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Some Numerical Methods For Solving Fractional Parabolic Partial Differential Equations

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Abstract

The aim of this paper isto approximate the solution offractional parabolicpartial differential equations using two numerical methods which are Bellman's method (make use of Lagrange interpolation formula) and the method of lines. Fractional Parabolic partial differential equations are transformed to a system of first order ordinary differential equations that are solved using a Runge-Kutta method. An illustrative example using thesemethodsare also presented and compared with the exact solution.

Keywords: Bellman's Method, Method of lines, Runge-Kutta Methods.

بعض الطرائق العددية لحل المعادلات التفاضلية الجزئية الكسورية المكافئة

الخلاصة

الهدف الرئيس من هذا البحث هو تقريب حل المعادلات التفاضلية الجزئية الكسورية المكافئة(fractional Parabolic partial differential equations) باستخدام طريقتان عدديتان هما طريقة بيلمان (وبالاعتماد على صيغة لاكرانج للاندراج) وطريقة الخطوط. المعادلات التفاضلية الجزئية الكسوريةستتحول الى منظومة معادلات تفاضلية إعتيادية من الرتبة الاولى والتي سوف تحل بوساطة أستخدام طريقة رنج-كتا. تم إعطاء مثال توضيحي لتلك الطرائق وتم مقارنة النتائج مع الحل المضبوط.

1. Introduction:

Fractional order partial differential equations are generalizations of classical partial differential equations .Increasingly these models are used in applications such as fluid flow, financeand others [Ghareeb,2007].

Fractional calculus is a field ofmathematical study that grows out of the traditional detentions of the calculus integral and derivative operators in the same way fractional exponent is an outgrowth of exponent with integer value,[Loverro,2004].

Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the word of fractional calculus are the Riemann-Liouville andGrünwald-Letnikov definition .Also caputo,[Podlubny,1999] reformulated the more "classic" definition of the Riemann Liouvillefractional derivative in order to use integer order initial conditions to solve his fractional order differential equations .

Recently [Kolowankar, 1996] reformulated again, the Riemann-Liouville fractional derivative, in

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order to differentiate no-where differentiable fractal functions.

Fractional partial differential equations have been studied and explicit solutions have been achieved by [Mainardi, 2003],[Mainardi, 2005], [Yu, 2005],[Langlands, 2006], [Mainardi, 2006]and several other research works can be found in the literature.

In this paper, we shall use some numerical methods which areBellman's method and method of lines to solve the fractional Parabolic Partial differential Equations of the form:

$$\frac{\partial u(x,t)}{\partial t}=c(x,t)\frac{\partial^a u(x,t)}{\partial x^a}+s(x,t)\ldots(1)$$

On a finite domain, $L < x < R$, $0 \le t \le T$ .Here we consider the case $1 \le a \le 2$, where the parameter a is the fractional order of the special derivative and The function s(x,t) is source/sink term, the function $c(x,t)$may be interpreted as transport related coefficient. We will also assume that $c(x,t) \ge 0$over the region $L < x < R$, $0 \le t \le T$. We assume an initial condition $u(x,0)=f(x)$ for $L < x < R$ and zero Dirichlet boundary conditions.

This paper consists of foursections, In section two Bellman’s methodwill be considered to solve equation (1), while in section three the method of lines will be presented to solve equation (1) this method was proposed by Richard Bellman for solving originally Partial differential equations, which has the general idea of evaluating the solution at certain lines of the independent variable of the Partial differential Equations.

An illustrative example was given in section four in order to compare these two methods with the exact solution.

