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# Strongly Nil-Clean Rings of Order Two Units

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**Abstract**

If every element of a ring $\Re$ is the sum of idempotent and nilpotent that commute, then the ring is said to be a strongly nil-clean. Further features of a strongly nil-clean ring are given in this paper. Furthermore, we present and investigate a special class of strongly nil-clean rings with order two units. Additionally, we examine a ring with each element $a$ in $\Re$, $a^2$ and $a^4$ is a strongly nil-clean with order two and order four units. Among other results, we prove that: If $\Re$ is a strongly nil-clean ring of order two units, then for all $a$ in $\Re$, existing $\mathfrak{b}$ in $\Re$, such that $a.\mathfrak{b} = \Psi$, $a - \mathfrak{b} - 1 = u$ and $u^2 = 1$, and the converse of this result is true if 2 is nilpotent.

***Keywords***:
strongly nil-clean, Idempotent element, Nilpotent element.

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## I. Introduction

All of the rings are associative with an identity throughout. A ring $\Re$ is defined as clean [1] if each member of $\Re$ is the sum of an idempotent and a unit. Additionally, according to [2], if the idempotent and the unit commute $\Re$ is considered to be a strongly clean ring .

The set of idempotents, units, nilpotents and the Jacobson radical of $\Re$, will be represented by the symbols $Id(\Re)$ , $U(\Re)$, $Nil(\Re)$ and $J(\Re)$ respectively.

A nil-clean ring, according to Diesl in [3], is one in which every element is the sum of an idempotent and a nilpotent. If the idempotent and nilpotent commute. $\Re$ is regarded as a strongly nil clean (or strongly NC for short) Kosan and Zhou [4] in 2016. The structure of strongly NC rings and rated subjects were provided in [5] and [6].

Chen and Sheibani [7], described a strongly 2-nil clean (or strongly 2-NC for short) ring as a ring $\Re$, where each member is the sum of two idempotents and a nilpotent that commute with one another .

The invo-clean ring was defined by Danchev [8] in 2017, as a ring $\Re$ where each member of $\Re$ is the sum of an idempotent and a unit of order two.

This work introduces the idea of a strongly NC ring with an order two units, and rings with every $a$, $a^2$ and $a^4$ are strongly NC with an order of two or four units.

## II. Background

In this part, we shall give some definitions and well-known results which may be needed in our work.

**Definition 2.1:** [1]

If a ring $\Re$ has an element $a = \Psi + \mathrm{u}$, where $\Psi \in \mathrm{Id}(\Re)$ and $\mathrm{u} \in \mathrm{U}(\Re)$, then the ring $\Re$ is considered clean, if each element of $\Re$ is clean .

**Example 2.2:**

The ring $Z_6$ is clean ring, obviously Id($Z_6$)={0,1,3,4} and U($Z_6$)={1,5}. Clearly, every element of $Z_6$ is the sum of element of $Id(\Re)$ and element of $U(\Re)$, so $Z_6$ is a clean ring .

**Definition 2.3:** [8]

A ring $\Re$ is said to be invo-clean, if every $a \in \Re$ , $a = v + \Psi$, where $\Psi \in Id(\Re)$ and $v^2 = 1$.
If $v\Psi = \Psi v$ is called strongly invo-clean.

**Example 2.4:**

The rings $Z_5$ and $Z_7$ are not invo-clean. Oppositely, the rings $Z_2, Z_3, Z_4, Z_6$ and $Z_8$ are all invo-clean rings.

**Definition 2.5 :** [9]

An element $n$ of a ring $\mathfrak{R}$ is called a nilpotent, if there is a positive integer $\mathfrak{r}$, such that $n^{\mathfrak{r}} = 0$.

**Example 2.6:**

In the ring $Z_8$ the nilpotent elements are $\{0,2,4,6\}$.

**Lemma 2.7:** [10]

If $n$ is a nilpotent and $u$ is a unit and $un = nu$ then;

1) $1 \pm n$ is a unit.

2) $u + n$ is a unit.

**Definition 2.8:**

Let $\mathfrak{R}$ be a ring and $a$ in $\mathfrak{R}$, define $\mathfrak{r}(a) = \{\mathfrak{b} \in \mathfrak{R}: a\mathfrak{b} = 0\}$ .

