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A Novel hybridization of CG-techniques for Solving Unconstrained Optimization Problems

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ABSTRACT: Conjugate gradient methods are an extremely helpful way for handling large scale non-linear optimization issues. In this paper, based on the three famous Dai-yuan (DY), Liu–Storey (LS)and Conjugate-Descent (CD) conjugate gradient methods, a new hybrid CG method is proposed. Under strong wolf line search, we prove the sufficient descent and global convergence features. The new formula is more efficient than other traditional conjugate gradient approaches, according to numerical results.

Keywords: Unconstrained optimization, hybrid conjugate gradient, global convergence, sufficient descent condition

1. INTRODUCTION

In the field of unconstrained optimization, we minimize an objective function that is dependent on real variables without imposing any limitations on the value of those variables. Thus, we take the following approach to the generic unconstrained optimization problem:

$$min\{f(x), x \in R^n\} \quad (1)$$

where f: $R^n \to R$ is continuously differentiable function, we defined the gradient as $g_k = \nabla f(x_k)$. The conjugate gradient methods are one of the best optimization techniques for solving large-scale problems.
Generally, for solving this problem, starting from an initial point $x_0 \in R^n$, a conjugate gradient algorithm generates a sequence of the points $\{x_k\}$

$$x_{k+1} = x_k + \alpha_k d_k \quad (2)$$

Where α_k is the stepsize selected by using line search and the directions d_k are generated as

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \qquad d_0 = -g_0 \quad (3)$$

β_k is known as the conjugate gradient coefficient, The different choices for this coefficient correspond to different conjugate gradient methods. Some of these methods, such as (HS) (Hestenes and Stiefel) [4], (FR) (Fletcher and Reeves) [5], (PRP) (Polak and Ribiere) [6], (DY) (Dai and Yaun) [7], (LS) (Liu and Storey)[8] and (CD) (Fletcher)[9].

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{y_k^T d_k} \qquad \beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|gk\|^2} \qquad \beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}$$

$$\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{y_k^T d_k} \qquad \beta_k^{LS} = \frac{g_{k+1}^T y_k}{-g_k^T d_k} \qquad \beta_k^{CD} = \frac{\|g_{k+1}\|^2}{-d_k^T g_k}$$

where $y_k = g_{k+1} - g_k$, $\|.\|$ denotes the Euclidean norm.
In this paper, we use strong Wolfe line search (SWC) which is determined by the sub sequent criteria:

$$f(x_k + \alpha_k d_k) - f(x_k) \le \rho \alpha_k g_k^T d_k \quad (4)$$

$$\sigma g_k^T d_k \le g_{k+1}^T d_k \le -\sigma g_k^T d_k \quad (5)$$

Where $0 < \rho < \sigma < 1$, A fundamental class of conjugate gradient techniques is the hybrid algorithm [10-12]. Moreover, because hybrid schemes capitalize in the factors that make them up, they outer form standard conjugate gradient approaches in term of computational performance and have more reliable convergence characteristics [13-15, 20].

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Because of this, academics were interested in hybrid or mix conjugate gradient approaches, for instance Sabrina et al. [1] proposed a new hybrid conjugate gradient method based on convex combination of HS and DY which is define as:

$$\beta_k^C = (1-\theta_k)\beta_k^{HS} + \theta_k \beta_k^{DY}, \quad 0 \le \theta_k \le 1$$

See more details [2-3]. In 2014, J.K. Liu and S.J. Li. [14], Suggested a hybrid CG method between LS and DY

$$\beta_k = (1-\theta_k)\beta_k^{LS} + \theta_k \beta_k^{DY}$$

Furthermore, in 2017, Snezana S. Djordjevic [19], proposed the following hybrid method:

$$\beta_k^{LSCD} = (1-\theta_k)\beta_k^{LS} + \theta_k \beta_k^{CD}, \quad 0 \le \theta_k \le 1$$

In this paper, we propose a new hybrid conjugate gradient method based on convex combination of DY, LS and CD conjugate gradient algorithms for solving unconstraint optimization problems.

Because of this, this paper is organized as follows: under section 2, we introduce the newly chosen hybrid conjugate gradient method and we obtained the parameters ψ and ϕ through a variety of techniques, and we demonstrate that under mild conditions, the chosen method with Wolfe line search produce directions that meet the sufficient descent criteria. The algorithm will be presented in section 3. Section 4 analyzes the new chosen method's descent condition and convergence features. We provided several numerical examples in section 5 to demonstrate the effectiveness of our approach, and section 6 concludes with a succinct analysis.

2. PROPOSED METHOD

In this paper, we propose a convex combination of DY, HS and HZ conjugate gradient algorithms. We use the following conjugate gradient parameter:

$$\beta_k^{hDYLSCD} = \psi_k \beta_k^{DY} + \phi_k \beta_k^{LS} + (1-\psi_k-\phi_k)\beta_k^{CD} \qquad (6)$$

Consequently, the direction d_k is given by:

$$d_{k+1} = \begin{cases} -g_{k+1} & if\ k=0 \\ -g_{k+1} + \beta_k^{hDYLSCD}\, d_k & if\ k \ge 1 \end{cases} \qquad (7)$$

The parameters ψ_k, ϕ_k satisfying $0 \le \psi_k, \phi_k \le 1$ which will be chosen in a certain manner that will be explained later. It ought to be mention that:

