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Fuzzy Hilbert Spaces

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Abstract

we introduce the definition of afuzzy inner product space and discuss some properties of this space,and we use the definition of fuzzy inner product space to introduced anew definitions such that the definition of fuzzy Hilbert space ,Fuzzy convergence, ,Fuzzy complete,and we studied the relation between ordinary inner product space and fuzzy inner product space.

Keywords: fuzzy set,fuzzy inner product space

فضاءات هلبرت الضبابية

الخلاصة

قدمنا تعريف الضرب الداخلي الضبابي وناقشنا بعض الخصائص المتعلقة بهذا الفضاء واستخدمنا تعريف الضرب الداخلي الضبابي لتقديم تعاريف جديدة مثلاً تعريف فضاء هلبرت الضبابي , التقارب الضبابي, التام الضبابي وقمنا بدراسة العلاقة بين فضاء الضرب الداخلي الاعتيادي وفضاء الضرب الداخلي الضبابي.

1-Introduction

J.R.Kiderwas introduced the definition of fuzzy inner product [2], which is using to study the fuzzy inner product spaces in detail and these gave us new results.

we use the definition of fuzzy inner product space appeared in [2] to study this space and investigate some properties of this space,we prove that for any fuzzy inner product space H there exists a fuzzy Hilbert space Y and an isomorphism T from H onto fuzzydensesubspace$\widetilde{W}$ $of\ Y$.Thespace Y is unique except for isomorphism's (See theorem 3.12).

2- Basic concepts:

Definition1.1:

Let X be any set of element.A fuzzy set $\tilde{A}\ in\ X$ is characterized by amembership function, $\mu_{\tilde{A}}(x): X \to I, where\ I$ is the closed unit interval [0,1].Then we can writ a fuzzy set $\tilde{A}$ as:

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$.

The collection of all fuzzy subsets of Xwill be denoted by I^X,$i.e$,

$I^X = \{\tilde{A}: \tilde{A}\ is\ fuzzy\ subset\ of\ X\}$

Remarks 1.2:

Following, some fundamental concept related to the basic operations and concepts of fuzzy subsets of X. Let $\tilde{A}\ and\ \tilde{B}$ be two fuzzy subsets of$X$with membershipfunction$\mu_{\tilde{A}}\ and\ \mu_{\tilde{B}}$ respectively, then:

1- $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x), \quad \forall x \in X$

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2- $\tilde{A} = \tilde{B}$ *if and only if* $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x),\ \forall x \in X.$

3- The complement of $\tilde{A}$ (denoted by $\tilde{A}^C$) is also a fuzzy set which has the membershipfunction $\mu_{\tilde{A}^C}(x) = 1 - \mu_{\tilde{A}}(x)$.

4- $\tilde{A} = \emptyset$ *if and only if* $\mu_{\tilde{A}}(x) = 0, \forall x \in X$, *where* $\emptyset$ *is the empty*

5- The height of fuzzy set is the Supremum valueof $\mu_{\tilde{A}}(x)$ *over all* $x \in X$. If the height is 1, then $\tilde{A}$ is normal, otherwise it is subnormal.

6- The support of $\tilde{A}$ is the set of all elements x *in* X *at which* $0 < \mu_{\tilde{A}}(x) \le 1$ and is denoted by supp $(\tilde{A})$.

7- A point $x \in X$ is said to be crossover point of $\tilde{A}$ *if* $\mu_{\tilde{A}}(x) = 0.5$.

8- $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function $\mu_{\tilde{C}}(x) = \min_{x\in X}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

9- $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function $\mu_{\tilde{D}}(x) = \max_{x\in X}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

10- If $\mu_{\tilde{A}\cap\tilde{B}}(x) = 0, \forall x \in X$, *then* $\tilde{A}$ *and* $\tilde{B}$ are said to be separated fuzzy sets.

11- The product of $\tilde{A}$ *and* $\tilde{B}$ (denoted by $\tilde{A}\tilde{B}$) is also fuzzy set which have a membershipfunction $\mu_{\tilde{A}\tilde{B}}(x) = \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)$.

12- n-th power of $\tilde{A}$(denoted by $\tilde{A}^n$) is also a fuzzy set which have a membership function $\mu_{\tilde{A}^n} = [\mu_{\tilde{A}}(x)]^n$ where n is a positive integer and consequently $\tilde{A}^n \subseteq \tilde{A}^m$ for all $n \ge m \ge 0$.

Theorem 1.3:

Let $\{\tilde{A}_j, j \in J$, *where* J *is any index set*$\}$ be a collection of fuzzy sets in X .

