On semi-delta-open sets in topological spaces

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Abstract -

The leading purposes of this chapter are how to use the concepts ($\triangle$-open and $semi$ $\triangle$-open sets) with certain types of functions via α-open sets, such as ($\triangle$-continuous, $\triangle^{*}$-continuous, $\triangle^{**}$-continuous, $semi$ $\triangle$-continuous, $semi$ $\triangle^{*}$-continuous, and $semi$ $\triangle^{**}$-continuous) functions .We shall characterize the relationships between the previous concepts types that we are going to relate of functions, and the continuity. Moreover, we shall introduce some examples, theorems, remarks, and properties, about these new concepts of functions.

Keywords : $semi$ $\triangle$-$continuous$, $semi$ $\triangle^{*}$-$continuous$, and $semi$ $\triangle^{**}$-$continuous$ $functions$.

حول المجموعات شبه دلتا المفتوحة في الفضاءات الطوبولوجية

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الملخص

الأغراض الرئيسية لهذا الفصل هي كيفية استخدام المفاهيم (المجموعات المفتوحة △ وشبه △ المفتوحة) مع أنواع معينة من الدوال عبر المجموعات المفتوحة α، مثل الدوال (△-المستمرة، △^*-المستمرة، △^(* *)-المستمرة، شبه △-المستمرة، شبه △^*-المستمرة، وشبه △^(* *)-المستمرة). سوف نحدد العلاقات بين أنواع المفاهيم السابقة التي سنربطها بالدوال، والاستمرارية. علاوة على ذلك، سوف نقدم بعض الأمثلة والنظريات والملاحظات والخصائص حول هذه المفاهيم الجديدة للدوال.

الكلمات المفتاحية: الدوال شبه المستمرة △، وشبه المستمرة △^*، وشبه المستمرة.△^(* *)

1. Introduction

When a problem occurs in the field of topological space, scientists and researchers race to obtain elective, solid results and apply them. The topological space has contributed to all sciences and penetrated mathematical research in a wider field, taking the point with the set, whether it was open or closed, and building its structure it with research through definition, proof, examples and its application.

2. Concepts

Definitions 2.1.

If (X, τ_x) and (Y, τ_y) are two topological spaces. Let $f: X \to Y$ is be a function

- Then f is called α-continuous function if and only if, for each A is open set in Y Thus f^{-1} (A) is α-open set in (X, τ).
- Or Then f is called α-continuous function if and only if, every open set A in Y, thus $f^{-1}(A) \subseteq Int\ Cl\ Int\ f^{-1}(A)$

Definitions 2.2.

If $f : (X, \tau_x) \rightarrow (Y, \tau_y)$ is a function.

- Thus, f is named $semi$-continuous function. When A is open set in Y, thus, $f^{-1}(A)$ is $semi$-open set in (X, τ_x), such that $f^{-1}(A) \subseteq Cl\ Int\ f^{-1}(A)$.
- Or Then, f is called $semi$-continuous for every open set U on Y, then, $f^{-1}(U)$ is $semi$-open in X.

Remark 2.3.

Every continuous function is α-continuous function, the reverse is not necessarily true. The example shows that:

Example 2.4.

Let $X = \{0,2,4,6\}$, $\tau_x = \{\emptyset, \{0\}, X\}$, $Y = \{1,3,5\}$, $\tau_y = \{\emptyset, \{1\}, Y\}$,

The sets of △-open; $\tau_x^{\triangle} = \tau_x \cup \{\{0,2\}, \{0,4\}, \{0,6\}, \{0,2,4\}, \{0,2,6\}, \{0,4,6\}\}$,

The sets of α-open on y; $\tau_y^{\triangle} = \tau_y \cup \{\{1,3\}, \{1,5\}\}$,

Let f define $f: X \rightarrow Y$, by $f(0) = f(2) = 1, f(4) = 3, f(6) = 5$,

We see f is △-continuous, but is not continuous. Since, $\{1\}$ open set on Y, and

$f^{-1}(\{1\}) = \{0,2\}$, on the other hand $\{0,2\}$ is not open set on X.

In the text result, we show that, the relation of the f is $semi$ α-continuous,

and every point $x \in X$, as in [14]

Theorem 2.5.

A function $f: X \rightarrow Y$. Then the following statement is equivalent .

a) f is $semi$ △-continuous.

b) f is $semi$ △-continuous at each point $x \in X$.