*2.*Bellman's Method for Solving Fractional Parabolic Partial Differential Equations:

Consider the fractional order parabolic partial differential equations of the form:

$$\frac{\partial u(x,t)}{\partial t}=c(x,t)\frac{\partial^a u(x,t)}{\partial x^a}+s(x,t)$$

$$L < x < R \ , 0 \le t \le T \ .....(2.a)$$

Together with the initial and zero Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= f(x) && , L \le x \le R \\ u(L,t) &= 0 && , 0 \le t \le T \\ u(R,t) &= 0 && , 0 \le t \le T \end{aligned} \quad ....(2.b)$$

where $\frac{\partial^a u(x,t)}{\partial x^a}$ denote the left handed partial fractional derivative of order α of the function u with respect to$x$ and $1 \le a \le 2$Now we solve problem (2.a, 2.b) using Bellman’s method[AbidMechi, 1991];let $u(x_i,t)=u_i(t)$ and suppose that

$$u(x,t)=\sum_{i=0}^{I} l_i(x)u_i(t) \qquad .....(3)$$

where$l_i(x)$ is the Lagrange interpolation functions satisfying

$$l_i(x_j)=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

After substituting (3) into (2.a) we have

$$\sum_{i=0}^{I} l_i(x)u_i'(t)=c(x,t)\sum_{i=0}^{I}\frac{\partial^a}{\partial x^a}l_i(x)u_i(t)+s(x,t) \quad ....(4)$$

And substituting $x = x_j$ in equation (4),$j=1,2, \ldots,n-1$ in order to get the number of unknowns equals to the number of equations therefore we have a system of first order differential equations as follows :

$$u_j'(t) = c(x_j,t)\sum_{i=0}^{I}\frac{\partial^a}{\partial x^a} l_i(x_j)u_i(t) + s(x_j,t)$$

$j=1,2,\ldots,n-1$ ………(5)

After using the Runge-Kutta method [Burden, 1981]the system (5) will be solved and then we get the values of $u_j(t)$,$j=1,2, \ldots, n-1$.which represent the approximate values of $u(x,t)$ at the pointsx_j and $0 \le t \le T$.

3. The Method of Lines for Solving Fractional Parabolic Partial Differential Equations:

Consider the fractional order parabolic partial differential equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = c(x,t)\frac{\partial^a u(x,t)}{\partial x^a} + s(x,t),$$

$L \le x \le R,\ 0 \le t \le T$ ……(6.a)

together with the initial and zero Dirichlet boundary conditions:

$$u(x,0) = f(x) \quad ,L \le x \le R$$
$$u(L,t) = 0 \quad ,0 \le t \le T$$
$$u(R,t) = 0 \quad ,0 \le t \le T$$
……(6.b)

where $\frac{\partial^a u(x,t)}{\partial x^a}$ denote the left-handed partial fractional derivative of order α of the function u with respect to x and $1 \le \alpha \le 2$.

In this section, we shall use explicit finite difference approximation for $\frac{\partial^a u(x,t)}{\partial x^a}$ to solve this initial-boundary value problem (6.a, 6.b) by the method of lines[Ames 1977]. To do this, suppose that $u(x_i,t)=u_i(t)$and substituting $x=x_i$ into equation (6.a) we shall get:

$$\frac{du_i(t)}{dt} = c(x_i,t)\frac{\partial^a u(x_i,t)}{\partial x^a} + s(x_i,t)\ .$$
…..(7)

Where $x_i=i\Delta x,\ i=0,1,\ldots,n,$ and n is the number of subintervals of the interval $[L,R]$.

The left-handed shifted Grünwald estimate to the left-handed derivativeas illustrated in [Ghareeb, 2007]is:

$$\frac{d^a f(x)}{dx^a} = \frac{1}{(\Delta x)^a}\sum_{k=0}^{n} g_k\ f(x-(k-1)\Delta x)$$

Where n is the number of subintervals of the interval $[L,R]$ and α is the fractional number.