1. If $\psi_k = 1$ and $\phi_k = 0$ then $\beta_k^{hDYLSCD} = \beta_k^{DY}$
2. If $\psi_k = 0$ and $\phi_k = 1$ then $\beta_k^{hDYLSCD} = \beta_k^{LS}$
3. If $\psi_k = 0$ and $\phi_k = 0$ then $\beta_k^{hDYLSCD} = \beta_k^{CD}$
4. If $\psi_k = 0$ and $0 < \phi_k < 1$ then $\beta_k^{hDYLSCD} = \phi_k\beta_k^{LS} + (1-\phi_k)\beta_k^{CD}$ which is convex combination of β_k^{LS} and β_k^{CD}
5. If $\phi_k = 0$ and $0 < \psi_k < 1$ then $\beta_k^{hDYLSCD} = \psi_k\beta_k^{DY} + (1-\psi_k)\beta_k^{CD}$ which is convex combination of β_k^{DY} and β_k^{CD}
6. If $(1-\psi_k-\phi) = 0$ and $0 \le \psi_k, \phi_k \le 1$ then $\phi_k = 1-\psi$, hence $\beta_k^{hDYLSCD} = \psi_k\beta_k^{DY} + (1-\psi_k)\beta_k^{LS}$ which is convex combination of β_k^{DY} and β_k^{LS}
7. If $\psi_k \in (0,1), \phi_k \in (0,1)$ and $0 < \psi_k + \phi_k < 1$ then we get a new hybrid conjugate gradient method as a convex combination of DY, LS and CD.

From (6) and (7) it is evident that we receive:

$$d_{k+1} = \begin{cases} -g_{k+1} & if\ k=0 \\ -g_{k+1} + \psi_k \frac{\|g_{k+1}\|^2}{d_k^T y_k} d_k + \phi_k \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k + (1-\psi_k-\phi_k)\left(\frac{\|g_{k+1}\|^2}{-g_k^T d_k}\right) d_k, & if\ k \ge 1 \end{cases} \qquad (8)$$

We apply the conventional conjugacy requirement to choose the parameters ψ, ϕ that is $(d_{k+1}^T y_k = 0)$. Hence, we have

the following lemma:

Lemma1: If the condition $d_{k+1}^T y_k = 0$ is satisfied at each iteration, we get:

$$\phi_k = \frac{g_{k+1}^T y_k d_k^T g_k - \|g_{k+1}\|^2 g_k^T d_k + \|g_{k+1}\|^2 g_{k+1}^T d_k (1-\psi_k)}{g_{k+1}^T g_k d_k^T y_k} \qquad 0 < \psi_k < 1$$

Proof:

from (8) we have:

$$d_{k+1} = -g_{k+1} + \psi_k \frac{\|g_{k+1}\|^2}{d_k^T y_k} d_k + \phi_k \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k + (1-\psi_k-\phi_k)\left(\frac{\|g_{k+1}\|^2}{-g_k^T d_k}\right) d_k$$

Multiply both sides by y_k we get:

$$d_{k+1}^T y_k = -g_{k+1}^T y_k + \psi_k \frac{\|g_{k+1}\|^2}{d_k^T y_k} d_k^T y_k + \phi_k \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k^T y_k + (1 - \psi_k - \phi_k)\left(\frac{\|g_{k+1}\|^2}{-g_k^T d_k}\right) d_k^T y_k$$

If $d_{k+1}^T y_k = 0$

$$0 = -g_{k+1}^T y_k + \psi_k \frac{\|g_{k+1}\|^2}{d_k^T y_k} d_k^T y_k + \phi_k \frac{g_{k+1}^T y_k}{-d_k^T g_k} d_k^T y_k + (1 - \psi_k - \phi_k)\left(\frac{\|g_{k+1}\|^2}{-g_k^T d_k}\right) d_k^T y_k$$

after some algebraic computations, we have:

$$\phi_k = \frac{g_{k+1}^T y_k d_k^T g_k - \|g_{k+1}\|^2 g_k^T d_k + \|g_{k+1}\|^2 g_{k+1}^T d_k (1-\psi_k)}{g_{k+1}^T g_k d_k^T y_k} \qquad 0 < \psi_k < 1 \qquad (9)$$

The parameter ϕ_k can be outside [0,1] then:

- If $\phi_k < 0$ then we set $\phi_k = 0$
- If $\phi_k > 1$ then we set $\phi_k = 1$
- If $\phi_k + \psi_k \geq 1$ then we set $\phi_k + \psi_k = 1$

2.2 Algorithm (hDYLSCD)

Step 1: Initialization: Given $x_0 \in R^n$ and the parameters $0 < \rho < \sigma < 1$. compute $f(x_0)$, $g_0 = \nabla f(x_0)$. Consider $d_0 = -g_0$, set the initial guess: $\psi_k = 0.5$

Step 2: If $\|g_k\| \leq 10^{-6}$, then stop. Else go to next

Step 3: Compute the step size α_k by using strong Wolfe condition (4) and (5)

Step 4: Generate $x_{k+1} = x_k + \alpha_k d_k$. Compute $f(x_{k+1})$, $g_{k+1} = \nabla f(x_{k+1})$ and $y_k = g_{k+1} - g_k$

Step 5: Compute ϕ_k as in equation (9)

Step 6: Calculate $\beta_k^{hDYLSCD}$ by equation (6)

Step 7: Search direction: $d = -g_k + \beta_k^{hDYLSCD} d_k$, If the restart criterion of Powell[17]

$$|g_{k+1}^T g_k| \geq 0.2\|g_{k+1}\|^2$$

is satisfied, then restart, i.e. set $d_{k+1} = -g_{k+1}$ otherwise define $d_{k+1} = d$

Step 8: Put $k = k + 1$ and continue with **Step 2**.

