Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM), Vol. 18, No. 2, 2024 (121-135)

Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM)

www.csmj.mosuljournals.com

Commutative Rings with Ideal Based Zero Divisor Graph of Orders 12,13 and 14

Shukur, Raad Saad[1,*] and Husam Q. Mohammad[2]

[1,2]*Department of Mathematics College of Computer Science and Mathematics University of Mosul, Mosul, Iraq*
Emails: raad.23csp29@student.uomosul.edu.iq, husamqm@uomosul.edu.iq

Article information

Article history:
Received:4/7/2024
Accepted:3/8/2024
Available online:15/12/2024

Abstract

A recent year, several studies have emerged on the graphs for commutative rings. Researchers have investigated ideal based zero-divisor graphs linked to commutative rings, delving into the characteristics of these graphs. Although significant progress has been made for rings with degrees up to 11, the exploration of this classification for degrees 12, 13, and 14 is still a subject of on-going study. In this work, we study the other type of graph of commutative ring called the ideal based zero divisor graph denoted by $\Gamma_I(R)$. J. Smith investigated the ideal based zero divisor graph of vertices less than or equal 7. In this work, also we used $\Gamma_I(R)$orders 12,13 and 14 to find all possible rings with respect to ideal I. To represent $\Gamma_I(R)$, utilize the characteristic
$|V(\Gamma_I(R))| = |I| \cdot |V(\Gamma(R/I)|$.

Keywords:
Ideal based zero divisor graph, zero divisor graph, non-local ring, finite direct product, order of a graph.

Correspondence:
Author: Shukur, Raad Saad
Email: raad.23csp29@student.uomosul.edu.iq

I. Introduction

In this paper, we consider R as a commutative ring with a unit element 1. In 2003, Redmond [1] defined a new kind of graph on R known as the ideal-based zero divisor graph, denoted by $\Gamma_I(R)$, whose vertices in $R - I$ and two vertices $r_1 \neq r_2$ are adjacent if $r_1 r_2 \in I$, this graph denoted by $\Gamma_I(R)$. This kind of graph generalizes the zero divisor graph defined in 1999 by Anderson and Livingston[2], whose vertices in $Z(R)^* = Z(R) - \{0\}$ and two vertices $r_1 \neq r_2$ are adjacent if and only if $r_1 r_2 = 0$, this graph denoted by $\Gamma(R)$. This field has been studied by numerous authors for instance, refer to sources [3],[4],[5],[6]. We denote to $|S|$ the cardinality of a set S and F_S to a field of order S, where S is a power of prime number p, a field with degree S is represented by F_S. An ideal I in a ring R is called a maximal ideal if $I \neq R$ and for any ideal A of R such that $I \subset A \subseteq R$, then $A = R$.

A ring R is called local ring if has unique maximal ideal. We denote $R[T]$ as a polynomial ring over coefficient T, which can be defined as $\{\sum_{i=0}^{\infty} a_i T^i : a_i \in R\}$, and R/I is denoted by a quotient ring or (factor ring) for more details see [7],[8]. The ring R is a direct product of the rings R_i for $i = 1,2,\dots,n$ if $R \cong R_1 \times R_2 \times \dots \times R_n = \{(r_1, r_2, \dots, r_n) : r_i \in R_i\}$ see [9].

Redmond proved the following relationship: $|V(\Gamma_I(R)| = |I| \cdot |V(\Gamma(R/I))|$. Using this mathematical expression, researchers in [10] were able to find all rings corresponding to the ideal I with the number of vertices n, where $1 \leq n \leq 7$ or n is a prime number. Additionally, researchers in [11] identified these rings when $n = 8, 9$, or 10.

In this study, we identify non-local rings corresponding to the ideal I with the number of vertices $12, 13$, or 14.

Ring with $|V(\Gamma_I(R))| = 12$:

In this section we give all possible ring with $|V(\Gamma_I(R))| = 12$ .First we give this observing.

Observing 2.1:

We consider when a non-trivial $\Gamma_I(R)$ is the graph on 12 vertices since $|V(\Gamma_I(R))| = |V(\Gamma(R/I)| \cdot |I|$ *and* $|I| \geq 2$ we get five possibilities :-

1- $|I| = 12$ *and* $|V(\Gamma(R/I)| = 1$.
2- $|I| = 6$ and $|V(\Gamma(R/I)| = 2$.
3- $|I| = 2$ and $|V(\Gamma(R/I)| = 6$.
4- $|I| = 4$ and $|V(\Gamma(R/I)| = 3$.
5- $|I| = 3$ *and* $|V(\Gamma(R/I)| = 4$.

Theorem 2.2:

Let R be a non local ring satisfied $|\Gamma_I(R)| = 12$ with $|I| = 12$. Then R corresponding the following rings in Table 1.