(a) If $\tilde{A}_i \subseteq \tilde{A}_j$ then:

(i) $\tilde{A}_i \cap \tilde{A}_j = \tilde{A}_i$

(ii) $\tilde{A}_i \cup \tilde{A}_j = \tilde{A}_j$

(iii)$\tilde{A}_j^{\,C} \subseteq \tilde{A}_i^{\,C}$ *where* $i, j \in J$

(b) Commutative laws: For $i, j \in J$

$\tilde{A}_i \cup \tilde{A}_j = \tilde{A}_j \cup \tilde{A}_i$ *and* $\tilde{A}_i \cap \tilde{A}_j = \tilde{A}_j \cap \tilde{A}_i$

(c) Associative laws: For $i, j, k \in J$

$\tilde{A}_i \cup (\tilde{A}_j \cup \tilde{A}_k) = (\tilde{A}_i \cup \tilde{A}_j) \cup \tilde{A}_k$ *and*

$\tilde{A}_i \cap (\tilde{A}_j \cap \tilde{A}_k) = (\tilde{A}_i \cap \tilde{A}_j) \cap \tilde{A}_k$

(d). Distributive laws: for $i, j, k \in J$

$\tilde{A}_i \cup (\tilde{A}_j \cap \tilde{A}_k) = (\tilde{A}_i \cup \tilde{A}_j) \cap (\tilde{A}_i \cup \tilde{A}_k)$ *and*

$\tilde{A}_i \cap (\tilde{A}_j \cup \tilde{A}_k) = (\tilde{A}_i \cap \tilde{A}_j) \cup (\tilde{A}_i \cap \tilde{A}_k)$

(e)De Morgan's laws: For $i, j, \in J$

$(\tilde{A}_i \cap \tilde{A}_j)^C = \tilde{A}_i^C \cup \tilde{A}_j^C$ *and*

$(\tilde{A}_i \cup \tilde{A}_j)^C = \tilde{A}_i^C \cap \tilde{A}_j^C$

Definition 1.4 :

If $\tilde{A}$ *and* $\tilde{B}$ are fuzzy sets in anon-emptysets X *and* Y, respectively then the Cartesian product

$\tilde{A} \times \tilde{B}$ of $\tilde{A}$ $and \tilde{B}$ is a fuzzy set with membership function

$\mu_{\tilde{A}\times\tilde{B}}(x,y) = min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$ $for\ all\ (x,y)\ in\ X \times Y$.

Definition 1.5:

A fuzzy point p in X is a fuzzy set with membership function

.

$$P(y) = \begin{cases} \alpha, & if\ y = x \\ 0, & otherwise \end{cases}$$

For all y in X where $0 < \alpha < 1$. P is said to have support x and value α. We denote this fuzzy point by x_α or (x, α).

$Two\ fuzzy\ points\ x_\alpha and\ y_\delta$ are said to be distinct if and only if $x \neq y$.

Remark 1.6

Two fuzzy points x_α and y_β are said to be equal if and only if $x = y$ and $\alpha = \beta$, $where$ $\alpha, \beta \in (0,1]$.

Definition 1.7

Let f be a function from a nonempty set X to a nonempty set Y. If $\tilde{B}$ is a fuzzy set in Y then $f^{-1}(\tilde{B})$ is a fuzzy set in X defined by:

$\mu_{f^{-1}(\tilde{B})}(x) = \mu_{\tilde{B}}(f(x))$ $for\ all\ x\ in\ X$

Equivalently $\mu_{f^{-1}(\tilde{B})} = \mu_{\tilde{B}}\ o\ f$???.

Also if $\hat{A}$ is a fuzzy set in X then $f(\hat{A})$ is a fuzzy set in Y defined by

$$\mu_{f(\tilde{A})}(y) = \begin{cases} sup\{\mu_{\tilde{A}}(x) | x \in f^{-1}(y)\}, & f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$for\ all\ y\ in\ Y\ where\ f^{-1}(y) = \{x \in X | f(x) = y\}$

Definition 1.8

Let x_α be a fuzzy point and $\hat{A}$ a fuzzy set in X. Then x_α is said to be in $\hat{A}$ or belongs to $\hat{A}$ denoted by $x_\alpha \in \hat{A}$ if and only if $\alpha \leq \mu_{\tilde{A}}(x)$. The set of all fuzzy points in X is denoted by $P(X)$.

Remark 1.9:

Let $\hat{A}$ and $\tilde{B}$ be fuzzy sets in X. Then:

(1) $\tilde{A} \subset \tilde{B}$ if and only if for every $x_\alpha \in \hat{A}$ implies $x_\alpha \in \tilde{B}$.

(2) $\tilde{A} = \tilde{B}$ if and only if $x_\alpha \in \tilde{A}$ if and only if $x_\alpha \in \tilde{B}$ for all $x_\alpha \in P(X)$.

(3) $x_\alpha \in \tilde{A} \cup \tilde{B}$ if and only if $x_\alpha \in \tilde{A}$ or $x_\alpha \in \tilde{B}$.

(4) $x_\alpha \in \tilde{A} \cap \tilde{B}$ if and only if $x_\alpha \in \tilde{A}$ and $x_\alpha \in \tilde{B}$.