3.THE SUFFICIENT DESCENT CONDITION AND CONVERGENCE

To show that the new method satisfies the sufficient descent condition, we need the following assumptions:

Assumption 1. The level set T= $\{x \in R^n: f(x) \leq f(x_0)\}$ is bounded, i.e. there is a constant B > 0 such that

$$\|x\| \leq B \text{ for all } x \in T \qquad (10)$$

Assumption 2. In a neighborhood N of T, f is continuously differentiable and its gradient is Lipschitz continuous, i.e. ∃ L ≥ 0 such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \text{ for all } x, y \in N \qquad (11)$$

According to the assumption 1 and 2 on $f(x)$, there is a constant Ʈ ≥ 0 such that

$$\|\nabla f(x)\| \leq \text{Ʈ} \text{ for all } x \in T$$

The search direction determined by the novel approach meets sufficient descent criterion, as demonstrated by the following theorem:

Theorem1: Let $\{g_k\}$ and $\{d_k\}$ generated by the new method, then d_k satisfies the sufficient descent condition:

$$g_k^T d_k \leq -c\|g_k\|^2 \quad \text{for all } k \geq 0, c > 0 \qquad (13)$$

Proof. By using mathematical induction, we show that the search direction d_k shall satisfy the sufficient descent condition when $k = 0$ that is $d_0 = -g_0$ hence $g_0^T d_0 = -\|g_0\|^2$ then the condition is hold when $k = 0$

Now if $k \geq 1$:

$$d_{k+1} = -g_{k+1} + \beta_k^{hDYLSCD} d_k$$

$$= -g_{k+1} + (\psi_k \beta_k^{DY} + \phi_k \beta_k^{LS} + (1 - \psi_k - \phi_k)\beta_k^{CD}) d_k$$

$$= -\psi_k g_{k+1} + \phi_k g_{k+1} + (1 - \psi_k - \phi_k)) + (\psi_k \beta_k^{DY} + \phi_k \beta_k^{LS} + (1 - \psi_k - \phi_k)\beta_k^{CD}) d_k$$

After some algebra:

$$d_{k+1} = \psi_k d_{k+1}^{DY} + \phi_k d_{k+1}^{LS} + +(1 - \psi_k - \phi_k) d_{k+1}^{CD} \qquad (14)$$

Multiply both sides by g_{k+1}^T we obtain:

$$g_{k+1}^T d_{k+1} = g_{k+1}^T d_{k+1}^{DY} + \phi_k g_{k+1}^T d_{k+1}^{LS} + +(1 - \psi_k - \phi_k) g_{k+1}^T d_{k+1}^{CD} \qquad (15)$$

We are going to prove the seven cases:

Case (1): if $\psi_k = 1$, $\phi_k = 0$

$$g_{k+1}^T d_{k+1} = g_{k+1}^T d_{k+1}^{DY}$$

We have to prove the sufficient descent condition for DY

$$d_{k+1} = -g_{k+1} + \beta_k^{DY} d_k$$

we want to show that

$$g_{k+1}^T d_{k+1} \leq -c_1 \|g_{k+1}\|^2$$

$$g_{k+1}^T d_{k+1} = g_{k+1}^T (-g_{k+1} + \beta_k^{DY} d_k)$$

$$g_{k+1}^T d_{k+1} \leq \|g_{k+1}\|^2 + \beta_k^{DY} g_{k+1}^T d_k$$

$$g_{k+1}^T d_{k+1} \leq \|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{y_k^T d_k} g_{k+1}^T d_k$$

$$d_k^T y_k = d_k^T g_{k+1} - d_k^T g_k \geq -(1-\sigma) d_k^T g_k \geq 0 \tag{16}$$

Then it is follows from (16)

$$g_{k+1}^T d_{k+1} \leq \|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{y_k^T d_k} |g_{k+1}^T d_k| \leq -\frac{1-2\sigma}{1-\sigma} \|g_{k+1}\|^2$$

$$g_{k+1}^T d_{k+1}^{DY} = -c_1 \|g_{k+1}\|^2,\ c_1 = \frac{1-2\sigma}{1-\sigma} \tag{17}$$

Case (2): if $\psi_k = 0$, $\phi_k = 1$

$$g_{k+1}^T d_{k+1} = g_{k+1}^T d_{k+1}^{LS}$$

We want to prove that the sufficient descent condition for LS satisfies, i.e.:

$$g_{k+1}^T d_{k+1} \leq -c_2 \|g_{k+1}\|^2$$

$$d_{k+1} = -g_{k+1} + \beta_k^{LS} d_k$$

Multiplying both sides by g_{k+1}^T

$$g_{k+1}^T d_{k+1} = g_{k+1}^T (-g_{k+1} + \beta_k^{LS} d_k)$$

$$= -\|g_{k+1}\|^2 + \beta_k^{LS} g_{k+1}^T d_k$$

By substituting β_k^{LS}

$$g_{k+1}^T d_{k+1}^{LS} = -\|g_{k+1}\|^2 + \frac{g_{k+1}^T y_k}{-g_k^T d_k} g_{k+1}^T d_k$$

In addition, we have

$$g_{k+1}^T d_{k+1}^{LS} = -\|g_{k+1}\|^2 + \frac{|g_{k+1}^T y_k||g_{k+1}^T d_k|}{|g_k^T d_k|} \leq -(1-1.2\sigma)\|g_{k+1}\|^2$$

Therefore

$$g_{k+1}^T d_{k+1}^{LS} \leq -c_2 \|g_{k+1}\|^2 \text{ , with } c_2 = (1-1.2\sigma) > 0 \tag{18}$$

Case (3): if $\psi_k = 0$, $\phi_k = 0$

$$g_{k+1}^T d_{k+1} = g_{k+1}^T d_{k+1}^{CD}$$

We want to prove that the sufficient descent condition for CD satisfies, i.e.:

$$g_{k+1}^T d_{k+1} \leq -c_3 \|g_{k+1}\|^2$$

$$d_{k+1} = -g_{k+1} + \beta_k^{CD} d_k$$

Multiplying both sides by g_{k+1}^T :

$$g_{k+1}^T d_{k+1} = g_{k+1}^T (-g_{k+1} + \beta_k^{CD} d_k)$$

$$= -\|g_{k+1}\|^2 + \beta_k^{CD} g_{k+1}^T d_k$$

By substituting β_k^{CD}

$$g_{k+1}^T d_{k+1}^{CD} = -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{-g_k^T d_k} g_{k+1}^T d_k$$

$$g_{k+1}^T d_{k+1}^{CD} = -\|g_{k+1}\|^2 (1 - \frac{g_{k+1}^T d_k}{-g_k^T d_k})$$

$$g_{k+1}^T d_{k+1}^{CD} = -\|g_{k+1}\|^2 (\frac{-g_k^T d_k - g_{k+1}^T d_k}{-g_k^T d_k})$$

