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Darboux and Analytic First Integrals of the Generalized Michelson System

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Abstract

The purpose of this work is to demonstrate that, for any value of a_1, a_2, and a_3, the generalized Michelson system u ̇= v,v ̇=w,w ̇= a_1 +a_2 v+ a_3 w-u^2/2 is neither a Darboux nor a rational first integral. Furthermore, we shall demonstrate that for a_3<0,√(2 a_1) >0 , and a_3 a_2-√(2 a_1) >0, this system has no global C^1 first integrals. Additionally, the analytic first integral of this system for a generic condition is investigated near the equilibrium point (√(2 a_1),0,0). Key words: Exponential factor, Darboux First Integral, Invariant Algebraic Surfaces, Analytic First Integral.

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I. Introduction

A 3-D independent ordinary asymmetric system known as the generalized Michelson system appears while studying the travelling wave solutions of the Kuramoto-Sivashinsky problem [1]. It may be conceptualized using a 3-D framework differential system.

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= w, \\ \dot{w} &= a_1 + a_2 v + a_3 w - \frac{u^2}{2}, \end{aligned} \qquad (1)$$

Where real arbitrary, arbitrary parameters a_1, a_2, and a_3 are used. The dot denotes to the derivative with respect to time t_1. This system has a symmetric equilibrium points $\left(\mp\sqrt{2\,a_1}, 0,0\right)$. The field of research that focuses on systems (1) is of great importance in the fields of physics and engineering, particularly in travelling waves [1].

In [2], triple-zero bifurcation in its normal form discussed. Through analytical techniques and numerical modeling, the homoclinic orbits for the 3-D continuous piecewise linear generalized Michelson system- have been investigated in [3]. Invariant manifold theory and the Poincaré map are discussed in the saddle-focus equilibrium which is connected by homoclinic orbits in [3].

In [4], the author studied IIR filter model for the generalized Michelson interferometer.

System (1) was discussed by more authors, although it provides no mention of the integrability.

In this paper, we study the analytic type and a Darboux first integrals of system (1). The Darboux integration is a property that a function may possess technique for finding a solution of system (1). For additional information, see [5, 8].

II. Preliminary Results

The topics discussed include the issue of integrability, the Darboux technique, and the auxiliary conclusions that have been offered briefly reviewed at the beginning of this section [9, 10]. First, we provide some fundamental definitions and theorems to set the scene for this study to demonstrate our significant findings. The vector field associated with system (1) is defined as follows:

$$\chi = v\,\frac{\partial}{\partial u} + w\,\frac{\partial}{\partial v} + \left(a_1 + a_2 v + a_3 w - \frac{u^2}{2}\right)\frac{\partial}{\partial w} \qquad (2)$$

Let D be an open subset of $\mathbb{R}^3$. If $H: D_1 \to \mathbb{R}$ is a constant on all orbits $\left(u(t_1), v(t_1), w(t_1)\right)$ of χ included in D, then it represents the polynomial of the vector field χ on D. That H is clearly referred to as a first integral of χ on D if and only if

$$\chi(H) = \mathrm{v}\ \frac{\partial H}{\partial u} + \mathrm{w}\ \frac{\partial H}{\partial v} + \left(a_1\ + a_2\ \mathrm{v} +\ a_3\ \mathrm{w} - \frac{\mathrm{u}^2}{2}\right)\frac{\partial H}{\partial w} = 0 \qquad (3)$$

A local (global) first integral H is a first integral whose domain of definition is a neighborhood of an equilibrium point (whose domain of definition is $\mathbb{R}^3$) of system (1). We recall that H is an analytic (rational) first integral if it is an analytic (rational) function.

An equilibrium points (u_0, v_0, w_0) of system (1) is said to be an attractor if all eigenvalues λ_i of the Jacobian matrix of (1) at (u_0, v_0, w_0) have negative real parts.

Theorem 2.1. Routh-Hurwitz Criterion [14]:

When $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$, then the zero of $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ has negative real components.

Theorem 2.2. System (1) doesn't have any C^1 initial integrals determined in the nearby neighborhood at (u_0, v_0, w_0), if system (1) has an equilibrium point (u_0, v_0, w_0) that is either an attractor or a repeller.