**Definition 2.9:** [11]

An element $\mathfrak{t} \in \mathfrak{R}$ is tripotent if $\mathfrak{t} = \mathfrak{t}^3$. If every element of a ring $\mathfrak{R}$ is tripotent, then $\mathfrak{R}$ is referred to be a tripotent ring. Clearly, $Z_6$ is a tripotent ring.

**Theorem 2.10:** [4]

An element $a \in \mathfrak{R}$ is strongly NC if and only if $a$ is strongly clean in R and $a - a^2$ is nilpotent .

**Definition 2.11:** [7]

A ring $\mathfrak{R}$ is considered to be a strongly 2-NC if every $a \in \mathfrak{R}$, $a = \Psi_1 + \Psi_2 + n$, where $\Psi_1, \Psi_2 \in Id(\mathfrak{R})$ , $n \in Nil(\mathfrak{R})$ that commute.

**Example 2.12:**

In the ring $Z_{12}$, then:

$Id(Z_{12})$={0,1,4,9}

$Nil(Z_{12})$={0,6}

$U(Z_{12}) = \{1,5,7,11\}$ .

It turns out that the ring $Z_{12}$ is strongly 2-NC.

**Lemma 2.13**: [7]

The following are equivalent for a ring $\mathfrak{R}$ :

1- $\mathfrak{R}$ is strongly 2-NC .

2- For any $a \in \mathfrak{R}$, $a^3 - a \in Nil(\mathfrak{R})$ .

3-For all $a \in \mathfrak{R}$ , $a^2 \in \mathfrak{R}$ is strongly NC.

**Definition 2.14**: [12]

A ring $\mathfrak{R}$ is a Zhou nil-clean, if every element in $\mathfrak{R}$ is the sum of two tripotents and a nilpotent that commute with one another.

**Example 2.15:** Consider the ring $Z_{25}$. Then

$Id(Z_{25}) = \{0,1\}$.

And the tripotent elements are $\{0,1,24\}$

$Nil(Z_{25}) = \{0,5,10,15,20\}$

$U(Z_{25}) = \{1,2,3,4,6,7,8,9,11,12,13,14,16,17,18,19,21,22,23,24\}$

Observe the ring is Zhou nil-clean.

**Lemma 2.16:** [13]

Assume that there are two commuting idempotents, $\Psi_1$ and $\Psi_2$, then :

1- $(\Psi_1 - \Psi_2)^2$ is an idempotent .

2- $(\Psi_1 + \Psi_2)^3$ is a tripotent .

3- $(\Psi_1 - \Psi_2)^2 + (\Psi_1 - \Psi_2) - 1$ is a unit of order two .

4- $2(\Psi_1 - \Psi_2)^2 - 1$ is a unit of order two .

## 3. A strongly NC ring of order two units

This section defines strongly NC ring of order two units, outlines some of its fundamental characteristics, and offers some examples.

**Definition 3.1:** [4]

A ring $\mathfrak{R}$ is said to be a nil-clean ring, if for every $a \in \mathfrak{R}$ such that $a = \Psi + n$, where $\Psi$ is idempotent element, and $n$ is a nilpotent element in $\mathfrak{R}$ , if $\Psi n = n\Psi$, a ring $\mathfrak{R}$ is called strongly NC if every element of $\mathfrak{R}$ is strongly NC.

**Example 3.2:**

In the ring $Z_2 \times Z_4$, note that

$Id(Z_2 \times Z_4) = \{(0,0), (0,1), (1,0), (1,1)\}$

$Nil(Z_2 \times Z_4) = \{(0,0), (0,2)\}$

By direct calculation, $\mathfrak{R}$ is strongly NC ring.

The next result gives further properties a strongly NC rings.

**Theorem 3.3 :**

If $\mathfrak{R}$ is strongly NC ring , then for every $a \in \mathfrak{R}$, $a = \Psi + n$

1) $a^m$is strongly NC .

2) $1 - a$ is strongly NC .

3) $\mathfrak{r}(a) \cap \Psi\mathfrak{R} = 0$

**Proof: (1)**

Let $a = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$ that commute, then $a^2 = (\Psi + n)^2 = \Psi + 2\Psi n + n^2$, observe that $2\Psi n + n^2 \in Nil(\mathfrak{R})$, say $n'$, $n' = 2\Psi n + n^2$, hence $a^2 = \Psi + n'$ . Assume that $a^{\mathfrak{r}} = \Psi + n''$, is true . Now, $a^{\mathfrak{r}+1} = a^{\mathfrak{r}}.a = (\Psi + n'')(\Psi + n) = \Psi + \Psi n + n''\Psi + n''n$, say $n_1 \in Nil(\mathfrak{R})$, $n_1 = \Psi n + n''\Psi + n''n \in Nil(\mathfrak{R})$, hence $a^{\mathfrak{r}+1} = \Psi + n_1$.