Proof :

(a) $\Longrightarrow$ (b)

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let $f : X \longrightarrow Y$ is a *semi* α-continuous.

And $x \in X$, N be open set of Y containing $f(x)$.

Then $x \in f^{-1}(N)$. also f is *semi* α-continuous.

So $M = f^{-1}(N)$ is *semi* α-open set in X having (x), therefore $f(M) \subset N$.

(b) $\Longrightarrow$ (a)

if $f : X \longrightarrow Y$ is a *semi* α-continuous for all point in X.

And N open set in Y. Let $x \in f^{-1}(N)$.

Then N is open set in Y containing $f(x)$.

By (b), at hand is *semi* α-open set M of X having x.

Since $f(x) \in f(M) \subseteq N$. Therefor $M \subseteq f^{-1}$(N).

Hence $f^{-1}(N) = \cup \{M : x \in f^{-1}(N)\}$.

Then $f^{-1}(N)$ is *semi* α-open in X.

Remark 2.6.

Each α-continuous function is *semi* $\triangle$-continuous function, The convers is not necessarily true. The example shows that.

Example 2.7.

Let $X = \{1,5,9\}$, $\tau = \{\emptyset, \{1\}, \{5\}, \{1,5\}, X\}$

The α-open sets on X, $\tau_x^{\triangle} = \tau_x$,

The *semi* α-open sets on X, $semi\ \alpha\ O\ (X) = \tau_x^{\triangle} \cup \{\{5,9\}, \{1,9\}\}$,

Let $f: X \rightarrow X$ is function define the following shape ,

by $f(1) = 1, f(5) = f(9) = 5$,

We see f is semi α-continuous function, but is not α-continuous function.

Since, $\{5\}$ is open set, however, $f^{-1}(\{5\}) = \{5,9\} \notin \tau_x^{\alpha}$

We get on a series relation between the types of functions

Let $f : (X, \tau_x) \rightarrow (Y, \tau_y)$ be a function on a topological space, then,

continuous → -continuous △ → △-continuous *semi*

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Figure 1: the relation of (continuous, △-continuous an *semi* △-continuous)

3. Concepts and Relationship via Function

We can prove the relation between the α-$c$ontinuous function, and f is *semi*-continuous by the following result,

Theorem 3.1.

If (X, τ_x) as well (Y, τ_y) are a topological spaces, and if $f: X \rightarrow Y$ be

α-$c$ontinuous function, then f is *semi*-continuous.

Proof :

Since, f is α-continuous function.

Thus, $f^{-1}(A) \subseteq Int\ Cl\ Int f^{-1}$

We have $Int\ Cl\ Int\ f^{-1}(A) \subseteq Cl\ Int\ f^{-1}(A)$,

So, $f^{-1}(A) \subseteq Cl\ Int\ f^{-1}(A)$. Therefore, f is semi-continuous.

Remark 3.2.

The convers of theorem is not necessary true in general. To get this,

We offer the previous counter example as mentioned above.

Example 3.3.

If X=Y={7,8,9}, τ_x={$\emptyset$, {7}, {8}, {7,8}, X}, $\tau_y = \{\emptyset, \{7\}, \{8,9\}, Y\}$,

Therefore, $f : X \rightarrow X$, since $X = Y$.

$f: (X, \tau_x) \rightarrow (X, \tau_y)$ be *semi*-continuous. But , f is not α-continuous.

Remark 3.4.

Every continuous function is α-continuous function, so it is *semi* △-continuous, on the other hand the convers is not necessarily true as shown by the following example.

Example 3.5.

If $X = \{0,1,3,5\}$, and $\tau_x = \{\emptyset, \{0\}, X\}$, let $y = \{2,4,6\}$, $\tau_y = \{\emptyset, \{2\}, Y\}$.

The α-open sets define on space X are,

$\tau_x^{\triangle} = \tau_x \cup \{\{0,1\}, \{0,3\}, \{0,5\}, \{0,1,3\}, \{0,1,5\}, \{0,3,5\}\}$, So, △-open sets defined on space Y are ; $\tau_y^{\triangle} = \tau_y \cup \{2,4\},\{2,6\}$

If $f: X \to Y$, defined are ; $f(0) = f(1) = 2$, $f(3) = 4, f(4) = 6$.

Since, f is △-continuous function, however it is not continuous function.

Because $\{2\}$ is open in space Y, but $f^{-1}(\{2\}) = \{0,1\}$,

When $\{0,1\}$ is not open in space X.