Theorem 1.10

Let $\{\tilde{A}_j | j \in J\}$ be a family of fuzzy sets in X where J is any index. Then $x_\alpha \in \cup_{j\in J} \tilde{A}_j$ if and only if $x_\alpha \in \tilde{A}_k$ for some $k \in$J

. **Proposition 1.11**

Let $f: X \to Y$ be a function. Then for a fuzzy point x_α in X, $f(x_\alpha)$ is a fuzzy point in Y and $(x_\alpha) = (f(x))_\alpha$.

Theorem 1.12

A fuzzy set $\tilde{A}$ in X is a union of all its fuzzy points.

2- Fuzzy Inner Product Spaces:

Fuzzy inner product spaces are special fuzzy normed spaces as we shall see. In fact fuzzy inner product spaces are probably the most natural generalization of fuzzy Euclidean space and the reader should note the great harmony and beauty of the concepts .

Definition 2.1:[1]

Let H be a vector space over field $\mathbb{K}$, an inner product on H is a mapping of $H \times H$ into the scalar field $\mathbb{K}$ of H, that is with every pair of vector x and

y there is associated a scalar which is written $\langle x,y\rangle$and is called the inner product of x and y such that for all vectors x,y,z and scalar r we have:

$(IP_1)\langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle$

$(IP_2)\langle rx,y\rangle = r\langle x,y\rangle$

$(IP_3)\langle x,y\rangle = \overline{\langle y,x\rangle}$

$(IP_4)\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$

Definition 2.2:[2]

A fuzzy inner product space on$H$, where $H$is a vector space over the field $\mathbb{K}$(where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) is a mapping of $H\times H$into the field$\mathbb{K}$, that is with every pair of fuzzy vectors x_α, y_β there is associated a scalar Which written $\langle x_\alpha, y_\beta\rangle or \langle x,y\rangle(\lambda)$, Where $\lambda = min\{\alpha,\beta\}, \alpha,\beta \in (o,1]$ and is called the fuzzy inner product of $x_\alpha\ and\ y_\beta$ Such that for all fuzzy vectors $x_\alpha, y_\beta, z_\sigma$ with $\lambda = min\{\alpha,\beta,\sigma\}$ and scalar r we have:

$(FIP_1)\langle x+z,y\rangle(\lambda) = \langle x,y\rangle(\lambda) + \langle z,y\rangle(\lambda)$

$(FIP_2)\langle rx,y\rangle(\lambda) = r\langle x,y\rangle(\lambda)$

$(FIP_3)\langle x,y\rangle(\lambda) = \langle \overline{y,x}\rangle(\lambda)$

$(FIP_4)\langle x,x\rangle(\alpha) \geq 0$ $and \langle x,x\rangle\ (\alpha) = 0 \Leftrightarrow x = 0$

(FIP_5) $if\ 0 < \beta \leq \alpha < 1$, then $\langle x,x\rangle(\alpha) \leq \langle x,x\rangle(\beta)$ and then there exists $0 < \alpha_n < \alpha$ such that $\lim_{n\to\infty}\langle x,x\rangle(\alpha_n) = \langle x,x\rangle(\alpha)$

Proposition 2.3:[2]

If $(H,\langle\cdot,\cdot\rangle)$ is an ordinary inner product space then $(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy inner product space, where $\langle x,y\rangle(\lambda) = \frac{1}{\lambda}\langle x,y\rangle$ for every $x_\alpha, y_\beta \in H,\ with\ \lambda = min\{\alpha,\beta\}\ and\ \alpha,\beta \in (0,1]$

Proof:

Let $x_\alpha, y_\beta, z_\sigma \in H\ and\ r \in \mathbb{K}\ \ where\ \alpha,\beta,\sigma \in (0,1]$

with $\lambda = min\{\alpha,\beta,\sigma\}$ then

$(FIP_1)\langle x+z,y\rangle(\lambda) = \frac{1}{\lambda}\langle x+z,y\rangle$

$= \frac{1}{\lambda}[\langle x,y\rangle + \langle z,y\rangle]$

$= \langle x,y\rangle(\lambda) + \langle z,y\rangle(\lambda)$

$= \frac{1}{\lambda}\langle x,y\rangle + \frac{1}{\lambda}\langle z,y\rangle$

$(FIP_2)\langle rx,y\rangle(\lambda) = \frac{1}{\lambda}\langle rx,y\rangle$

$= \frac{1}{\lambda}[r\langle x,y\rangle]$

$= r\left[\frac{1}{\lambda}\langle x,y\rangle\right] = r\langle x,y\rangle(\lambda)$

$(FIP_3)\langle x,y\rangle(\lambda) = \frac{1}{\lambda}\langle x,y\rangle = \frac{1}{\lambda}\langle\overline{y,x}\rangle = \overline{\langle y,x\rangle}(\lambda)$

$(FIP_4) since\ \langle x,x\rangle \geq 0$ so $\frac{1}{\alpha}\langle x,x\rangle \geq 0$ thus $\langle x,x\rangle(\alpha) \geq 0$

Also

$\langle x,x\rangle(\alpha) = 0 = \frac{1}{\alpha}\langle x,x\rangle \Leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0$

(FIP_5) If $0 < \beta \leq \alpha < 1$ then $\frac{1}{\alpha} \leq \frac{1}{\beta} or \frac{1}{\alpha}\langle x,x\rangle \leq \frac{1}{\beta}\langle x,x\rangle$

Which implies that

$\langle x,x\rangle(\alpha) \leq \langle x,x\rangle(\beta)$.