**(2):** Let $a \in \mathfrak{R}$, then $a = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$, $n \in Nil(\mathfrak{R})$ and $\Psi n = \Psi n$, then $1 - a = 1 - \Psi - n$, so $1 - a = (1 - \Psi) + (-n)$, since $(1 - \Psi)$ is idempotent and $-n$ is a nilpotent. Therefore $1 - a$ is strongly NC .

**(3):** Let $x \in \mathfrak{r}(a) \cap \Psi\mathfrak{R}$, then $ax = 0$, and $x = \Psi\mathfrak{r}$, for some $\mathfrak{r} \in \mathfrak{R}$.Hence $a(\Psi\mathfrak{r}) = 0$, so $(\Psi + n)\Psi\mathfrak{r} = \Psi^2\mathfrak{r} + n\Psi\mathfrak{r} = 0$, implies $\Psi\mathfrak{r} + n\Psi\mathfrak{r} = 0$, since $\Psi^2 = \Psi$ is idempotent element. So $(1 + n)\Psi\mathfrak{r} = 0$, since $(1 + n) \in U(\mathfrak{R})$, then $\Psi\mathfrak{r} = x = 0$. Therefore $\mathfrak{r}(a) \cap \Psi\mathfrak{R} = 0 \cdot \blacksquare$

Next, will look at a strongly NC ring of order two units .

**Proposition 3.4 :**

If $\mathfrak{R}$ is Strongly NC ring of order two units and $n^2 + 2n = 0$, for every nilpotent $n$. Then $\mathfrak{R}$ is strongly invo – clean ring.

**Proof :** Let $a \in \mathfrak{R}$, then $a - 1 = \Psi + n$, where $\Psi$ is an idempotent element in $\mathfrak{R}$ and $n$ is a nilpotent element in $\mathfrak{R}$ with $\Psi n = n\Psi$. Then $a = \Psi + n + 1$, but $n + 1$ is a unit (Lemma 2.7, part 1 ) we get $a = \Psi + u$, where $u = n + 1$. Now $u^2 = (n+1)^2 = n^2 + 2n + 1 = 0 + 1 = 1$. ■

**Proposition 3.5:**

Let $\mathfrak{R}$ be a strongly invo-clean ring with $2 \in Nil(\mathfrak{R})$. Then $\mathfrak{R}$ is a strongly NC ring.

**Proof:** Let $a$ in $\mathfrak{R}$, then $a - 1 = \Psi + u$, where $\Psi \in Id(\mathfrak{R})$ and $u^2 = 1$, implies $a = \Psi + \mathrm{u} + 1$, where $u^2 = 1$, we get $(u+1)^2 = u^2 + 2u + 1 = 1 + 2u + 1 = 2 + 2u = 2(1+u)$, since $2 \in Nil(\mathfrak{R})$, then $(1+u) \in Nil(\mathfrak{R})$, say $n$, $n = 1 + u$. Hence $a = \Psi + n$. ■

**Theorem 3.6 :**

If $\mathfrak{R}$ is strongly NC ring of order two units, then for all $a \in \mathfrak{R}$, existing $\mathfrak{b} \in \mathfrak{R}$ such that $a.\mathfrak{b} = \Psi$ and $a - \mathfrak{b} - 1 = u$, $u^2 = 1$.