Using the strong Wolfe line search, now it holds

$$\frac{-g_k^T d_k - g_{k+1}^T d_k}{-g_k^T d_k} \geq \frac{(\sigma - 1) g_k^T d_k}{-g_k^T d_k} = 1 - \sigma > 0$$

Now we have

$$g_{k+1}^T d_{k+1}^{CD} = -(1 - \sigma)\|g_{k+1}\|^2$$

$$g_{k+1}^T d_{k+1}^{CD} = -c_3 \|g_{k+1}\|^2, \text{ with } c_3 = 1 - \sigma > 0 \qquad (19)$$

Case (4): if $\psi_k = 0 \; and \; 0 < \phi_k < 1$

$$\beta_k^{hDYLSCD} = \phi_k \beta_k^{LS} + (1 - \phi_k)\, \beta_k^{CD} \text{ where } \phi_k \in [0,1]$$

Now suppose that $0 < \phi_k < 1$, i.e., $0 < a_1 < \phi_k < a_2 < 1$.
now we conclude

$$g_{k+1}^T d_{k+1} \leq a_1 g_{k+1}^T d_{k+1}^{CD} + (1 - a_2) g_{k+1}^T d_{k+1}^{LS}$$

By using (18) and (19), we get

$$c_4 = a_1 c_3 + (1 - a_2) c_2$$

then we finally get

$$g_{k+1}^T d_{k+1}^{CDLS} \leq -c_4 \|g_{k+1}\|^2$$

Case (5): if $\phi_k = 0 \; and \; 0 < \psi_k < 1$ then:

$$\beta_k^{hDYLSCD} = \psi_k \beta_k^{DY} + (1 - \psi_k)\, \beta_k^{CD}, \text{ where } \psi_k \in [0,1]$$

We are going to prove the sufficient descent condition for the convex combination of DY and CD
Now suppose that $0 < \psi_k < 1$, i.e., $0 < b_1 < \psi_k < b_2 < 1$.
now we conclude

$$g_{k+1}^T d_{k+1} \leq b_1 g_{k+1}^T d_{k+1}^{DY} + (1 - b_2) g_{k+1}^T d_{k+1}^{CD}$$

By using (17) and (19), we get

$$c_5 = b_1 c_1 + (1 - b_2) c_3$$

then we finally get

$$g_{k+1}^T d_{k+1}^{DYCD} \leq -c_5 \|g_{k+1}\|^2$$

Case (6): If $(1 - \psi_k - \phi_k) = 0$ when $0 < \psi_k, \phi_k < 1$ then $\phi_k = 1 - \psi_k$

$$\beta_k^{hDYLSCD} = \beta_k^{DYLS} = \psi_k \beta_k^{DY} + (1 - \psi_k)\, \beta_k^{LS} \qquad (20)$$

where the second inequality follows from (5), the Triangular inequality and step (7) for Powell restart. Finally, when $\psi_k \in (0,1)$, the parameter $\beta_k^{hDYLSCD}$ is computed by (20). Then it follows from (3) that

$$g_{k+1}^T d_{k+1}^{DYLS} \leq -\|g_{k+1}\|^2 + |\beta_k^{DY}| \cdot |g_{k+1}^T d_k| + |g_{k+1}^T d_k| \cdot |\beta_k^{LS}|$$

$$\leq -\|g_{k+1}\|^2 + \sigma|\beta_k^{DY}| \cdot |g_k^T d_k| + \sigma|g_k^T d_k| \cdot |\beta_k^{LS}|$$

$$= -\|g_{k+1}\|^2 + \sigma|\beta_k^{DY}| \cdot |g_{k+1}^T d_{k+1}| + \sigma|g_{k+1}^T y_k| \cdot |\beta_k^{LS}|$$

$$\leq -\|g_{k+1}\|^2 + \sigma\|g_{k+1}\|^2 + \sigma|g_{k+1}^T g_k| + \sigma|g_{k+1}^T d_{k+1}|$$

$$g_{k+1}^T d_{k+1}^{DYLS} \leq -\|g_{k+1}\|^2 + 1.2\sigma\|g_{k+1}\|^2 + \sigma|g_{k+1}^T d_{k+1}|$$

From the above inequality, we have

$$g_{k+1}^T d_{k+1}^{DYLS} - \sigma|g_{k+1}^T d_{k+1}| \leq -(1 - 1.2\sigma)\|g_{k+1}\|^2$$

Since $\sigma < 0.5$, the symbol of the left side of the above inequality is consistent with the symbol of $d_{k+1}^T g_{k+1}$. So there always exists a constant $u > 0$ such that

$$g_{k+1}^T d_{k+1}^{DYLS} - \sigma|g_{k+1}^T d_{k+1}| = u d_{k+1}^T g_{k+1}$$

Then we have

$$g_{k+1}^T d_{k+1}^{DYLS} \leq -c_6 \|g_{k+1}\|^2$$

Where $c_6 = \frac{(1-1.2\sigma)}{u}, u = 1 + \sigma \; or \; 1 - \sigma$, this inequality with (17) and (18) leads to case (6) holds for $k + 1$.■

Case (7): If $0 < \psi_k < 1$ and $0 < \phi_k < 1$ and $0 < \psi_k + \phi_k < 1$ then we get a new hybrid conjugate gradient method as a convex combination of DY, LS and CD

We have to prove the direction satisfies the sufficient descent condition at each iteration i.e.