$\left[\text{With}\ \ \alpha_n = \left(1-\frac{1}{n}\right)\alpha\right]$ Such that

$\lim_{n\to\infty}\langle x,x\rangle(\alpha_n) = \lim_{n\to\infty}\frac{1}{\alpha_n}\langle x,x\rangle$

$= \frac{1}{\alpha}\langle x,x\rangle = \langle x,x\rangle(\alpha)$

Thus $(H,\langle\cdot,\cdot\rangle(\cdot))$ is fuzzy inner product space.

Example2.4:

The space $\mathbb{C}^n$ is an inner product space with inner product defined by

$\langle x,y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$

Where x $= (x_1,x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n)$

Hence $(\mathbb{C}^n,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy inner product space with fuzzy inner product define by

$\langle x,y\rangle(\lambda) = \frac{1}{\lambda}\langle x,y\rangle$

Forevery

$x_\alpha = (x_1,x_2,\ldots,x_n)_\alpha\ , y_\beta = (y_1,y_2,\ldots,y_n)_\beta$

where$\alpha,\beta \in (0,1]$with$\lambda = min\{\alpha,\beta\}$ by proposition 2.3

Proposition 2.5:

If $(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy inner product space then $(H,\langle\cdot,\cdot\rangle)$ is an ordinary inner product space with,

$\langle x,y\rangle = \langle x,y\rangle(\lambda) \quad for\ every \quad x,y \in H,\ \lambda \in (0,1]$

Proof :

Let

$x,y,z \in H \ and\ r \in \mathbb{K},\ \lambda \in (0,1],$

then

$(IP_1)\langle x+z,y\rangle = \langle x+z,y\rangle(\lambda)$

$= \langle x,y\rangle(\lambda) + \langle z,y\rangle(\lambda)$

$= \langle x,y\rangle + \langle z,y\rangle$

$(IP_2)\langle rx,y\rangle = \langle rx,y\rangle(\lambda)$

$= r\langle x,y\rangle(\lambda) = r\langle x,y\rangle$

$(IP_3)\langle x,y\rangle = \langle x,y\rangle(\lambda) = \overline{\langle y,x\rangle(\lambda)} = \overline{\langle y,x\rangle}$

$(IP_4)\langle x,x\rangle = \langle x,x\rangle(\lambda) \geq 0$ and

$\langle x,x\rangle = \langle x,x\rangle(\lambda) = 0 \quad \Leftrightarrow x = 0$

Thus $(H,\langle\cdot,\cdot\rangle)$ is an arbitrary inner product space.

Remark 2.6:

If $(H\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy inner product space then

(i)

$\langle rx+sy,z\rangle(\lambda) = r\langle x,z\rangle(\lambda) + s\langle y,z\rangle(\lambda)$

(ii)

$\langle x,ry\rangle(\lambda) = \bar{r}\langle x,y\rangle(\lambda)$

(iii)

$\langle x,ry+sz\rangle(\lambda) = \bar{r}\langle x,y\rangle(\lambda) + \bar{s}\langle x,z\rangle(\lambda)$

For all $x_\alpha, y_\beta \in H$, $r,s \in \mathbb{K}$ where $\alpha,\beta,\sigma \in (0,1]$ with $\lambda = min\{\alpha,\beta,\sigma\}.,z_\sigma$

Proof:

(i)

$\langle rx+sy,z\rangle(\lambda) = \langle rx,z\rangle(\lambda) + \langle sy,z\rangle(\lambda)$

by (FIP_1)

$= r\langle x,z\rangle(\lambda) + s\langle y,z\rangle(\lambda)$ by (FIP_2)

(ii)

$\langle x,ry\rangle(\lambda) = \overline{\langle ry,x\rangle}(\lambda)$

by (FIP_3)

$= \bar{r}\,\overline{\langle y,x\rangle}(\lambda) = \bar{r}\langle x,y\rangle(\lambda)$

(iii)

$\langle x,ry+sz\rangle(\lambda) = \overline{\langle ry+sz,x\rangle}$

by (FIP_3)

$= \bar{r}\,\overline{\langle y,x\rangle}(\lambda) + \bar{s}\,\overline{\langle z,x\rangle}(\lambda)$

By (FIP_1)

$= \bar{r}\langle x,y\rangle(\lambda) + \bar{s}\langle x,z\rangle(\lambda)$

By (FIP_3)

Theorem 2.7:[2]

If $x_\alpha, y_\beta \in H$ where $(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy inner product space then