**Proof :** Let $a \in \mathfrak{R}$, then $a = \Psi + n$, $\Psi n = n\Psi$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$. If $n^{\mathrm{r}} = 0$, if we set $\mathfrak{b} = \Psi - \Psi n + \Psi n^2 - \Psi n^3 + \cdots \ldots \ldots + (-1)^{\mathrm{r}-1} n^{\mathrm{r}-1}$, where $\mathrm{r} \in z^+$. Hence $a.\mathfrak{b} = (\Psi + n)(\Psi - \Psi n + \Psi n^2 - \Psi n^3 + \cdots \ldots \ldots + (-1)^{\mathrm{r}-1} n^{\mathrm{r}-1}) = \Psi^2 - \Psi^2 n + \Psi^2 n^2 - \Psi^2 n^3 + \cdots \ldots \ldots + (-1)^{\mathrm{r}-1}\Psi n^{\mathrm{r}-1} + \Psi n - \Psi n^2 + \Psi n^3 - \Psi n^4 + \cdots \ldots \ldots + (-1)^{\mathrm{r}-1} n^{\mathrm{r}} = \Psi - \Psi n + \Psi n^2 - \Psi n^3 + \cdots \ldots \ldots + (-1)^{\mathrm{r}-1} \Psi n^{\mathrm{r}-1} + \Psi n - \Psi n^2 + \Psi n^3 + \cdots \ldots \ldots + (-1)^{\mathrm{r}-1} n^{\mathrm{r}} = \Psi$. Consider $a - \mathfrak{b} - 1 = (\Psi + n) - (\Psi - \Psi n + \Psi n^2 - \Psi n^3 + \cdots \ldots + (-1)^{\mathrm{r}-1} n^{\mathrm{r}-1}) - 1 = \Psi + n - \Psi + \Psi n - \Psi n^2 + \Psi n^3 - \cdots \ldots \ldots . -(-1)^{\mathrm{r}-1} n^{\mathrm{r}-1} - 1 = n + \Psi n - \Psi n^2 + \cdots \ldots \ldots . -(-1)^{\mathrm{r}-1} n^{\mathrm{r}-1} - 1 = n(1 + \Psi - \Psi n + \cdots \ldots \ldots - (-1)^{\mathrm{r}-1} n^{\mathrm{r}-2}) - 1$. Let $n' \in Nil(\mathfrak{R})$, $n' = 1 + \Psi - \Psi n + \cdots \ldots \ldots - (-1)^{\mathrm{r}-1} n^{\mathrm{r}-2}$, hence $a - \mathfrak{b} - 1 = nn' - 1$. Since $n \in Nil(\mathfrak{R})$, then $nn' Nil(\mathfrak{R}) and\ nn' = n'n$. Hence $a - \mathfrak{b} - 1 = n'' - 1 = u$, $where\ n'' = nn'$. ■

The converse this theorem is not true. As shown in the following example :

**Example 3.7:**

In the ring $Z_{24}$

$Id(Z_{24}) = \{0,1,9,16\}$

$Nil(Z_{24}) = \{0,6,12,18\}$

$U(Z_{24}) = \{1,5,7,11,13,17,19,23\}$

And all the unit elements of $Z_{24}$ are of order two. By using ( Theorem 3.6) the ring $Z_{24}$ is not strongly NC, since 2,5 and 8 does not satisfy (Theorem 2.10) , but $a.\mathfrak{b} = \Psi\ and$ $a - \mathfrak{b} - 1 = u$.

**Theorem 3.8 :**

If $a.\mathfrak{b} = \Psi$ and $a - \mathfrak{b} - 1 = u$, $u^2 = 1$, with $u\mathfrak{b} = \mathfrak{b}u$ and $2 \in Nil(\mathfrak{R})$. Then $\mathfrak{R}$ is strongly NC ring .

**Proof:** If $a - \mathfrak{b} - 1 = u$, where $u^2 = 1$, then $a - \mathfrak{b} = u + 1$, note that $(u+1)^2 = u^2 + 2u + 1 = 2u + 2 = 2(u+1)$, since $2 \in Nil(\mathfrak{R})$, then$(u+1) \in Nil(\mathfrak{R})$, say $n$, $n = u + 1$. Hence $a - \mathfrak{b} = n$. Since $u\mathfrak{b} = \mathfrak{b}u$, then $(u+1)\mathfrak{b} = \mathfrak{b}(u+1)$, implies that to $n\mathfrak{b} = \mathfrak{b}n$ .Now let $a \in \mathfrak{R}$, $n \in Nil(\mathfrak{R})$, then $an \in Nil(\mathfrak{R})$. Since $a - \mathfrak{b} = n$ , implies $a = \mathfrak{b} + n$ multiply by $n$ we get $an = \mathfrak{b}n + n^2 = n\mathfrak{b} + n^2 = na$, so $an = na$. Now since $a.\mathfrak{b} = \Psi\ and\ a - \mathfrak{b} = n$, multiply this by $a$, we get $a^2 - \Psi = an$, where $an \in Nil(\mathfrak{R})$, say $n_1$, $an = n_1$, hence $a^2 = \Psi + n_1$ . By( Lemma 2.13) , $a = \Psi_1 + \Psi_2 + n_1$, is strongly 2-NC .