A Darboux theory of integrability has a best method to determine that systems have a first integral or not. Now, we will discuss some fundamental notations [10, 11]. Suppose that $f = f(u, v, w) \in \mathbb{R}[u, v, w]$, then $f = 0$ is said to be an invariant algebraic surface or it is called a Darboux polynomial of χ if there exist a polynomial $K_f \in \mathbb{R}[u, v, w]$ such that

$$\chi(f) = \mathrm{v}\ \frac{\partial f}{\partial u} + \mathrm{w}\ \frac{\partial f}{\partial v} + \left(a_1\ + a_2\ \mathrm{v} +\ a_3\ \mathrm{w} - \frac{\mathrm{u}^2}{2}\right)\frac{\partial f}{\partial w} = f\ K_f, \qquad (4)$$

We say that K_f is the cofactor of f and it has a maximum degree of 1.

Proposition 2.3. System (1) has a rational first integral if it has two distinct invariant algebraic surfaces with the similar non zero cofactor.

We denote an exponential factor of system (1) by E which defined by a non-constant function of the form $E = e^{\frac{g}{f}}$ with greatest common divisor between g and f is equal to one. That means $(g, f) = 1$, where $g, f \in \mathbb{R}[u, v, w]$ and it is satisfied

$$\chi(E) = \mathrm{v}\ \frac{\partial E}{\partial u} + \mathrm{w}\ \frac{\partial E}{\partial v} + \left(a_1\ + a_2\ \mathrm{v} +\ a_3\ \mathrm{w} - \frac{\mathrm{u}^2}{2}\right)\frac{\partial E}{\partial w} = E\ L, \qquad (5)$$

for some polynomial $L = L(u, v, w) \in \mathbb{R}[u, v, w]$ of degree at most 1 which is called the cofactor of E.

Proposition 2.4. i) If f is a non-constant polynomial and the function $E = e^{\frac{g}{f}}$ is an exponential factor of the polynomial differential system (1), then $f = 0$ is an invariant algebraic surface.

ii) Lastly, e^g, which derives from the multiplicity of the infinity invariant plane, can be an exponential factor.

Theorem 2.5. The Darboux Theory [11]

Let a polynomial vector field χ of degree d in $\mathbb{R}^3$ have p_1 irreducible invariant algebraic surfaces $f_i = 0$ such that the f_i are pairwise relatively prime with cofactors K_i for $i = 1, \dots, p_1$ and q_1 exponential factors $e^{\frac{g_j}{f_j}}$ combined with cofactors L_j for $j = 1, \dots, q_1$. There exist $\alpha_i, \beta_j \in \mathbb{R}$ are not all zero such that.

$$\sum_{i=1}^{p_1} \alpha_i\ K_i + \sum_{j=1}^{q} \beta_j\ L_j = 0, \qquad (6)$$

If and only if the function

$$f_1^{\alpha_1} \dots f_p^{\alpha_p}\left(\left[e^{\frac{g_1}{f_1}}\right]^{\beta_1} \dots \left[e^{\frac{g_q}{f_q}}\right]^{\beta_q}\right) \qquad (7)$$

is the first integral of the system (1).

III. Basic Results and Their Proving

In this part, the existence of rational first integrals (see Theorem 3.3), Darboux first integrals (see Theorem 3.5) and an analytic first integral (Theorem 3.8) are the main results of system (1) are described. Moreover, some other results relative to this topic are studied in this work such as a polynomial first integral, invariant algebraic surfaces, exponential factors and C^1 first integrals of system (1).

The following proposition is the first result in this work.

Proposition 3.1. System (1) has no polynomial first integrals.

Proof. Let $H = \sum_{i=1}^{n} H_i(u, v, w)$ be a polynomial first integral of the system (1), where each $\mathrm{H_i}$ is a homogeneous polynomial in its u, v and w variables of degree i. by definition we have

$$v\ \frac{\partial H}{\partial u} + w\ \frac{\partial H}{\partial v} + \left(a_1\ + a_2\ v +\ a_3\ w - \frac{u^2}{2}\right)\frac{\partial H}{\partial w} = 0 \qquad (8)$$