$$g_{k+1}^T d_{k+1} \le -c_7 \| g_{k+1} \|^2 \text{ when } (0 < \psi_k < 1 \text{ and } 0 < \phi_k < 1)$$

We have

$$d_{k+1} = -g_{k+1} + \beta_k^{hDYLSCD} d_k \quad (21)$$

when $\beta_k^{hDYLSCD}$ is convex combination of the parameters of DY, LS and CD

$$\beta_k^{hDYLSCD} = \lambda_1 \beta_k^{DY} + \lambda_2 \beta_k^{LS} + \lambda_3 \beta_k^{CD} \quad (22)$$

When $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$

This ensures that $\beta_k^{hDYLSCD}$ is a weighted average of the individual conjugate gradient parameters inheriting properties from each of them. Since $\beta_k^{hDYLSCD}$ is convex combination of the parameters of β_k^{DY}, β_k^{LS} and β_k^{CD} and since each individual methods have well known descent properties and under standard condition (such as using line search that satisfies Wolfe conditions), each method satisfies sufficient descent condition

Now from (21) and (22) we get:

$$g_{k+1}^T d_{k+1} = -\| g_{k+1} \|^2 + \beta_k^{hDYLSCD} g_{k+1}^T d_k$$

$$g_{k+1}^T d_{k+1} \le -\| g_{k+1} \|^2 + (\lambda_1 \beta_k^{DY} + \lambda_2 \beta_k^{LS} + \lambda_3 \beta_k^{CD}) g_{k+1}^T d_k$$

we have:

$$g_{k+1}^T d_{k+1} \le \lambda_1(-c_1 \| g_{k+1} \|^2) + \lambda_2(-c_2 \| g_{k+1} \|^2) + \lambda_3(-c_3 \| g_{k+1} \|^2)$$

hence

$$g_{k+1}^T d_{k+1} \le -(\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3) \| g_{k+1} \|^2$$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 1$

Let $c_7 = \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3$ then

$$g_{k+1}^T d_{k+1} \le -c_7 \| g_{k+1} \|^2 \quad (23)$$

When $c_7 > 0$ and c_7 is a constant derived from the convex combination of the individual descent conditions.

4. CONVERGENCE ANALYSIS

The conjugate gradient method's global convergence is frequently demonstrated using the Zoutendijk criterion[18]. Furthermore, the Zoutendijk requirement is met by the new approach under the strong Wolfe condition, as demonstrated by the following lemma:

LEMMA 2: consider that Assumptions (1) and (2) hold and $x_{k+1} = x_k + \alpha_k d_k$ where d_k is the descent direction and α_k is the step size determined by strong Wolfe conditions. Then, the Zoutendijk condition

$$\sum_{k \ge 0} \frac{(g_{k+1}^T d_{k+1})^2}{\| d_{k+1} \|^2} < \infty \quad (24)$$

holds.

The novel method's global convergence is provided by the following theorem:

THEOREM 2: suppose the assumption (1) and (2) hold and $\{x_k\}$be generated by the new algorithm, then

$$\lim_{k \to \infty} \inf \| g_k \| = 0 \quad (25)$$

Proof: we use contradiction for proving this theorem. Suppose $\lim_{k \to \infty} \inf \| g_k \| = 0$ is not true, then there exist C > 0 s.t. $\| g_k \| \ge C$ for all $k \ge 1$ from theorem (1) we have:

$$g_k^T d_k \le -K \| g_k \|^2 \text{ for all } K > 0$$

Since we have from Lipschitz rule:

$$\| y_k \| = \| g_{k+1} - g_k \| \le L \| x_{k+1} - x_k \| \le LD$$

Where D = max $\{\| x - y \| : x, y \in N\}$ is the diameter of N

We have

$$d_{k+1} = -g_{k+1} + \beta_k^{hDYLSCD} d_k$$

$$\| d_{k+1} \| \le \| g_{k+1} \| + |\beta_k^{hDYLSCD}| \| d_k \|$$

And since we have

$$\beta_k^{hDYLSCD} = \psi_k \beta_k^{DY} + \phi_k \beta_k^{HS} + (1 - \psi_k - \phi_k) \beta_k^{HZ}$$

where $0 < \psi_k, \phi_k < 1$ and $0 < 1 - \psi_k - \phi_k < 1$

we get

$$|\beta_k^{hDYLSCD}| \le |\beta_k^{DY}| + |\beta_k^{LS}| + |\beta_k^{CD}|$$

$$= \frac{\| g_{k+1} \|^2}{y_k^T d_k} + \frac{g_{k+1}^T y_k}{-g_k^T d_k} + \frac{\| g_{k+1} \|^2}{-d_k^T g_k}$$

$$\le \frac{\| g_{k+1} \|^2}{\| y_k^T \| \| d_k \|} + \frac{\| g_{k+1}^T \| \| y_k \|}{\| g_k^T \| \| d_k \|} + \frac{\| g_{k+1} \|^2}{\| d_k^T \| \| g_k \|}$$

And since we have $g_k^T d_k \leq -K\|g_k\|^2$, $\|\nabla f(x)\| \leq Ʈ$ and $\|y_k\| \leq LD$

$$|\beta_k^{hDYLSCD}| \leq \frac{Ʈ^2}{KLD} + \frac{ƮLD}{KLD} + \frac{ƮLD}{KLD} = M$$

From (7) and since $\alpha_k \geq \lambda$(where $\lambda > 0, k \geq 0$) , then

$$\frac{1}{\alpha_k} \leq \frac{1}{\lambda}$$

Hence

$$\|d_{k+1}\| \leq \|g_{k+1}\| + |\beta_k^{hDYLSCD}|\|d_k\|$$
$$\leq \|g_{k+1}\| + \frac{|\beta_k^{hDYLSCD}|\|x_{k+1} - x_k\|}{\alpha_k} \leq Ʈ + \frac{MD}{\lambda} = W$$