$|\langle x,y\rangle(\lambda)|^2 \leq \langle x,x\rangle(\lambda)\cdot\langle y,y\rangle(\lambda)$

is called fuzzy Schwarz inquality,

$where \quad \alpha,\beta \in (0,1]\ with\ \lambda - min\{\alpha,\beta\}$

Proof:

If $x = 0$ then (1) is obvious. Put $\delta = \langle x,y\rangle(\lambda)$ then

$0 \leq \langle \gamma x+y, \gamma x+y\rangle(\lambda)$

$= \langle \gamma x, \gamma x+y\rangle(\lambda) + \langle y, \gamma x+y\rangle(\lambda)$

$= \overline{\langle gx+y, gx\rangle}(l) + \overline{\langle gx+y, y\rangle}(l)$

$|g|^2\langle x,x\rangle(l) + \overline{\langle y,gx\rangle}(l) + g\overline{\langle x,y\rangle}(l) + \overline{\langle y,y\rangle}(l)$

$|g|^2\langle x,x\rangle(l) + g\langle x,y\rangle(l) + g\overline{\langle x,y\rangle}(l) + \langle y,y\rangle(l)$

$0 \leq |\gamma|^2\langle x,x\rangle(\lambda) + 2Re(\gamma\delta) + \langle y,y\rangle(\lambda)$

Take $\gamma = \frac{-\bar{\delta}}{\langle x,x\rangle(\lambda)}$

Now with this γ (2) becomes

$0 \leq \langle y,y\rangle(\lambda) - \frac{|\delta|^2}{\langle x,x\rangle(\lambda)}$, or

$\frac{|\delta|^2}{<x,x>(\lambda)} \leq \langle y,y\rangle(\lambda)$, or

$|\delta|^2 \leq \langle x,x\rangle(\lambda)\cdot\langle y,y\rangle(\lambda)$

Thus

$|\langle x,y\rangle(\lambda)|^2 \leq \langle x,x\rangle(\lambda)\cdot\langle y,y\rangle(\lambda)$

Theorem 2.8:[2]

Every fuzzy inner product space is a fuzzy normed space.

Proof:

Let $(H,\langle\cdot,\cdot\rangle(\cdot))$ be a fuzzy inner product space. Define

$\|x\|(\alpha) = [\langle x,x\rangle(\alpha)]^{1/2}$

Foreach $x_\alpha \in H,\ where\ \alpha \in (0,1],$

let $x_\alpha, y_\beta, z_\sigma \in H$

where $\alpha, \beta, \sigma \in (0,1]$ with $\lambda = min\{\alpha, \beta, \sigma\}$ *and* $r \in \mathbb{K}$.

$(FN_1) \|x\|(\alpha) = 0 = [\langle x,x\rangle(\alpha)]^{\frac{1}{2}} \Leftrightarrow \langle x,x\rangle(\alpha) = 0 \Leftrightarrow x = 0$

$$(FN_2) \|rx\|(\alpha) = [\langle rx, rx\rangle(\alpha)]^{\frac{1}{2}}$$
$$= [r\bar{r}\ \langle x,x\rangle(\alpha)]^{\frac{1}{2}}$$
$$= [|r|^2 \langle x,x\rangle(\alpha)]^{\frac{1}{2}}$$
$$= |r|[\langle x,x\rangle(\alpha)]^{\frac{1}{2}}$$
$$= |r|\|x\|(\alpha)$$

$(FN_3) \|x+y\|^2(\lambda) = \langle x+y, x+y\rangle(\lambda)$
$= \overline{\langle x, x+y\rangle(\lambda)} + \overline{\langle y, x+y\rangle(\lambda)}$
$= \overline{\langle x+y, x\rangle(\lambda)} + \overline{\langle x+y, y\rangle(\lambda)}$

$\overline{\langle x,x\rangle}(l) + \overline{\langle y,x\rangle}(l) + \overline{\langle x,y\rangle}(l) + \overline{\langle y,y\rangle}(l)$
$=$
$\langle x,x\rangle(l) + \langle x,y\rangle(l) + \overline{\langle x,y\rangle}(l) + \langle y,y\rangle(l)$

$= \|x\|^2(\alpha) + 2Re\ \langle x,y\rangle(\lambda) + \|y\|^2(\lambda)$

$\leq \|x\|^2(\alpha) + 2|\langle x,y\rangle(\lambda)|^2 + \|y\|^2(\lambda)$

$\leq \|x\|^2(\alpha) + 2\|x\|(\lambda)\|y\|(\lambda) + \|y\|^2(\lambda)$

Thus
$\|x+y\|^2(\lambda) \leq [\|x\|(\lambda) + \|y\|(\lambda)]^2$
Hence
$\|x+y\|(\lambda) \leq \|x\|(\lambda) + \|y\|(\lambda)$
(FN_4) if $0 < \beta \leq \alpha < 1$ then
$\|x\|(\alpha) = [\langle x,x\rangle(\alpha)]^{\frac{1}{2}} \leq [\langle x,x\rangle(\beta)]^{\frac{1}{2}} = \|x\|(\beta)$
And there exist $0 < \alpha_n < \alpha$ such that

$$\lim_{n\to\infty} \|x\|(\alpha_n) = \lim_{n\to\infty} [\langle x,x\rangle(\alpha_n)]^{\frac{1}{2}} = [\langle x,x\rangle(\alpha)]^{\frac{1}{2}} = \|x\|(\alpha)$$