Taking terms of degree $\mathrm{n} + 1$, we obtain

$$\left(-\frac{\mathrm{u}^2}{2}\right)\frac{\partial \mathrm{H_n}(u,v,w)}{\partial \mathrm{w}} = 0, \qquad (9)$$

that is

$$H_n(u, v, w) = F_1(u, v),$$

where $\mathrm{H_n}$ is a polynomial of degree n. Additionally, the terms of degree n in equation (8), we get

$$v\left(\frac{\partial}{\partial u}F_1\right) + w\left(\frac{\partial}{\partial v}F_1\right) + (a_2\ v +\ a_3\ w\ )\left(\frac{\partial}{\partial w}F_1\right) - \frac{u^2}{2}\left(\frac{\partial}{\partial w}H_{n-1}\right) = 0,$$

this gives

$$H_{n-1} = \frac{2\,wv\ \left(\frac{\partial}{\partial u}F_1(u,v)\right) + z^2\left(\frac{\partial}{\partial v}F_1(u,v)\right)}{u^2} +\ G_1(u, v). \qquad (10)$$

In [13], The process of altering variables is carried out by using the weight change.

Let $u = \mu_1\ U,\ v = V,\ w = W$ and $t_1 = \mu_1\ T_1$, (11)

with $\mu_1 \in \mathbb{R}^+$. Then, system (8) becomes

$$\begin{aligned} U' &= V \\ V' &= \mu_1\ W \\ W' &= a_1\ \mu_1 + a_2\ \mu_1\ V + a_3\ \mu_1\ W - \frac{{\mu_1}^3 U^2}{2}, \end{aligned} \qquad (12)$$

where the dots denote the derivative of the variables U, V and W with respect to T_1. Set $F(U,V,W) = \mu_1^n f(\mu_1 U, V, W) = \sum_{j=0}^{n} \mu^j F_j(U,V,W)$, where F_j is the weight homogeneous part with weight degree $n-j$ of F, and n is the weight degree of F with weight exponent $s = (1,0,0)$. And $K(U,V,W) = k_0$. Then, by invariant algebraic surfaces, we have

$$V \sum_{j=0}^{n} \mu_1^j \frac{\partial F_j(U,V,W)}{\partial U} + \mu_1 W \sum_{j=0}^{n} \mu_1^j \frac{\partial F_j(U,V,W)}{\partial V} + (a_1 \mu_1 + a_2 \mu_1 V + a_3 \mu_1 W - \frac{1}{2}\mu_1^3 U^2) \sum_{j=0}^{n} \mu_1^j \frac{\partial F_j(U,V,W)}{\partial W} = 0 \qquad (13)$$

We compute the terms that include μ_1^0 to obtain's

$$\frac{\partial F_0}{\partial U} V = 0,$$

that is

$$F_0 = F_0(V,W),$$

(14)

then from (9), we have

$$H_n(u,v,w) = H_n(v),$$

also, from (10), H_{n-1} becomes

$$H_{n-1}(u,v,w) = \frac{w^2\left(\frac{\partial}{\partial v} H_n(v)\right)}{u^2} + G_1(u,v)$$

(15)

Given that $H_{n-1}(u,v,w)$ is a degree $n-1$ polynomial, it must be

$\frac{\partial}{\partial v} H_n(v) = 0$, then $H_n(v) = K$,

where K is an arbitrary constant. Since H_n is a homogeneous polynomial of degree n, then it must be $H_n = 0$.

Hence system (8) has no polynomial first integrals. Then there is no polynomial first integral of system (1).

■

Proposition 3.2. System (1) does not have invariant algebraic surfaces with non-zero cofactors.

Proof. Suppose that $f = \sum_{i=1}^{n} f_i(u,v,w)$ is an invariant algebraic surface of system (1) with the cofactor $K = k_0 + k_1 u + k_2 v + k_3 w$, where $k_i \in \mathbb{R}$ for i $= 0, \dots, 3$, and each f_i is a homogeneous polynomial in its variables of degree i. Assume that $f_n \neq 0$ for $n > 1$, then by definition of invariant algebraic surface, we obtain

$$v \frac{\partial f}{\partial u} + w \frac{\partial f}{\partial v} + \left(a_1 + a_2 v + a_3 w - \frac{u^2}{2}\right) \frac{\partial f}{\partial w} = K f$$

(16)

We first compute the terms of degree $n+1$, to obtain

$$-\frac{u^2}{2}\left(\frac{\partial f_n(u,v,w)}{\partial w}\right) = (k_1 u + k_2 v + k_3 w) f_n(u,v,w),$$