Now $a^2 = (\Psi_1 - \Psi_2)^2 + 2\Psi_1\Psi_2 + n_2$, since $2 \in Nil(\mathfrak{R})$. Then $2\Psi_1\Psi_2 + n_2 \in Nil(\mathfrak{R})$, write $n_3 \in Nil(\mathfrak{R})$. So $2\Psi_1\Psi_2 + n_2 = n_3$, implies $a = (\Psi_1 - \Psi_2)^2 + n_3$. Since $(\Psi_1 - \Psi_2)^2$ is idempotent, then $(\Psi_1 - \Psi_2)^2 = \Psi_3$. Hence $a = \Psi_3 + n_3$ is strongly NC element . So $\mathfrak{R}$ is strongly NC ring. ■

**Theorem 3.9 :**

If $\mathfrak{R}$ is strongly NC ring of order two units. Then for all $a \in J(\mathfrak{R})$, $a^3 = 4a = 0$ .

**Proof:** Let $a \in J(\mathfrak{R})$ then $1 - a \in U(\mathfrak{R})$. since $\mathfrak{R}$ is strongly NC, then $a = \Psi + n$. Then $1 - a = 1 - \Psi - n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$, implies $u = 1 - \Psi - n$, then $u + n = 1 - \Psi$, since $un = nu$, then $u + n \in U(\mathfrak{R})$, by (Lemma 2.7, part 2), so $1 - \Psi = u_1$ implies $1 - \Psi = 1$, hence $\Psi = 0$. So $a = \Psi + n = 0 + n = n$, where $n$ is nilpotent and $1 + n$ is unit. Then $a \in Nil(\mathfrak{R})$, so $1 + a \in U(\mathfrak{R})$, where $u^2 = 1$. Hence $1 + a = 1 + n$, implies $1 + a = u$, then $a = u - 1$. Now $a^2 = (u-1)^2 = u^2 - 2u + 1 = 1 - 2u + 1 = 2 - 2u = 2(1-u) = 2a$. Also $a^3 = (u-1)^3 = 2(u-1)^2 = 2^2(u-1) = 4a$. So $a^2 = 2a$, which yields $a^3 = 4a$. Put $a = 2\mathfrak{b}$, since $2\mathfrak{b} \in J(\mathfrak{R})$. Implies that to $(2\mathfrak{b})^2 = 2(2\mathfrak{b}) = 4\mathfrak{b}$. So $(2\mathfrak{b})^3 = 4(2\mathfrak{b}) = 8\mathfrak{b}$, Hence $8\mathfrak{b}^3 = 8\mathfrak{b}$, so $8\mathfrak{b}^3 - 8\mathfrak{b} = 0$, we get $8\mathfrak{b}(\mathfrak{b}^2 - 1) = 0$, since $\mathfrak{b} \in J(\mathfrak{R})$, then $1 - \mathfrak{b}^2 \in U(\mathfrak{R})$, then $8\mathfrak{b} = 0$. So $4a = 0 = a^3$. ■

**Theorem 3.10 :**

If $\mathfrak{R}$ is a strongly NC ring of order two units. Then $\mathfrak{R}/I$ is strongly NC ring of order two units .

**Proof:** Assume $\mathfrak{R}$ is a strongly NC of order two units and $I$ be an ideal of $\mathfrak{R}$ and let $a \in \mathfrak{R}$, then $a + I \in \mathfrak{R}/I$. Then $a = \Psi + n$, where $\Psi$ is an idempotent element and $n$ is a nilpotent element, with $\Psi n = n\Psi$, hence $a + I = (\Psi + n + I) = \Psi + I + n + I$.

Since$(\Psi + I)(n + I) = \Psi n + I = n\Psi + I = (n + I)(\Psi + I)$. Let $u \in U(\mathfrak{R})$, $u^2 = 1$, then $(u + I)^2 = (u + I)(u + I) = u^2 + I = 1 + I$. Hence $\mathfrak{R}/I$ is strongly NC ring with order two units. ■

**Proposition 3.11:**

Suppose $\mathfrak{R}$ is a strongly NC ring and for every $n \in Nil(\mathfrak{R})$, $n^2 + 2n = 0$ , then $|\mathfrak{R}| = 8$ and $u^2 = 1$ .