Theorem 2.9:

Let $(H, \langle\cdot,\cdot\rangle(\cdot))$ be a fuzzy inner product space then the fuzzy norm $\|x\|(\alpha) = [\langle x,x\rangle(\alpha)]^{\frac{1}{2}}$ Satisfies the fuzzy parallelogram equality

$$\|x+y\|^2(\lambda) + \|x-y\|^2(\lambda) = 2[\|x\|^2(\lambda) + \|y\|^2(\lambda)]$$

for each $x_\alpha, y_\beta \in H$ where $\alpha, \beta \in (0,1]$ with $\lambda = min\{\alpha, \beta\}$

$\|x+y\|^2(\lambda) = \langle x+y, x+y\rangle(\lambda)$
$= \langle x, x+y\rangle(\lambda) + \langle y, x+y\rangle(\lambda)$
$=$
$\overline{\langle x+y, x\rangle}(l) + \overline{\langle x+y, y\rangle}(l)$
$\overline{\langle x,x\rangle}(l) + \overline{\langle y,x\rangle}(l) + \overline{\langle x,y\rangle}(l) + \overline{\langle y,y\rangle}(l)$

$= \langle x,x\rangle(\lambda) + \langle x,y\rangle(\lambda) + \overline{\langle x,y\rangle(\lambda)} + \langle y,y\rangle(\lambda)$

$\overline{\langle x+y, x\rangle}(l) + \overline{\langle x+y, y\rangle}(l)$

$\overline{\langle x,x\rangle}(l) + \overline{\langle y,x\rangle}(l) + \overline{\langle x,y\rangle}(l) + \overline{\langle y,y\rangle}(l)$
$= \|x\|^2(l) + \langle x,y\rangle(l) + \overline{\langle x,y\rangle}(l) + \|y\|^2(l)$ ……(1)

Similarly, we have

$\|x-y\|^2(l) = \|x\|^2(l) - \langle x,y\rangle(l) - \overline{\langle x,y\rangle}(l) - \|y\|^2(l)$ ….. (2)

Thus by adding these two equations we get
$\|x+y\|^2(\lambda) + \|x-y\|^2(\lambda) = 2[\|x\|^2(\lambda) + \|y\|^2(\lambda)]$

Remark 2.10:

Not all fuzzy normed spaces are fuzzy inner product spaces, for example, the space $C[a,b]$ is a fuzzy normed space with fuzzy norm defined by

$$\|x\|(\alpha) = \frac{1}{\alpha} \max_{t \in J} |x(t)|\ , \qquad J = [a,b]$$

See [1] but it is not a fuzzy inner product space

We shows that the fuzzy norm cannot be obtained from a fuzzy inner product since this fuzzy norm does not satisfy the fuzzy parallelo-gram equality.

Indeed, if we take $(x(t) = 1, \alpha)$ *and* $\left(y(t) = \frac{t-a}{b-a}, \beta\right)$ *where* $\alpha, \beta \in (0,1]$ *with* $\lambda = min\{\alpha, \beta\}$

We have

$\|x\|(\lambda) = \frac{1}{\lambda}$ $\|y\|(\lambda) = \frac{1}{\lambda}$

$x(t) + y(t) = 1 + \frac{t - a}{b - a}$

$x(t) - y(t) = 1 - \frac{t-a}{b-a}$

Hence

$\|x + y\|(\lambda) = \frac{2}{\lambda}$

$\|x - y\|(\lambda) = \frac{1}{\lambda}$

$\|x + y\|^2(\lambda) + \|x - y\|^2(\lambda) = \frac{5}{\lambda^2}$,

but

$2[\|x\|^2(\lambda) + \|y\|^2(\lambda)] = \frac{4}{\lambda^2}$

Thus

$\|x + y\|^2(\lambda) + \|x - y\|^2(\lambda) = \frac{5}{\lambda^2} \neq \frac{4}{\lambda^2} = 2[\|x\|^2(\lambda) + \|y\|^2(\lambda)]$

Definition 2.11:

A fuzzy element x_α of a fuzzy inner product space H is said to be orthogonal to a fuzzy element $y_\beta \in H$ if $\langle x,y\rangle(\lambda) = 0$ where $\alpha, \beta \in (0,1]$ with $\lambda = min\{\alpha, \beta\}$.