Table 1. $|\Gamma_I(R)| = 12$ where $|I| = 12$

Ring	Ideal	Figure
$Z_4 \times Z_2 \times Z_2 \times Z_3$ or $Z_2[T_1]/(T_1{}^2) \times Z_2 \times Z_2 \times Z_3$	$(0) \times Z_2 \times Z_2 \times Z_3$	K_{12}
$Z_4 \times Z_4 \times Z_3$ or $Z_2[T_1]/(T_1{}^2) \times Z_2[T_1]/(T_1{}^2) \times Z_3$ or $Z_4 \times Z_2[T_1]/(T_1{}^2) \times Z_3$	$(0) \times Z_4 \times Z_3$ or $(0) \times Z_2[T_1]/(T_1{}^2) \times Z_3$	K_{12}
$Z_8 \times Z_2 \times Z_3$	$(4) \times Z_2 \times Z_3$	K_{12}
$Z_2[T_1]/(T_1{}^3) \times Z_2 \times Z_3$	$(T_1{}^2) \times Z_2 \times Z_3$	K_{12}
$Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2 \times Z_2 \times Z_3$	$(T_1) \times Z_2 \times Z_3$ or $(T_2) \times Z_2 \times Z_3$ or $(T_1 + T_2) \times Z_2 \times Z_3$	K_{12}
$Z_4[T_1]/(2T_1,T_1{}^2) \times Z_2 \times Z_3$	$(2) \times Z_2 \times Z_3$ or $(T_1) \times Z_2 \times Z_3$ or $(2 + T_1) \times Z_2 \times Z_3$	K_{12}
$Z_4[T_1]/(2T_1,T_1{}^2 - 2) \times Z_2 \times Z_3$	$(2) \times Z_2 \times Z_3$	K_{12}
$Z_{16} \times Z_3$	$(4) \times Z_3$	K_{12}
$Z_2[T_1]/(T_1{}^4) \times Z_3$	$(T_1^2) \times Z_3$	K_{12}
$Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2) \times Z_3$	$(T_1) \times Z_3$ or $(T_2) \times Z_3$ or $(T_1 + T_2) \times Z_3$	K_{12}
$Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2 - T_1 \times Z_3$	$(T_1) \times Z_3$ or $(T_2) \times Z_3$ or $(T_1 + T_2) \times Z_3$	K_{12}
$Z_2[T_1,T_2]/(T_1{}^3,T_1T_2,T_2{}^2 \times Z_3$	$(T_1) \times Z_3$ or $(T_1 + T_2) \times Z_3$	K_{12}
$Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3) \times Z_3$	$(T_1,T_2) \times Z_3$ or $(T_1,T_3) \times Z_3$ or $(T_2,T_3) \times Z_3$	K_{12}
$Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3) \times Z_3$	$(T_1,T_2 + T_3)$or$(T_2,T_1 + T_3)$or$(T_3,T_1 + T_2)$	K_{12}
$Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3) \times Z_3$	$(T_1 + T_2,T_{1+}T_3) \times Z_3$	K_{12}
$Z_4[T_1]/(T_1{}^2) \times Z_3$	$(2) \times Z_3$or$(T_1) \times Z_3$or$(2 + T_1) \times Z_3$	K_{12}
$Z_4[T_1]/(T_1{}^2 + 2) \times Z_3$	$(2) \times Z_3$	K_{12}
$Z_4[T_1]/(T_1{}^2 + 1) \times Z_3$	$(2) \times Z_3$	K_{12}
$Z_4[T_1]/(T_1{}^2 + 3) \times Z_3$	$(2) \times Z_3$or$(1 + T_1) \times Z_3$or$(1 + 3T_1) \times Z_3$	K_{12}
$Z_4[T_1]/(T_1{}^2 + T_1 + 1) \times Z_3$	$(2) \times Z_3$	K_{12}
$Z_4[T_1]/(T_1{}^3,2T_1) \times Z_3$	$(T_1) \times Z_3$or$(2 + T_1) \times Z_3$or$(2,T_1) \times Z_3$	K_{12}
$Z_4[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2 \times Z_3$	$(2,T_1) \times Z_3$or$(2,T_2) \times Z_3$or$(T_1,T_2) \times Z_3$or$(2,T_1 + T_2) \times Z_3$or$(T_1,2 + T_2) \times Z_3$or$(T_2,2 + T_1) \times Z_3$or$(2 + T_1,2 + T_2) \times Z_3$	K_{12}
$(2,T_1) \times Z_3$or$(2,T_2) \times Z_3$or$(T_1,T_2) \times Z_3$or$(2,T_1 + T_2) \times Z_3$or$(T_1,2 + T_2) \times Z_3$or$(T_2,2 + T_1) \times Z_3$or$(2 + T_1,2 + T_2) \times Z_3$	$Z_4[T_1,T_2]/(T_1{}^2 - 2,T_2^2,T_1T \times Z_3$	K_{12}
$Z_8[T_1]/(2T_1,T_1{}^2) \times Z_3$	$(2) \times Z_3$or$(4,T_1) \times Z_3$	K_{12}
$Z_8[T_1]/(2T_1,T_1{}^2 + 4) \times Z_3$	$(2) \times Z_3$	K_{12}

