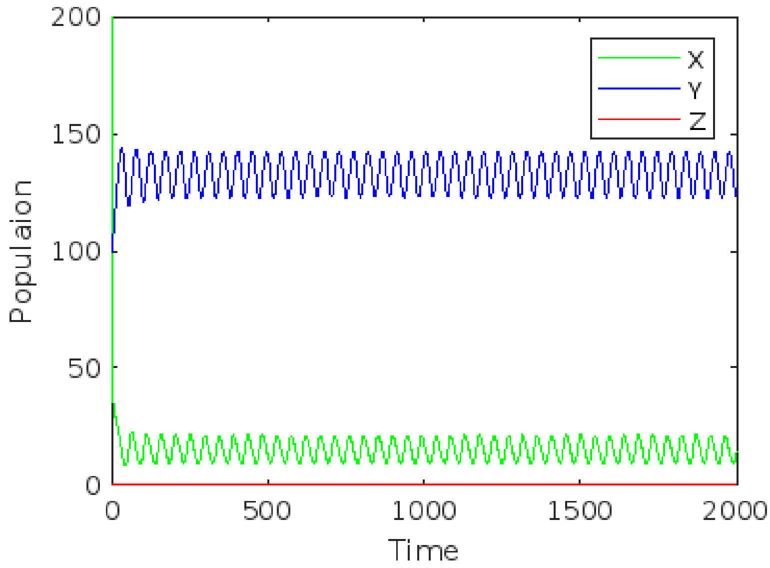

Figure 6: The time shows periodic oscillations around top predator steady state.

In general, above figures confirm the analytical results regarding to stability and Hopf- bifurcation, further it is observed that when the value of a and K (the amount of extra food) the dynamics of the system (1) induced a transition from a stability situation to the state where the prey species and apex predators oscillate periodically.

6. CONCLUSION

In this article, a food-web is that includes three species of prey, intermediate predators and top predators, an amount of additional food supplies to intermediate predators. the intermediate predators predating the prey according to the Holling type-II functional response, while the apex predators predating both prey and intermediate predators according to extended Holling type II functional response for two prey species. it is proved that under condition (2), Six biologically possible steady states are explored and it is discovered that both trivial and prey-free steady states are unstable, however, in theorem2 it is proved that, only prey existence, intermediate predators-free, top predator-free and coexistence steady states are LAS under on the sample parameter conditions (3). (4), (5), and (6), respectively. And also, it is proved that critical values K_1 and K_2, make the occurrence of Hopf-bifurcation of the model near intermediate predators-free and top predator-free steady states, respectively. Based on choosing suitable values as given in (9), the model solved numerically. In the numerical solutions, analytical results regarding to the stability and Hopf- bifurcation are confirmed. It is illustrated in Fig. (1-6), if we increase the values of a and K, that is the amount of extra food, the dynamics of the system (1) induced a transition from the stability situation to the state where the prey species and apex predators oscillate periodically.

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