We also say that x_α and y_β are orthogonal, and we write $x_\alpha \perp y_\beta$. Similarly for a fuzzy two sets $\tilde{A}, \tilde{B}$ in H we write $x_\alpha \perp \tilde{B}$ if $x_\alpha \perp y_\beta$ for all $y_\beta \in \tilde{B}$ and $\tilde{A} \perp \tilde{B}$

If $x_\alpha \perp y_\beta$ for all x_α in $\tilde{A}$ and y_β in $\tilde{B}$

3- Fuzzy Convergence, Fuzzy Cauchy Sequence, Fuzzy Complete

Definition 3.1:

A fuzzy sequence $\{(x_n, \alpha_n)\}$ in a fuzzy inner product space $(H, \langle\cdot,\cdot\rangle(\cdot))$ is said to be fuzzy convergent if there is a fuzzy vector x_α in H such that $\lim_{n\to\infty}\|x_n - x\|(\lambda) = 0$, where $\|x\|(\alpha) = [\langle x, x\rangle(\alpha)]^{\frac{1}{2}}$ and $\{\alpha, \alpha_i \in (0,1] | i \in \mathbb{N}\}$ with $\lambda = inf\{\alpha, \alpha_n | n \in \mathbb{N}\}$

Definition 3.2:

A fuzzy sequence $\{(x_n, \alpha_n)\}$ in a fuzzy inner product spac $(H, \langle\cdot,\cdot\rangle(\cdot))$ is said to be a fuzzy Cauchy if for every $\varepsilon > 0$, there is an $M > 0$ such that $\|x_m - x_n\| < \varepsilon$

For every $m, n > M$ where $\|x\|(\alpha) = [\langle x, x\rangle(\alpha)]^{\frac{1}{2}}$ with $\lambda = inf\{\alpha_i \in (0,1] | i \in \mathbb{N}\}$

Definition 3.3:

A fuzzy inner product space $(H, \langle\cdot,\cdot\rangle(\cdot))$ is said to be fuzzy complete if every fuzzy Cauchy sequence in H converges, that is, has a fuzzy limit which is a fuzzy vector of H.

Proposition 3.4:

Let $(H, \langle\cdot,\cdot\rangle(\cdot))$ be a fuzzy inner product space, if $\{(x_n, \alpha_n)\}$ fuzzy sequence converges to x_α and $\{(y_n, \beta_n)\}$ fuzzy sequence converges to y_β then $\{\langle x_n, y_n\rangle(\lambda)\}$ fuzzy sequence converges to $\langle x, y\rangle(\lambda)$ where $\lambda = inf\{\alpha, \alpha_i, \beta, \beta_i \in (0,1] | i \in \mathbb{N}\}$

Proof:

Subtracting and adding a term, using triangle inequality for numbers and finally, the fuzzy Schwarz in equality,

We obtain

$|\langle x_n, y_n\rangle(\lambda) - \langle x, y\rangle(\lambda)| = |\langle x_n, y_n\rangle(\lambda) - \langle x_n, y\rangle(\lambda) + \langle x_n, y\rangle(\lambda) - \langle x, y\rangle(\lambda)|$

$\leq |\langle x_n, y_n - y\rangle(\lambda)| + |\langle x_n - x, y\rangle(\lambda)|$

$\leq \|x_n\|(\lambda)\|y_n - y\|(\lambda) + \|x_n - x\|(\lambda)\|y\|(\lambda) \longrightarrow 0$

Since $\{(y_n - y, \lambda)\}$ converges to zero and $\{(x_n - x, \lambda)\}$ converges to zero as $n \longrightarrow \infty$

Definition 3.5:

A fuzzy inner product space $(H,\langle\cdot,\cdot\rangle(\cdot))$ is said to be a fuzzy Hilbert space if it is fuzzy complete with respect to the fuzzy normed $\|x\|(\alpha)=[\langle x,x\rangle(\alpha)]^{\frac{1}{2}}$ where $x_\alpha \in H$

Theorem 3.6:

If $(H,\langle\cdot,\cdot\rangle)$ is an ordinary Hilbert space, then $(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy Hilbert space where $\langle x,y\rangle(\lambda)=\frac{1}{\lambda}\langle x,y\rangle$, for every $x_\alpha,y_\beta \in H$ with $\lambda=min\{\alpha,\beta \in (0,1]\}$

Proof:

$(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy inner product space by proposition 2.3 since $(H,\|\cdot\|)$ is complete normed space where $\|x\|=[\langle x,x\rangle]^{\frac{1}{2}}$, so $(H,\|\cdot\|(\cdot))$ is a fuzzy complete normed space where: $\|x\|(\alpha)=\frac{1}{\alpha}\|x\|$ for all $x_\alpha \in H$. Thus $(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy Hilbert space

Theorem 3.7:

If $(H,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy Hilbert space then $(H,\langle\cdot,\cdot\rangle)$ is an ordinary Hilbert space where $\langle x,y\rangle=\langle x,y\rangle(\lambda)$ where $\lambda \in (0,1]$

Proof:

First $(H,\langle\cdot,\cdot\rangle)$ is an ordinary inner product space by proposition 2.6, since $(H,\|\cdot\|(\cdot))$ is fuzzy complete normed space where $\|x\|(\alpha)=[\langle x,x\rangle(\alpha)]^{\frac{1}{2}}$ so $(H,\|\cdot\|)$ is complete normed space where $\|x\|=\|x\|(\alpha)$ where $\alpha \in (0,1]$ Thus $(H,\langle\cdot,\cdot\rangle)$ is an ordinary Hilbert space.