Hence

$$\|d_{k+1}\| \leq W$$

Then

$$\sum_{k\geq 1} \frac{1}{\|d_k\|^2} = \infty, k \geq 0$$

Since we have from Zoutendijk condition

$$\sum_{k\geq 0} \frac{(g_{k+1}^T d_{k+1})^2}{\|d_{k+1}\|^2} < \infty$$

and since

$$\|g_{k+1}\| \geq C$$

And

$$g_{k+1}^T d_{k+1} \leq -K\|g_{k+1}\|^2$$

$$k^2 c^4 \sum_{k\geq 0} \frac{1}{\|d_{k+1}\|^2} \leq \sum_{k\geq 0} \frac{k^2\|g_{k+1}\|^4}{\|d_{k+1}\|^2} \leq \infty$$

Which is contradiction with $\sum_{k\geq 0} \frac{1}{\|d_{k+1}\|^2} = \infty$

Hence $$\lim_{k\to\infty} inf\|g_{k+1}\| = 0$$

5. NUMERICAL RESULTS

In this section, we present numerical experiment results obtained by testing our new algorithm hDYLSCD with DY, LS and CD conjugate gradient algorithms on a set of 81 unconstrained optimization test problems. in which the problems 1-39 are taken from the CUTE library [21] and the others come from the unconstrained problem collections [22, 23]. The dimensions of the test problems vary from 500 to 500000. For the sake of fairness, all the comparison methods use the strong Wolfe line search to compute the step-length α_k, and the relevant parameters are set to $\rho = 0.0001$ and $\sigma = 0.9$ and the hybridization parameter $\psi_k = 0.5$. For our methods, we adopt the strategy described in [40] to compute the initial step length. The termination criterion is (1) $\|g_k\|_\infty \leq 10^{-6}$ or (2) Itr > 2000, where “Itr” represents the number of iterations. When (2) does happen, we claim that the relevant algorithm is invalid for the corresponding test problem, and denote it by “F”. All codes are written in Matlab 2024b, and run on a Lenovo PC with 3.60 GHz CPU processor and 8 GB RAM memory as well as Windows 10 operation system. comparisons of these methods are given in the following context. Let f_i^{H1} and f_i^{H2} be the optimal value found by H1 and H2, for problem i=1,...81, respectively. We say that in the particular problem i the performance of H1 was better than the performance of H2 if

$$|f_i^{H1} - f_i^{H2}| < 10^{-3} \qquad (26)$$

and the number of iterations (NOI), or the number of function-gradient evaluations (NOF), or the CPU time of H1 methods is less than those of H2 methods, respectively. to obtain complete comparisons we used the profile of Dolan and Moré [17] to evaluate and compare the performance of the set of methods.

In this set of numerical experiments, we compare the performance of our new algorithm to the HS, DY and HZ conjugate gradient algorithms. Figures 1, 2 and 3 represent the performance profiles of the new method hDYLSCD *versus* DY, LS and CD based on the NOI, NOF and CPU time, respectively.

Table 1. Show that the compare the numerical results of the new algorithm (hDYLSCD) versus DY, LS and CD, and show that our new algorithm more effective and faster than DY, LS and CD.