(17)

this gives

$$f_n(u,v,w) = G_1(u,v)\, e^{A(u,v)},$$

(18)

where

$$A(u,v) = \frac{(2k_1 u + 2k_2 v + k_3 w)\, w}{u^2}.$$

Since $f_n(U,V,W)$ is a polynomial function of degree n, this implies that $k_1 = k_2 = k_3 = 0$. Then equation (18) becomes

$$f_n(u,v,w) = G_1(u,v),$$

where G_1 a polynomial of degree n is expressed in terms of the variables u and v. Furthermore, calculating the terms with a degree of n in the equation (10), we obtain

$$v\left(\frac{\partial}{\partial u} f_n\right) + w\left(\frac{\partial}{\partial v} f_n\right) + (a_2 v + a_3 w)\left(\frac{\partial}{\partial w} f_n\right) - \frac{1}{u^2}\left(\frac{\partial f_n}{\partial w}\right) = k_0 f_n,$$

this gives

$$f_{n-1} = \frac{2 v w \left(\frac{\partial}{\partial u} G_1(U,V)\right) + w^2 \left(\frac{\partial}{\partial v} G_1(U,V)\right)}{u^2} - \frac{2 k_0 w G_1(U,V)}{u^2} + G_2(U,V).$$

Since f_{n-1} is a polynomial then

$$\frac{2 v w \left(\frac{\partial}{\partial u} G_1(U,V)\right) + w^2 \left(\frac{\partial}{\partial v} G_1(U,V)\right)}{u^2} - \frac{2 k_0 w G_1(U,V)}{u^2} = 0.$$

(19)

The solution of equation (19), is

$$G_1(u,v) = G_2(-u w + v^2)\, e^{\frac{2 k_0 v}{w}},$$

since, G_1 is the polynomial of n-th degree then it must be $k_0 = 0$,

This implies that system (1) has no invariant algebraic surfaces with non-zero cofactors. ■

Theorem 3.3. System (1) has no rational first integrals.

Proof. From Proposition 3.2, system (1) has no Darboux polynomials. Then by Proposition 2.3, system (1) has no proper rational first integral. ■

We proved that in Proposition 3.2, system (1) does not have invariant algebraic surfaces. So, by Proposition 2.4, an exponential function must be in the following

$$E = e^{g(u,v,w)},$$

for the more information see the [12].

Proposition 3.4. System (1) has two exponential factors e^u and e^v with cofactors v and w, respectively.

Proof. Let $E = e^{g(u,v,w)}$, $g(u,v,w) = \sum_{k=0}^{n} g_k(u,v,w)$ be an exponential factor with non-zero cofactor $L = L_0 + L_1 u + L_2 v + L_3 w$, where each g_k is a homogeneous polynomial in its variables of degree k. Then, we have

$$v\left(\frac{\partial}{\partial u} e^{g(u,v,w)}\right) + w\left(\frac{\partial}{\partial v} e^{g(u,v,w)}\right) + \left(a_1 + a_2 v + a_3 w - \frac{u^2}{2}\right)\left(\frac{\partial}{\partial w} e^{g(u,v,w)}\right) = L\, e^{g(u,v,w)}. \qquad (20)$$

Simplifying

$$v\left(\frac{\partial}{\partial u} g(u,v,w)\right) + w\left(\frac{\partial}{\partial v} g(u,v,w)\right) + \left(a_1 + a_2 v + a_3 w - \frac{u^2}{2}\right)\left(\frac{\partial}{\partial w} g(u,v,w)\right) = L. \qquad (21)$$

We first assume that n is greatest than one, taking the terms of degree $n+1$ in equation (21), we obtain

$$-\frac{u^2}{2}\left(\frac{\partial}{\partial w} g_n(u,v,w)\right) = 0,$$

(22)

that is

$$g_n(u,v,w) = G_1(u,v),$$

where g_n is a polynomial of degree n. Also, computing the terms of degree n in equation (21), we obtain

$$v\left(\frac{\partial}{\partial u} G_1(u,v)\right) + w\left(\frac{\partial}{\partial v} G_1(u,v)\right) + (a_2 v +$$