Example3.8:

The space $\mathbb{C}^n$ is a Hilbert space with inner product defined by $\langle x,y\rangle=x_1\overline{y_1}+x_2\overline{y_2}+\cdots+x_n\overline{y_n}$ Where $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n)$

See [1]

Hence $(\mathbb{C}^n,\langle\cdot,\cdot\rangle(\cdot))$ is a fuzzy Hilbert space with fuzzy inner product defined by $\langle x,y\rangle(\lambda)=\frac{1}{\lambda}\langle x,y\rangle$ Where $\lambda=min\{\alpha,\beta \in (0,1]\}$ by theorem 3.6

Theorem 3.9: [3]

Let $(X,\|\cdot\|(\cdot))$ be a fuzzy normed space. Then there is a fuzzy Banach space Y and a fuzzy isometry T from X onto a fuzzy subspace $\widetilde{W}$ of Y which is fuzzy dense in Y. The space Y is unique except for fuzzy isometries.

Definition 3.10:

An isomorphism $T$of a fuzzy inner product Honto a fuzzy inner product Yover the same field is abijective linear operator $T:H\longrightarrow Y$ which preserves the fuzzy inner product that is for all $x_\alpha,y_\beta \in H$ with $\lambda=min\{\alpha,\beta \in (0,1]\}$ $<T(x),T(y)>(\lambda)=\langle x,y\rangle(\lambda)$ Where we denote the fuzzy inner product on H and Y by the same symbol, for simplicity,Y is then called isomorphism with H and H,Y are called isomorphic fuzzy inner product space.

Note that the bijectivity and linearity guarantees that T is a vector space isomorphism of H onto Y, so that T preserves the whole structure of fuzzy inner product. T is also an isometry of H onto Y because fuzzy distances in H and Y are determined by the fuzzy norms defined by the fuzzy inner products on H and Y.

The theorem about the fuzzy completion of a fuzzy inner product can now stated as follows:

Theorem 3.11:

For any fuzzy inner product space H there exists a fuzzy Hilbert space Y an isomorphism T from H onto a fuzzy dense fuzzy subspace $\widetilde{W}$ of Y. the space Y is unique except for isomorphisms.

Proof:

By theorem 3.10 there exists a fuzzy Banach space Y and an isometry T from H onto a fuzzy subspace $\widetilde{W}$ of Y which is fuzzy dense in Y. for reasons of continuity, under such an isometry, sums and scalar multiples of elements in H and w correspond to each other, so that T is even an isomorphism of H onto $\widetilde{W}$, both regarded as a fuzzy normed spaces. Proposition 3.4 shows that we can define a fuzzy inner product on Y by setting:

$$< [\tilde{x}],[\tilde{y}] > (\lambda) = \lim_{n\to\infty}\langle x_n, y_n\rangle(\lambda)$$

The notation being as in theorem 3.10, that is $\{(x_n,\alpha_n)\}$ and $\{(y_n,\beta_n)\}$ are representative of $[\tilde{x}] \in Y$ *and* $[\tilde{y}] \in Y$ respectively we see that T is an isomorphism of H onto $\widetilde{W}$, both regarded as fuzzy inner product spaces.

Theorem 3.10 also guarantees that Y is unique except for isometries that is, two completions Y and Z of H are related by an isometry $A: Y \longrightarrow Z$. Reasoning as in the case of T, we conclude that A must be an isomorphism of the fuzzy Hilbert space Y onto the fuzzy Hilbert space Z.

Lemma 3.12:

Let $(H,\langle\cdot,\cdot\rangle(\cdot))$ be a fuzzy inner product space.

If

$$x_\alpha \perp y_\beta \; then \; \|x+y\|^2(\lambda) = \|x\|^2(\lambda) + \|y\|^2(\lambda)$$

where $\lambda = min\{\alpha,\beta \in (0,1]\}$.

Proof:

$$\|x+y\|^2(\lambda) = \langle x+y, x+y\rangle(\lambda)$$

$$= \langle x, x+y\rangle(\lambda) + \langle y, x+y\rangle\lambda$$

$$= \overline{\langle x+y, x\rangle}(\lambda) + \overline{\langle x+y, y\rangle}(\lambda)$$

$$= \langle x, x\rangle(\lambda) + \langle x, y\rangle(\lambda) + \overline{\langle x, y\rangle}(\lambda) + \langle y, y\rangle(\lambda)$$

Then

$$\|x+y\|^2(\lambda) = \|x\|^2(\lambda) + \|y\|^2(\lambda)$$

Since $\langle x, y\rangle(\lambda) = 0$.

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