Table 1. Numerical Results

Function/size	DY	LS	CD	hDYLSCD
	Ite/Tcpu/Grad.	Ite/Tcpu/Grad.	Ite/Tcpu/Grad.	Ite/Tcpu/Grad.
cosine/5000	11/0.189/4.68e-07	13/0.067/4.07e-07	10/0.049/6.23e-07	10/0.060/3.03e-07
cosine/50000	12/0.429/2.19e-07	12/0.404/4.86e-07	10/0.348/5.62e-07	11/0.402/8.04e-07
cosine/500000	NaN/NaN/NaN	245/57.812/9.73e-07	206/44.696/8.56e-07	776/184.20/9.22e-07
dixmaana/15000	9/0.533/5.77e-07	16/0.803/8.02e-08	9/0.465/8.59e-07	10/0.482/4.97e-07
dixmaana/150000	11/5.067/3.14e-07	15/6.154/7.54e-07	11/4.944/5.55e-07	10/4.675/1.43e-07
dixmaanb/15000	10/0.534/2.82e-07	13/0.589/6.83e-07	10/0.482/2.54e-07	9/0.440/8.51e-08
dixmaanb/150000	11/4.956/5.74e-07	23/10.043/6.27e-07	27/10.732/8.26e-07	10/4.338/9.83e-07
dixmaanc/15000	10/0.562/3.55e-07	17/0.786/6.59e-07	10/0.493/5.61e-07	11/0.492/1.50e-07
dixmaanc/150000	11/5.010/1.98e-07	14/5.880/1.69e-07	11/5.125/1.41e-07	10/4.775/1.47e-07
dixmaand/15000	9/0.556/3.27e-07	16/0.756/6.52e-07	10/0.495/6.94e-07	9/0.467/8.65e-07
dixmaand/150000	10/4.730/8.70e-07	25/10.206/5.14e-08	12/5.675/2.98e-07	11/5.231/1.13e-07
dixmaane/15000	534/15.877/9.88e-07	591/17.178/9.97e-07	610/18.089/9.97e-07	535/15.947/9.81e-07
dixmaane/150000	1393/281.0/9.91e-07	1488/306./9.94e-07	1460/267.0/9.97e-07	1381/293.5/9.94e-07
dixmaanf/15000	443/11.990/9.82e-07	447/12.139/9.77e-07	446/12.039/9.99e-07	455/12.529/9.91e-07
dixmaanf/150000	453/118.01/9.92e-07	NaN/NaN/NaN	613/150.17/9.96e-07	977/319.84/9.90e-07
dixmaang/15000	434/24.578/9.87e-07	405/23.265/9.93e-07	438/24.584/9.81e-07	443/24.472/9.65e-07
dixmaang/150000	812/2275.4/9.99e-07	790/170.38/9.92e-07	1037/209.4/1.00e-06	926/193.02/9.96e-07
dixmaanh/15000	449/12.731/9.99e-07	434/11.455/9.83e-07	NaN/NaN/NaN	437/11.749/9.83e-07
dixmaanh/150000	659/160.56/9.88e-07	NaN/NaN/NaN	996/201.30/9.98e-07	804/175.89/9.98e-07
dixmaani/15000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
dixmaanj/15000	1067/21.69/9.97e-07	1019/21.39/9.94e-07	1389/26.40/9.99e-07	1037/21.63/9.98e-07
dixmaanj/150000	1484/253.7/9.96e-07	1519/276.1/9.92e-07	1820/312.0/9.98e-07	1429/239.3/9.92e-07
dixmaank/15000	NaN/NaN/NaN	980/21.128/9.99e-07	884/19.816/9.99e-07	929/19.723/9.87e-07
dixmaank/150000	1603/259.0/9.91e-07	1426/311.1/9.95e-07	NaN/NaN/NaN	1290/224.7/1.00e-06
dixmaanl/15000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	825/18.451/9.98e-07
dixmaanl/150000	1111/199.3/9.97e-07	NaN/NaN/NaN	1206/213.6/9.96e-07	1137/204.4/1.00e-06
dixon3dq/500	1079/0.389/9.72e-07	1551/0.477/9.95e-07	1848/0.598/9.94e-07	1418/0.457/9.69e-07
dqdrtic/5000	35/0.058/4.28e-07	54/0.051/4.68e-07	30/0.029/6.29e-07	25/0.030/3.30e-07
dqdrtic/50000	NaN/NaN/NaN	47/0.211/5.46e-07	40/0.183/5.15e-07	27/0.142/7.21e-07
dqrtic/5000	48/0.641/8.47e-07	39/0.529/6.40e-07	49/0.688/8.36e-07	15/0.238/5.36e-07
dqrtic/50000	93/10.567/9.67e-07	78/7.840/4.02e-07	88/10.199/9.38e-07	19/2.574/1.00e-07
edensch/5000	NaN/NaN/NaN	35/0.508/2.98e-07	38/0.520/5.06e-07	27/0.397/8.99e-07
edensch/50000	NaN/NaN/NaN	42/5.625/7.25e-07	63/7.347/8.81e-07	38/5.309/7.12e-07
edensch/500000	67/83.755/9.79e-07	34/39.110/7.48e-07	NaN/NaN/NaN	40/53.203/9.82e-07
eg2/500	NaN/NaN/NaN	130/0.119/2.57e-07	NaN/NaN/NaN	NaN/NaN/NaN
fletchcr/5000	178/0.267/7.08e-07	83/0.098/6.83e-07	93/0.091/9.73e-07	40/0.056/8.91e-07
fletcher/50000	NaN/NaN/NaN	34/0.217/7.89e-07	77/0.522/6.28e-07	76/0.552/9.98e-07
fletchcr/500000	333/22.406/9.57e-07	71/4.882/3.27e-07	738/35.295/3.69e-07	102/8.045/4.55e-08
freuroth/500	NaN/NaN/NaN	851/0.483/5.08e-07	464/0.232/5.95e-07	NaN/NaN/NaN
genrose/5000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
himmelbg/50000	2/0.056/7.15e-295	2/0.028/3.21e-280	2/0.029/7.15e-295	2/0.029/7.43e-295
himmelbg/500000	3/0.279/0.00e+00	3/0.290/6.82e-70	3/0.274/0.00e+00	3/0.295/0.00e+00
liarwhd/5000	88/0.115/3.78e-07	24/0.022/4.72e-07	75/0.076/9.45e-07	61/0.060/2.69e-08
liarwhd/50000	616/2.272/8.53e-07	54/0.265/8.28e-07	317/1.260/8.63e-07	131/0.689/8.18e-07
penalty 1/500	29/0.088/2.88e-08	NaN/NaN/NaN	166/0.386/3.80e-07	NaN/NaN/NaN
penalty 1/5000	136/18.474/9.18e-07	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
quartc/5000	48/0.639/8.47e-07	39/0.507/6.40e-07	49/0.669/8.36e-07	15/0.229/5.36e-07
quartc/50000	93/10.560/9.67e-07	78/7.862/4.02e-07	88/10.201/9.38e-07	19/2.581/1.00e-07
quartc/500000	174/187.03/7.90e-07	150/149.75/8.46e-07	178/192.26/7.82e-07	76/81.55/8.69e-07
tridia/500	323/0.170/8.69e-07	406/0.149/9.74e-07	484/0.167/9.45e-07	391/0.140/9.65e-07
tridia/5000	1105/0.861/9.94e-07	1600/1.204/9.73e-07	1590/1.176/9.57e-07	1431/1.144/9.84e-07
woods/50000	NaN/NaN/NaN	148/0.517/9.40e-07	NaN/NaN/NaN	45/0.205/6.56e-07
woods/500000	NaN/NaN/NaN	199/7.607/9.97e-07	NaN/NaN/NaN	53/3.066/7.22e-07
bdexp/5000	2/0.044/3.35e-37	NaN/NaN/NaN	2/0.013/3.35e-37	2/0.013/3.51e-37
bdexp/50000	2/0.062/0.00e+00	2/0.061/0.00d+00	2/0.061/0.00e+00	2/0.062/0.00e+00