To fined *semi* α-continuous, $semi\ \alpha O(X) = \tau_x^{\triangle}$, and $semi\ \alpha\ O(Y) = \tau_y^{\triangle}$

Then, f is *semi* △-continuous function, but it is not continuous,

Since, $\{2\}$ is open, however $f^{-1}(\{2\}) = \{0,1\}$, is not open in X.

Remark 3.6.

every α-continuous function is *semi* △-continuous, however convers is not True in general. As, in the example.

Example 3.7.

Let $X = \{4,6,8\}$, $\tau_x = \{\emptyset, \{4\}, \{6\}, \{4,6\}, X\}$.

Then, the α-open sets in space X, $\tau_x^{\triangle} = \tau_x$.

As, the *semi* α-open sets in space X, $semi\ \alpha O(X) = \tau_x^{\triangle} \cup \{\{6,8\},\{4,8\}\}$.

Thus, $f : X \to X$, such that $f(4) = 4$, $f(6) = f(8) = 6$,

Therefore, the f is *semi* α-continuous, but it is not α-continuous.

Since, $\{6\}$ is open set, but $f^{-1}(\{6\}) = \{6,8\} \notin \tau_x^{\triangle}$.

Remark 3.8

Let $f : X \to Y$ *and* $g : Y \to Z$ are two functions, thus f as well g exist△-continuous, Thus $fog: X \to Z$, we don't need to prove α-continuous as example shows.

Example 3.9.

If $X = \{4,5,6,7\}$, $\tau_x = \{\emptyset, \{6\}, \{4,6\}, \{4,5,6\}, X\}$,

$\tau_x^{\triangle} = \tau_x \cup \{\{5,6\},\{6,7\},\{5,6,7\},\{4,6,7\}\}$, And $Y = \{0,1,2\}$, $\tau_y = \{\emptyset, \{2\}, Y\}$, $\tau_y^{\alpha} = \tau_y \cup \{\{0,2\},\{1,2\}\}$,

By, $f: X \to Y, f(x_1) = f(x_2) = 0, f(x_3) = f(x_4) = 1$.

Also, $g: Y \to Z$, $g(y_1) = g(y_3) = 6$, $g(y_2) = 4$.

Then f and g are α-continuous, but $gof: X \to X$,

Where $gof(x_1) = gof(x_2) = 6$, $gof(x_3) = gof(x_4) = 4$.

Then, gof is not $\triangle$-continuous,

Since, $\{6\}$ be open set of X, but $(gof)^{-1}\{6\} = \{4,5\}$, be not $\triangle$-open set of X.

In the next result, we show that , the relation of $f : X \to Y$ and $g : Y \to Z$ are continuous Functions, then the composition gof is continuous functions, as in [1]

Theorem 3.10.

let $f : (\mathrm{X}, \tau_x) \to (\mathrm{Y}, \tau_y)$ and $g : (\mathrm{Y}, \tau_y) \to (\mathrm{Z}, \tau_z)$ are equally continuous Function, then the composition $gof : (X, \tau_x) \to (Z, \tau_z)$ is continuous function.

Proof :

If $M \in \tau_Z$. Then, $g^{-1}(M) \in \tau_y$, (by g is continuous).

Since, $g^{-1}(M) \subseteq Y$

Therefore, $f^{-1}(g^{-1}(M)) \in \tau_x$. (by f be continuous)

And $(f^{-1} o\, g^{-1})(M) \in \tau_X$,

Thus, $(gof)^{-1}(M) \in \tau_x$, (by $(gof)^{-1} = f^{-1} o\, g^{-1}$).

Then, gof is continuous.

Remarks 3.11.

- The composition of finite number of continuous function is continuous.

To explain this. The composition of four or seven or fifty continuous functions is continuous (if f, g, h, k are continuous, so $kohogof$ is continuous).

If f:$X \to Y$, $g: Y \to Z$, are α-continuous, and the arrangement function, gof is not necessary α-continuous.

- Let $f: X \to Y$ and $g: Y \to Z$ be two functions, then:

If f and g are α-continuous,

Then, $fog: X \to Z$ need not to be α-continuous

as the following example shows:

Example 3.12.

Let $X = \{0,3,5,7\}$, $\tau_x = \{\emptyset, \{5\}, \{0,5\}, \{0,3,5\}, X\}$.

And let $Y = \{2,4,6\}$, $\tau_y = \{\emptyset, \{6\}, Y\}$.