$a_3\, w\,)\left(\frac{\partial}{\partial w} G_1(u,v)\right) - \frac{u^2}{2}\left(\frac{\partial}{\partial w} g_{n-1}(u,v,w)\right) = 0,$
this gives

$$g_{n-1} = \frac{2\, v\, w\left(\frac{\partial}{\partial u} G_1(u,v)\right) + w^2\left(\frac{\partial}{\partial v} G_1(u,v)\right)}{u^2} + G_2(u,v), \quad (23)$$

The technique of weight modification of variables is used as explained in [13].
Let $u = \mu_1\ U,\ v = V,\ w = W$ and $t_1 = \mu_1\, T_1$, with $\mu_1 \in \mathbb{R}^+$. Then, system (1) becomes

$$U' = V\,, \quad V' = \mu_1\, W\,, \quad (24) \quad W' = a_1\,\mu_1 + a_2\,\mu_1\, V + a_3\,\mu_1\, W - \frac{1}{2}\,\mu_1{}^3 U^2.$$

Set $F(U,V,W) = \mu_1{}^n\, f(\mu_1\, U, V, W) = \sum_{j=0}^{n} \mu_1{}^j\, F_j(U,V,W)$, where F_j is the weight homogeneous part with weight degree $n-j$ of F and n is the weight degree of F with weight exponent $s = (1,0,0)$ and $L = \mu_1\, L(\mu_1\, U + V + W) = L_1\,\mu_1{}^2\, U + L_2\,\mu_1\, V + L_3\,\mu_1\, W$.
Then, by definition of exponential factor, we have

$$V \sum_{j=0}^{n} \mu_1{}^j\, \frac{\partial F_j(U,V,W)}{\partial U} + \mu_1\, W \sum_{j=0}^{n} \mu_1{}^j\, \frac{\partial F_j(U,V,W)}{\partial V} + (\,a_1\,\mu_1 + a_2\,\mu_1\, V + a_3\,\mu_1\, W - \frac{1}{2}\,\mu_1{}^3 U^2) \sum_{j=0}^{n} \mu_1{}^j\, \frac{\partial F_j(U,V,W)}{\partial W} = 0. \quad (25)$$

We take the terms which contain $\mu_1{}^0$ to obtain

$$\frac{\partial F_0(U,V,W)}{\partial U} V = 0,$$

that is

$$F_0\big((U,V,W)\big) = F_0(V,W), \quad (26)$$

then from (16), we have
$g_n(u,v,w) = g_n(v)$,
also, from (23), g_{n-1} becomes

$$g_{n-1}(u,v,w) = \frac{w^2\left(\frac{\partial}{\partial v} g_n(v)\right)}{u^2} + G_1(u,v), \quad (27)$$

Since $g_{n-1}(u,v,w)$ is a polynomial of degree $n-1$, then it must be
$\frac{\partial}{\partial v} g_n(v) = 0$, then $g_n(v) = K$,
where K is an arbitrary constant. Since g_n is a homogeneous polynomial of degree n, Then, for $n > 1$ we obtain $g = 0$.
Now, let $g = c_1 u + c_2\, v + c_3\, w$. Then, by equation (21), we have

$$v\, c_1 + w\, c_2 + \left(a_1 + a_2\ V + a_3\ W - \frac{1}{2}\, U^2\right) c_3 = L_1\, u + L_2\, v + L_3\, w.$$

Comparing the coefficient, we obtain $c_3 = L_1 = 0,\ c_2 = L_3$ and $c_1 = L_2$.
That is

$$g(u,v,w) = L_2\, u + L_3\, v.$$

This implies that $e^{L_2\, u + L_3\, v}$ is the exponential factor with cofactor $L_2\, v + L_3\, w$. Hence, the only two independent exponential factors of system (1) are e^u and e^v with cofactors v and w, respectively. ■

Theorem 3.5. System (1) has not Darboux first integrals.
Proof. Considering that the unique exponential factors e^u and e^v, have cofactors v and w, respectively. Next, using Darboux Theorem 2.5, we obtain

$$\beta_1(v) + \beta_2(w) = 0, \quad (28)$$

The constants $\beta_1,\ \beta_2 \in \mathbb{R}$ are non-zero. There is not a non-trivial solution to the above equation. Consequently, there is no Darboux first integral in system (1). ■