**Proof:** Let $a \in \mathfrak{R}$, then $a = \Psi + n$, and $\Psi n = n\Psi$ where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$, take $u \in U(\mathfrak{R})$, yielding to $u = \Psi + n$, implies that to $\Psi = u - n = v \in U(R)$. So $\Psi = 1$, hence $u = 1 + n$. Now $u^2 = (1 + n)^2 = 1 + 2n + n^2$, since $n^2 + 2n = 0$ given, implies $u^2 = 1$ . By ( Theorem 2.10) $a^2 - a \in Nil(\mathfrak{R})$, this gives $2^2 - 2 = 2 \in Nil(\mathfrak{R})$. Using the hypothesis $n^2 + 2n = 0$, yielding $2^2 + 2(2) = 8 = 0$.■

**Example 3.12:**
In the ring $Z_8$, $Id(Z_8)$={0,1}, $Nil(Z_8)$={0,2,4 ,6}, $U(Z_8)$={1,3,5,7}. Note that $1^2 = 3^2 = 5^2 = 7^2 = 1$.

### Proposition 3.13:
Let $\mathfrak{R}$ be a strongly NC ring and for each $n \in Nil(\mathfrak{R})$, $2n^2 + 4n = 0$. Then $|\mathfrak{R}| = 16$ and $u^4 = 1$ .
**Proof:**
Let $a = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$ with $\Psi n = n\Psi$. By hypothesis, $2n^2 + 4n = 0$, since $2 \in Nil(\mathfrak{R})$ yielding $2(2)^2 + 4(2) = 2(4) + 8 = 16 = 0$.
Take $u \in U(\mathfrak{R})$, then $u = \Psi + n$, so $\Psi = u - n = v \in U(\mathfrak{R})$. Hence $\Psi = 1$, hence $u = 1 + n$, then $u^4 = (1 + n)^4 = (1 + 2)^4 = 81 = 1$. ■

### Proposition 3.14:
Let $\mathfrak{R}$ be a strongly NC ring with $4n^2 + 8n = 0$ for each $n \in Nil(\mathfrak{R})$. Then $|\mathfrak{R}| = 32$ and $u^8 = 1$ .
**Proof:** Let $a$ in $\mathfrak{R}$, then $a = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ , $n \in Nil(\mathfrak{R})$ and $\Psi n = n\Psi$ . Since $2 \in Nil(\mathfrak{R})$, then $4(2)^2 + 8(2) = 4(4) + 16 = 32 = 0$. So $|\mathfrak{R}| = 32$.
Assume that $u \in U(\mathfrak{R})$, then $u = \Psi + n$, this gives $\Psi = u - n$, since $un = nu$, then by (Lemma 2.7), $u - n \in U(\mathfrak{R})$, so $\Psi = 1$. Thus $u = 1 + n$. Now consider
$u^8 = (1 + n)^8 = (1 + 2)^8 = (3)^8 = 6561 = 1$.■

### Theorem 3.15 :
If $\mathfrak{R}$ is strongly NC ring and $u^{2^{r-2}} = 1$, for all $r > 2$ and $n^2 + 2n = 0$. Then $|\mathfrak{R}| = 2^r$ .
**Proof:** If $r = 3$ the unit element $u^{2^{3-2}} = u^2 = 1$ . Then $|\mathfrak{R}| = 2^3 = 8$ . Assume that for some positive integer $s$ , $r = s$ the unit element $u^{2^{s-2}} = 1$ , so $|\mathfrak{R}| = 2^s$ .
Now if $r = s + 1$ then $u^{2^{(s+1)-2}} = 1$ . We have $u^{2^{s-1}} = (u^{2^{s-2}})^2$, since $u^{2^{s-2}} = 1$ we obtain $u^{2^{s-1}} = (u^{2^{s-2}})^2 = 1^2 = 1$ . The statement holds for $s + 1$.■
Hence the $|\mathfrak{R}| = 2^r$.

## 4. Rings with every elements $a$ in $\mathfrak{R}$, $a^2$ and $a^4$ are strongly nil-clean .

In this section, we consider , rings with every elements $a$ in $\mathfrak{R}$ , $a^2$ and $a^4$ are a strongly NC of order two units .