bdexp/500000	2/0.741/0.00e+00	2/0.872/3.52e-18	2/0.739/0.00e+00	2/0.759/0.00e+00
exdenschnf/5000	14/0.059/7.19e-07	11/0.013/3.89e-07	16/0.020/3.62e-08	17/0.025/1.33e-08
exdenschnf/50000	48/0.266/8.38e-07	26/0.174/2.97e-07	50/0.252/8.06e-07	18/0.130/4.29e-08
exdenschnf/500000	531/24.436/8.86e-07	26/1.836/6.74e-08	1519/72.52/9.53e-07	23/1.741/5.20e-08
exdenschnb/5000	11/0.042/2.60e-07	15/0.015/6.19e-08	10/0.008/9.80e-07	16/0.015/3.98e-07
exdenschnb/50000	276/0.845/9.59e-07	15/0.064/6.25e-07	15/0.069/4.05e-07	18/0.098/2.67e-07
exdenschnb/500000	13/0.679/9.00e-07	22/1.178/2.68e-08	12/0.676/7.44e-07	15/0.866/5.22e-09
genquartic/5000	865/0.648/9.80e-07	60/0.064/4.69e-07	32/0.033/9.21e-07	18/0.022/1.36e-07
genquartic/50000	NaN/NaN/NaN	468/2.156/5.20e-08	NaN/NaN/NaN	251/1.440/9.41e-07
genquartic/500000	16/1.026/6.44e-07	15/1.033/6.11e-07	16/1.031/5.53e-07	11/0.784/3.80e-07
biggsb1/500	551/0.213/8.75e-07	700/0.220/8.35e-07	904/0.291/9.76e-07	813/0.266/8.97e-07
sine/5000	NaN/NaN/NaN	NaN/NaN/NaN	939/3.622/5.34e-07	22/0.094/2.69e-07
sine/50000	266/7.916/9.77e-07	NaN/NaN/NaN	1391/44.47/7.12e-07	30/1.277/4.20e-07
sine/500000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	143/57.181/8.98e-07
fletcbv3/500	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
nonscomp/5000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
nonscomp/50000	NaN/NaN/NaN	79/0.386/5.87e-07	48/0.234/8.75e-07	56/0.271/8.71e-07
power 1/500	1853/0.580/9.96e-07	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
raydan 1/500	160/0.130/9.08e-07	159/0.093/8.91e-07	156/0.079/9.69e-07	160/0.088/9.08e-07
raydan1/5000	522/1.125/9.98e-07	540/1.144/9.75e-07	528/1.100/9.89e-07	535/1.092/9.88e-07
raydan2/5000	12/0.058/5.21e-07	72/0.302/1.16e-08	42/0.195/7.26e-08	27/0.118/1.16e-08
raydan 2/50000	13/0.446/1.05e-07	844/28.881/2.11e-07	19/0.681/7.84e-07	26/0.931/3.61e-07
raydan2/500000	14/4.607/8.24e-07	72/22.972/5.48e-07	22/7.654/7.15e-07	19/6.527/6.82e-08
diagonal1/500	986/0.960/9.62e-07	491/0.469/9.86e-07	719/0.678/9.71e-07	410/0.414/2.63e-07
diagonal1/5000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
diagonal2/5000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN
diagonal1/5000	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN	NaN/NaN/NaN

Figure 1. show that the performance profiles based on number of iterations, we can conclude hDYLSCD method is also better than DY, LS, DY methods

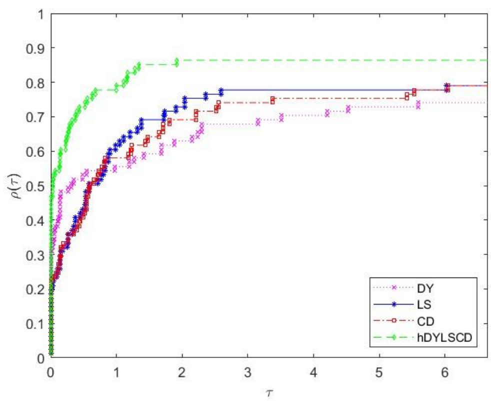

FIGURE 1. - Performance profiles using the iteration number

Figure 2. show that the performance profiles based on number of the function evaluation, we can conclude hDYLSCD method is also better than DY, LS, DY methods

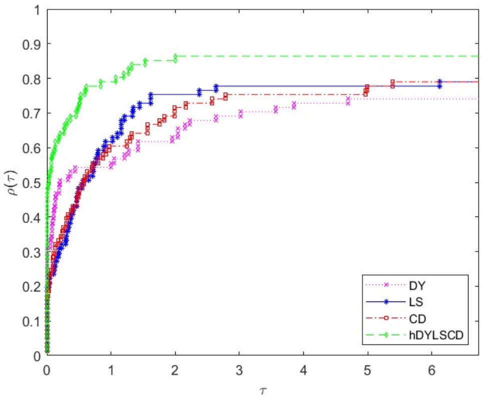

FIGURE 2. - Performance profiles using function evaluation

Figure 3. show that the performance profiles based on CPU time, we can conclude hDYLSCD method is also faster than DY, LS, DY methods

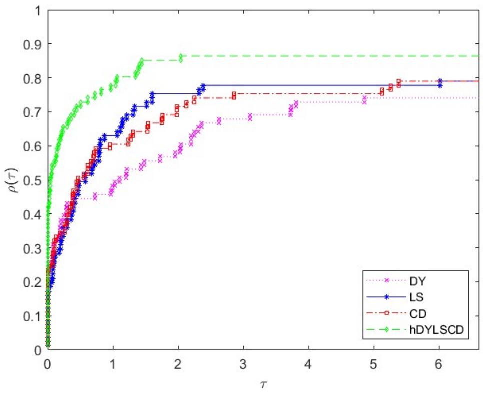

FIGURE 3. - Performance profiles using CPU time

6. CONCLUSION

Conjugate gradient techniques are widely used to solve unconstrained optimization problems, particularly of a large scale. The hybrid approach, which combines traditional methodologies is one of the most beneficial techniques. In order to develop a novel, effective technique we have presented a new hybrid approach in this research that computes parameter β as a convex combination of three parameters DY, LS and CD.

The practical results demonstrate that the chosen strategy is faster and more effective than alternative approaches. The sufficient descent and global convergence have been demonstrated.

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