If $\triangle$-open sets in space $X, \tau_x^{\triangle} = \tau_x \cup \{\{3,5\}, \{5,7\}, \{5,3,7\}, \{0,5,7\}\}$.

And the $\triangle$-open sets in space Y, $\tau_y^{\triangle} = \tau_y \cup \{\{2,6\}, \{4,6\}\}$.

define $f: X \to Y$; $f(0) = f(3) = 2, f(5) = f(7) = 4$.

And $g: Y \to X$; $g(2) = g(4) = 5$, $g(6) = 0$. Thus, f, g are $\triangle$-continuous.

But, $gof : X \to X$, $gof(0) = gof(3) = 5$, $gof(5) = gof(7) = 0$.

Then, gof, is not α- continuous since $\{5\}$ is open set of space X,

$(gof)^{-1}(5) = \{0,3\}$, but $\{0,3\}$ be not α-open set of space X.

Definition 3.13. [2]

If $f: X \to Y$. Then, f is called α^*-continuous, every N is $\triangle$-open set of Y**,**

thus $f^{-1}(N)$ be α-open set of X.

Definition 3.14.

Let (X, τ_x) and (Y, τ_y) be two topological spaces, and if $f: X \to Y$ is called

$semi$ α^*-continuous. If and only if each N $semi$ $\triangle$ -open set of Y. Thus,

$f^{-1}(N)$ be a $semi$ $\triangle$-open set of X.

Proposition 3.15.

A function $f : (X, \tau_x) \to (Y, \tau_y)$ be a function on topological space,

1. an open, continuous and bijective. Then, f is $\triangle^*$-continuous function.
2. Then, $\triangle^*$-continuous if and only if, $f : (X, \tau_x^{\triangle}) \to (Y, \tau_y^{\triangle})$ are continuous functions.

Proof :

Let $E \in \tau_x^{\triangle}$, to prove $f^{-1}(E) \in \tau_x^{\triangle}$. Then, $f^{-1}(E) \subseteq Int\,Cl\,Int\,f^{-1}(E)$

If $x \in f^{-1}(E) \Rightarrow f(x) \in (E)$. And $f(x) \in Int\,Cl\,Int\,E$ (since, $E \in \tau_y^{\triangle}$) .

And so, there follows N open set of Y. Since, $f(x) \in N \subseteq Cl\ Int\ E$.

And $x \in f^{-1}(N) \subseteq f^{-1}(Cl\ Int\ E)$. Then, $f^{-1}(Cl\ Int\ E) \subseteq Cl\big(f^{-1}(IntE)\big)$.

(then $f^{-1}$is continuous, which is same to f is open and bijective)

Thus, $x \in f^{-1}(N) \subseteq Cl\big(f^{-1}(Int\ E)\big)$.

Since, $x \in f^{-1}(N) \subseteq Cl(f^{-1}(IntE)) \subseteq Cl(Int(f^{-1}(E)))$, ($f$ is continuous)

Therefore, $x \in f^{-1}$(N) $\subseteq Cl(Intf^{-1}$(N)). But, $f^{-1}(N)$ is open set in X,

Thus, $x \in Int\ Cl$ (Int. $\big(f^{-1}(N)\big)$. As a result, $f^{-1}(N) \subseteq Int\ Cl$ Int$\big(f^{-1}(N)\big)$,

Then, f^{-1}(N) $\in \tau_x^{\triangle}$. therefore, f is $\triangle^*$-continuous function.

In the same way, we prove (2).

Remark 3.16.

The concepts of continuity and $\triangle^*$-continuity functions are independent. as shows example below.

Example 3.17.

If $X = \{0,2,4,6\}$, $\tau_x = \{\emptyset, \{0\}, \{2,4\}, \{0,2,4\}, X\}$. Then, $\tau_x^{\triangle} = \tau_x$.

And $Y = \{5,6\ 7\}, \tau_y = \{\emptyset, \{5\}, Y\}$. Thus, $\tau_y^{\triangle} = \tau_y \cup \{\{5,6\},\{5,7\}\}$.

Define $f : X \to Y$ by f (0)= 5 , f (2)= 6 , f (4) = f (6) = 7.

Then, f is continuous, however it is not $\triangle^*$-continuous.

Since, $\{5,6\} \in \tau_{(Y)}^{\triangle}$, but $f^{-1}\{5,6\} = \{0,2\} \notin \tau_x^{\triangle}$

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