Theorem 3.6. If $a_3 < 0, \sqrt{2\, a_1} > 0$ and $a_3\, a_2 - \sqrt{2\, a_1} > 0$ then system (1) has no a global C^1 first integral.
Proof. Since $\left(\sqrt{2\, a_1}, 0,0\right)$ is the equilibrium point of system (1), then the Jacobian matrix of system (1), at $\left(\sqrt{2\, a_1}, 0,0\right)$ is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{2\, a_1} & a_2 & a_3 \end{bmatrix}.$$

The characteristic equation of the above matrix is

$$J(\gamma) = \gamma^3 - a_3\,\gamma^2 - a_2\,\gamma + \sqrt{2\, a_1} = 0.$$

The eigenvalues are

$$\gamma_1 = \frac{1}{6}\, A^{\frac{1}{3}} - \frac{6\, B}{A^{\frac{1}{3}}} + \frac{1}{3}\, a_3 \quad \text{and} \quad \gamma_{2,3} = -\frac{1}{12}\, A^{\frac{1}{3}} + \frac{3\, B}{A^{\frac{1}{3}}} + \frac{1}{3}\, a_3 \pm \frac{\sqrt{3}\, i}{2}\left(\frac{1}{6} A^{\frac{1}{3}} + \frac{6\, B}{A^{\frac{1}{3}}}\right),$$

where

$A = 36\, a_2\, a_3 - 108\sqrt{2\, a_1} + 8\, a_3{}^3 + 12\sqrt{-12\, a_2{}^3 - 3\, a_2{}^2\, a_3{}^2 - 54\sqrt{2\, a_1}\, a_2 a_3 + 162\, a_1 - 12\sqrt{2\, a_1}\, a_3{}^2}$ and
$B = -\frac{1}{3}\, a_2 - \frac{1}{9}\, a_3{}^2$.
The eigenvalues have non-zero negative real portions, according to Theorem 2.1, if and only if
$a_3 < 0, \sqrt{2\, a_1} > 0$ and $a_3\, a_2 - \sqrt{2\, a_1} > 0$.

Then, by Theorem 2.2 system (1) has no global C^1 first integrals in the neighborhood of $\left(\sqrt{2\, a_1}, 0,0\right)$. ■

Proposition 3.7. Suppose that $H_1 = -a_2\, u - a_3\, v + w$ is a polynomial first integral of the linear part of system (1).
Proof. By definition if H_1 is a polynomial first integral of the linear part of system (1), then it must be satisfied

$$\dot{u}\,\frac{\partial H_1}{\partial u} + \dot{v}\,\frac{\partial H_1}{\partial v} + \dot{w}\,\frac{\partial H_1}{\partial w} = 0$$

This gives that

$$v\,\frac{\partial(-a_2\, u - a_3\, v + w)}{\partial u} + w\,\frac{\partial(-a_2\, u - a_3\, v + w)}{\partial v} + (a_2\, v + a_3\, w)\,\frac{\partial(-a_2\, u - a_3\, v + w)}{\partial w} = 0$$

Then H_1 is a first integral of the linear part of (1). ∎

Theorem 3.8. System (1) has no analytic first integrals at $\left(\sqrt{2\,\mathrm{a}_1}, 0,0\right)$.

Proof. Let the analytic first integral of system (1) is $H = \sum_{i\geq 1} H_i(u,v,w)$, where each homogeneous polynomial of degree i, $\forall\ i \geq 1$, is represented by H_{i}. We will demonstrate via induction that

$H_i = 0,\ \forall\ i \geq 1.$

Note that, H is a first integral of (1), Consequently, by definition, we have

$$v\,\frac{\partial H}{\partial u} + w\,\frac{\partial H}{\partial v} + \left(a_1 + a_2\,v + a_3\,w - \frac{u^2}{2}\right)\frac{\partial H}{\partial w} = 0 \quad (29)$$

Using the degree 1 terms in equation (29), we obtain

$$v\,\frac{\partial H_1(u,v,w)}{\partial u} + w\,\frac{\partial H_1(u,v,w)}{\partial v} + (a_1 + a_2\,v + a_3\,w)\,\frac{\partial H_1(u,v,w)}{\partial w} = 0. \quad (30)$$