### Proposition 4.1:
Let $\mathfrak{R}$ be a ring with every $a$ in $\mathfrak{R}$, $a^2 = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$, $n \in Nil(\mathfrak{R})$ and $\Psi n = n\Psi$. Then
1- $a$ and $-a$ are strongly clean .
2- $r(a) \cap \Psi\mathfrak{R} = 0$.
3- $6 \in Nil(\mathfrak{R})$.
**Proof:(1)**
Let $a \in \mathfrak{R}$. Then $a^2 = \Psi + n$, we may write $a^2$, as $a^2 = (1 - \Psi) + (2\Psi - 1) + n$. Since $2\Psi - 1 \in U(\mathfrak{R})$, then $2\Psi - 1 + n \in U(\mathfrak{R})$, by (Lemma 2.7).
So $a^2 = 1 - \Psi + u$, where $u = 2\Psi - 1 + n$. On the other hand $a^2 - (1 - \Psi) = u$, but $1 - \Psi = (1 - \Psi)^2$, then $\big(a - (1 - \Psi)\big)\big(a + (1 - \Psi)\big) = u$. Hence $a - (1 - \Psi)$ and $a + (1 - \Psi) \in U(\mathfrak{R})$.
**(2)** Let $b \in r(a) \cap \Psi\mathfrak{R}$, then $ab = 0$ and $b = \Psi r$. So $a^2 b = 0$, gives $(\Psi + n)\Psi r = 0$,
thus $\Psi r + n\Psi r = 0$, yielding $(1 + n)\Psi r = 0$. But $1 + n \in U(\mathfrak{R})$, then $\Psi r = x = 0$. Therefore $r(a) \cap \Psi\mathfrak{R} = 0$.
**(3)** Let $a \in \mathfrak{R}$, then by (Theorem 2.10) $a^4 - a^2 \in Nil(\mathfrak{R})$, so $a(a^3 - a) \in Nil(\mathfrak{R})$, multiply by $a^2 - 1$ we get, $(a^2 - 1)a(a^3 - a) \in Nil(\mathfrak{R})$, thus $(a^3 - a)^2 \in Nil(\mathfrak{R})$, so $a^3 - a \in Nil(\mathfrak{R})$, this implies $2^3 - 2 = 8 - 2 = 6 \in Nil(\mathfrak{R})$. ■

### Theorem 4.2 :
If $\mathfrak{R}$ is a ring with every $a \in \mathfrak{R}$, $a^2$ is a strongly NC, and $n^2 + 2n = 0$ for each $n \in Nil(\mathfrak{R})$.Then $48 = 0$ and $u^4 = 1$.
**Proof:** Given $a \in \mathfrak{R}$, $a^2$ is a strongly NC, then $a^2 = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$ with $\Psi n = n\Psi$. Assume $u \in U(\mathfrak{R})$, then $u^2 = \Psi + n$, gives $\Psi = u^2 - n = v \in U(\mathfrak{R})$, thus $\Psi = 1$ .
Hence $u^2 = 1 + n$, implies $u^4 = (1 + n)^2 = 1 + 2n + n^2$, by assumption $n^2 + 2n = 0$, so $u^4 = 1$. By (Theorem 4.1, part 3), we get $6 \in Nil(\mathfrak{R})$. Hence $6^2 + 2(6) = 36 + 12 = 48 = 0$. ■

**Example 4.3:**
In the ring $Z_{48}$. Note that:
$Id(Z_{48})$ ={0,1,16,33}
$Nil(Z_{48})$={0,6,12,18,24,30,36,42}
$U(Z_{48})$={1,5,7,11,13,17,19,23,25,29,31,35,37,41,43,47}
All the unit elements of $Z_{48}$ are of order 4 .

### Proposition 4.4:
Let $\mathfrak{R}$ be a ring, with every $a$ in $\mathfrak{R}$, $a^2$is a strongly NC and if $2n^2 + 4n = 0$ for each $n \in Nil(\mathfrak{R})$. Then $|\mathfrak{R}| = 96$ and $u^8 = 1$ .
**Proof:** Let $a \in \mathfrak{R}$, then $a^2 = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$ with $\Psi n = n\Psi$. Appling (Lemma 2.13), we have $a^3 - a \in Nil(\mathfrak{R})$, so $2^3 - 2 = 6 \in Nil(\mathfrak{R})$. Since $2n^2 + 4n = 0$, by hypothesis, then $2(6)^2 + 4(6) = 72 + 24 = 96$. Therefore $|\mathfrak{R}| = 96$.
For any $u \in U(\mathfrak{R})$, $u^2 = \Psi + n$, gives $\Psi = n - u^2 = v \in U(\mathfrak{R})$. Hence $\Psi = 1$. Now $u^8 = (1 + n)^8 = (1 + 6)^8 = 5764801 = 1$. ■

We next turn to consider rings with every $a$ in $\mathfrak{R}$, $a^4$ is a strongly NC .