Therefore:

$$\frac{\partial^a u(x_i,t)}{\partial x_i^a} = \frac{1}{(\Delta x)^a}\sum_{k=0}^{i+1} g_k\, u(x_i-(k-1)\Delta x,t)$$

$$\frac{\partial^a u(x_i,t)}{\partial x_i^a} = \frac{1}{(\Delta x)^a}\sum_{k=0}^{i+1} g_k\ u_{i-k+1}(t)$$
…….(8)

Where$g_0=1$ and

$$g_k = (-1)^k\frac{a(a-1)\ldots(a-k+1)}{k!},\ k=1,2,\ldots$$

the derivation of equation (8) is given by details in [Ghareeb, 2007], by substituting equation(8) into equation(7) and seek the values of i from 1 to $n-1$in order to get the number of unknowns equals to the numbers of equations as it get in section two, one can have:

$$\frac{du_i(t)}{dt} = \frac{c(x_i,t)}{(\Delta x)^a}\sum_{k=0}^{i+1} g_k u_{i-k+1}(t) + s(x_i,t),\quad i=1,2,\ldots,n-1\cdot$$
….(9)

then the system(9) is of first order differential equation and can be solved using Runge-Kutta method in order to get an approximate value of $u_i(t)$, $i=1,2,\ldots,n-1$ at the point x_i and $0 \le t \le T$ of problem (6.a, 6.b).

4. Illustrative Example:

In the present section, the result of the Bellman's method and the method of lines which were discussed in section two and three respectively will be given and implemented on the same example. The exact solutionis also given for comparison purpose.

The next example appeared in [Ghareeb, 2007] which is solved by using finite difference method.

Example:

Consider the initial-boundary value problem:

$$\frac{\partial u}{\partial t} = x^{\frac{4}{5}} \frac{\partial^{1.8} u}{\partial x^{1.8}} + x(x-1) - \frac{t}{\Gamma(0.2)}(10x-1),$$

$0 \le x \le 1,\ 0 \le t \le 1$

$$u(x,0) = 0$$

$$u(0,t) = 0 \quad 0 \le x \le 1,\ 0 \le t \le 1$$

$$u(1,t) = 0$$

Following table (1) and table (2) prescribed the result of the Bellman's method and the method of lines respectively for $h=0.2$ with the exact solution of the above example which is $u(x,t)=x(x-1)t$.

5. Conclusions:

From the results of table (1) and table (2) respectively it seems that the method of lines is more accurate than Bellman's method. Taking h large for the Bellman's method will reduces the number of basis $l_i(x)$ and hence reduce the calculation of $\frac{\partial^a l(x)}{\partial x^a}$ which also gives reasonable solution.

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Table (1) The approximate solution of Example (1) using Bellman's method

	t=0	*t=0.2*	*t=0.4*	*t=0.6*	*t=0.8*
Approximate solution x= 0.2	0	-0.033	-0.07	-0.108	-0.148
Exact solution x = 0.2	0	-0.032	-0.064	-0.096	-0.128
Approximate solution x= 0.4	0	-0.059	-0.137	-0.235	-0.349
Exact solution x = 0.4	0	-0.048	-0.096	-0.144	-0.192
Approximate solution x= 0.6	0	-0.068	-0.174	-0.317	-0.494
Exact solution x = 0.6	0	-0.048	-0.096	-0.144	-0.192
Approximate solution x= 0.8	0	-0.061	-0.18	-0.353	-0.578
Exact solution x = 0.8	0	-0.032	-0.064	-0.096	-0.128

Table (2) The approximate solution of Example (1) using the method of lines

	t=0	*t=0.2*	*t=0.4*	*t=0.6*	*t=0.8*
Approximate solution x=0.2	0	-0.0317	-0.0629	-0.0940	-0.1250
Exactsolution x=0.2	0	-0.0320	-0.0640	-0.0960	-0.1280
Approximate solution x=0.4	0	-0.0477	-0.0950	-0.1421	-0.1891
Exact solution x=0.4	0	-0.0480	-0.0960	-0.1440	-0.1920
Approximate solution x=0.6	0	-0.0478	-0.0953	-0.1426	-0.1898
Exact solution x=0.6	0	-0.0480	-0.0960	-0.1440	-0.1920
Approximate solution x=0.8	0	-0.0319	-0.0636	-0.0952	-0.1267
Exact solution x=0.8	0	-0.0320	-0.0640	-0.0960	-0.1280