By Proposition 3.7 we get that H_1 is a polynomial first integral of the linear part of system (1), this gives $H_1 = \alpha_1(-a_2\,u - a_3\,v + w\,)^{l_1}$, for some $l_1 \in \mathbb{N}$ and $\alpha_1 \in \mathbb{R}$. Since H_1 is the homogeneous polynomial of degree 1, we obtain

$H_1 = \alpha_1(a_2\,u - a_3\,v + w).$

Taking the terms of degree 2 in equation (29),

$$v\,\frac{\partial H_2(u,v,w)}{\partial u} + w\,\frac{\partial H_2(u,v,w)}{\partial v} + (a_2\,v + a_3\,w\,)\,\frac{\partial H_2(u,v,w)}{\partial w} - \frac{u^2}{2}\frac{\partial H_1(u,v,w)}{\partial w} = 0.$$

That is

$$v\,\frac{\partial H_2(u,v,w)}{\partial u} + w\,\frac{\partial H_2(u,v,w)}{\partial v} + (a_2\,v + a_3\,w\,)\,\frac{\partial H_2(u,v,w)}{\partial w} - \alpha_1\frac{u^2}{2} = 0. \quad (31)$$

By Proposition 3.7 equation (31) has a polynomial solution if $\alpha_1 = 0$, this implies that $H_1 = 0$. In this case $H_2 = \alpha_2(a_2\,u - a_3\,v + w)^{l_2}$, for some $l_2 \in \mathbb{N}$ and $\alpha_2 \in \mathbb{R}$. Note that H_2 is the homogeneous polynomial of the second degree, then

$H_2 = \alpha_2(a_2\,u - a_3\,v + w)^2.$

Taking the terms of degree 3 of equation (29), then

$$v\,\frac{\partial H_3(u,v,w)}{\partial u} + w\,\frac{\partial H_3(u,v,w)}{\partial v} + (a_2\,v + a_3\,w\,)\,\frac{\partial H_3(u,v,w)}{\partial w} - \frac{u^2}{2}\frac{\partial H_2(u,v,w)}{\partial w} = 0. \quad (32)$$

By Proposition 2.7 equation (32) has a polynomial solution if $\alpha_2 = 0$, this implies that $H_2 = 0$. Then $H_3 = \alpha_3(a_2\,u - a_3\,v + w)^{l_3}$, for some $\mathrm{l}_3 \in \mathbb{N}$ and $\alpha_3 \in \mathbb{R}$. Now, assume that $H_i = 0$ for $i = m$ with $2 \leq m \leq n-1$, and we will prove that $H_i = \alpha_i(a_2\,u - a_3\,v + w)^i = 0$, with $\alpha_i \in \mathbb{R}$ for $i = n$. By the induction hypothesis, taking the terms of degree $n+1$ in equation (29), we obtain

$$v\,\frac{\partial H_{n+1}(u,v,w)}{\partial u} + w\,\frac{\partial H_{n+1}(u,v,w)}{\partial v} + (a_2\,v + a_3\,w\,)\,\frac{\partial H_{n+1}(u,v,w)}{\partial w} - \frac{u^2}{2}\frac{\partial H_n(u,v,w)}{\partial w} = 0.$$

That is

$$v\,\frac{\partial}{\partial u}\,H_{n+1}(u,v,w) + w\,\frac{\partial}{\partial v}H_{n+1}(u,v,w) + (a_2\,v + a_3\,w\,)\,\frac{\partial}{\partial w}H_{n+1}(u,v,w) - \frac{u^2}{2}\alpha_n\,(-a_2\,u - a_3\,v + w)^{\,n-1} = 0, \quad (33)$$

By Proposition 3.7 equation (33) has a polynomial solution if $\alpha_n = 0$. Then by induction we obtain that $H_i = 0$ for all $i \geq 1$. Hence, system (1) has not analytic first integrals at the neighborhood of $\left(\sqrt{2\,a_1}, 0,0\right)$.

IV. Conclusion

In this paper, we proved that the Generalized Michelson system has no Darboux first integrals. Also, this system has no analytic first integrals at the neighborhood of the equilibrium point and we obtained that the system has no global C^1 first integrals for $a_3 < 0, \sqrt{2\,a_1} > 0$, and $a_3\,a_2 - \sqrt{2\,a_1} > 0$.

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