### Lemma 4.5: [14]
The following are equivalent for a ring $\mathfrak{R}$ :
1- $\mathfrak{R}$ is Zhou nil-clean .

2- For any $a \in \mathfrak{R}$ , $a^5 - a \in Nil(\mathfrak{R})$ .

**Theorem 4.6 :**

Let $\mathfrak{R}$ be a ring, then $a^4$ is strongly NC for each $a$ in $\mathfrak{R}$ if and only if $\mathfrak{R}$ is Zhou nil-clean ring.

**Proof:** Let $a$ in $\mathfrak{R}$, with $a^4$ is a strongly NC, then $a^4 = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$, with $\Psi n = n\Psi$. Then $(a^4)^2 - a^4 \in Nil(\mathfrak{R})$, (Theorem 2.10), so $a^2(a^6 - a^2) \in Nil(\mathfrak{R})$, multiply by $(a^4 - 1)$, we get $(a^4 - 1)a^2(a^6 - a^2) = (a^6 - a^2)^2 \in Nil(\mathfrak{R})$, this gives $a(a^5 - a) \in Nil(\mathfrak{R})$, again multiply by $(a^4 - 1)$, we get $(a^5 - a)^2 \in Nil(\mathfrak{R})$, hence $a^5 - a \in Nil(\mathfrak{R})$. So $\mathfrak{R}$ is Zhou nil-clean ring (Lemma 4.5) .

**Conversely**, let $a \in \mathfrak{R}$, then by (Lemma 4.5). $a^5 - a \in Nil(\mathfrak{R})$, multiply by $a^3$, we get $(a^8 - a^4) \in Nil(\mathfrak{R})$. Therefore $a^4$ is a strongly NC, (Lemma 2.10).■

**Corollary 4.7:**

If $\mathfrak{R}$ is a ring with every $a^4$ is strongly NC, then $30 \in Nil(\mathfrak{R})$ .

**Proof:** Let $2 \in \mathfrak{R}$, then by (Theorem 4.6), $a^5 - a \in Nil(\mathfrak{R})$, so $2^5 - 2 = 32 - 2 = 30 \in Nil(\mathfrak{R})$.■

**Theorem 4.8 :**

If $a^4$ is Strongly NC for every $a$ in $\mathfrak{R}$ and if $n^2 + 2n = 0$, for every nilpotent $n$ in $\mathfrak{R}$, then $|\mathfrak{R}| = 960$ and every unit is of order 16 .

**Proof:** Since $a^4$ is strongly NC, then $a^4 = \Psi + n$, where $\Psi \in Id(\mathfrak{R})$ and $n \in Nil(\mathfrak{R})$

With $\Psi n = n\Psi$, let $u \in U(\mathfrak{R})$ so $u^4 = \Psi + n$, we obtain $\Psi = u^4 - n \in U(\mathfrak{R})$ .

Hence $\Psi = 1$, so $u^4 = 1 + n$.

Now $(u^4)^4 = (1 + n)^4 = 1 + 4n + 6n^2 + 4n^3 + n^4$

From assumption where $n^2 = -2n$, we get

$$\begin{aligned} u^{16} &= 1 + 4n + 6(-2n) + 4n(-2n) + (-2n)^2 \\ &= 1 + 4n - 12n - 8n^2 + 4n^2 \\ &= 1 - 8n - 8(-2n) + 4(-2n) \\ &= 1 - 8n + 16n - 8n = 1. \end{aligned}$$

From (Corollary 4.7), we get $30 \in Nil(\mathfrak{R})$, so $30^2 + 2(30) = 900 + 60 = 960 = 0$.■

# III. Conclusion

In this article, new properties of a strongly nil-clean ring are given. We consider a strongly nil-clean ring with units of order two, four and eight. Additionally, we consider rings with $a^2$ and $a^4$ are a strongly nil-clean elements with units of order two and four. We discuss some of the fundamental properties of such rings.


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