Proof:

Since $|\Gamma_I(R)| = 12$ and $|I| = 12$, then $|V(\Gamma(R/I))| = 1$, by [12],[13] , $R/I \cong Z_4$ or $Z_2[T_1]/(T_1{}^2)$and so $|R/I| = 4$
Since $|R| = |I|.|R/I|$, which implies that $|R| = 4.12 = 48$.
We note that R direct product of local rings.
So $R \cong R_1 \times R_2 \times R_3 \times \ldots \times R_n$, where R_i local ring .
Since $|R| = 48$, and R non local ring , then $2 \leq n \leq 5$.
If $n = 5$, then $R \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$. which implies that $R \cong Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3$. Additionally $|I| = 12$, then $I = (0) \times (0) \times Z_2 \times Z_2 \times Z_3$
$R/I \cong Z_2 \times Z_2 \ncong Z_4$or $Z_2[T_1]/(T_1{}^2)$. which is a contradiction .
If $n = 4$, then
$R \cong R_1 \times R_2 \times R_3 \times R_4$, and we have $R \cong Z_4 \times Z_2 \times Z_2 \times Z_3$or $Z_2[T_1]/(T_1{}^2) \times Z_2 \times Z_2 \times Z_3$ or $F_4 \times Z_2 \times Z_2 \times Z_3$
If $I \cong 0 \times Z_2 \times Z_2 \times Z_3 \subseteq Z_4 \times Z_2 \times Z_2 \times Z_3$,then $R/I \cong Z_4$. Similarly if $I \subseteq Z_2[T_1]/(T_1{}^2) \times Z_2 \times Z_2 \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1{}^2)$.
If $I \subseteq F_4 \times Z_2 \times Z_2 \times Z_3$, then $R/I \cong F_4$, which is a contradiction .
If $I \cong (2) \times Z_2 \times 0 \times Z_3$, then $R/I = Z_2 \times Z_2$,which is a contradiction .

Similarly when $I \cong (T_1) \times Z_2 \times (0) \times Z_3$, then $R/I = Z_2 \times Z_2$, which is a contradiction.
If $n = 3$, then $R \cong R_1 \times R_2 \times R_3$, so that two sub-cases
Sub case(a) $R \cong A_1 \times A_2 \times Z_3$,where $A_1, A_2 \cong Z_4$ or $Z_2[T_1]/(T_1^2)$, if $I_1 \cong (0) \times A_2 \times Z_3$, then $R/I \cong A_1$
If $I_2 \cong (2) \times (2) \times Z_3$, then $R/I \cong Z_2 \times Z_2$, which is a contradiction .
Sub case(b) $R \cong A \times Z_2 \times Z_3$, where $A \cong Z_8$ or $Z_2[T_1]/(T_1^3)$or F_8 or $Z_2[T_1,T_2]/(T_1^2, T_1T_2, T_2^2)$ or $Z_4[T_1]/(2T_1, T_1^2)$ or $Z_4[T_1]/(2T_1, T_1^2 - 2)$.
We note that . If $A \cong Z_8$, then $R \cong A \times Z_2 \times Z_3$
If $I \cong (4) \times Z_2 \times Z_3$, then $R/I \cong Z_4$
$R \cong F_8 \times Z_2 \times Z_3$, which is a contradiction .
Similar rings if $R \cong Z_2[T_1]/(T_1^3) \times Z_2 \times Z_3$, $Z_2[T_1,T_2]/(T_1^2, T_1T_2, T_1^2) \times Z_2 \times Z_3$,
then $I \cong (T_1^2) \times Z_2 \times Z_3$, $(T_1) \times Z_2 \times Z_3$ or $(T_2) \times Z_2 \times Z_3$ or $(T_1 + T_2) \times Z_2 \times Z_3$, respectively , we get ,
$R/I \cong Z_2[T_1]/(T_1^2)$, $Z_2[T_2]/(T_2^2)$ or $Z_2[T_1]/(T_1^2)$.
If $A \cong Z_4[T_1]/(2T_1, T_1^2)$, then $R \cong A \times Z_2 \times Z_3$
If $I \cong (2) \times Z_2 \times Z_3$, $(T_1) \times Z_2 \times Z_3$, $(2 + T_1) \times Z_2 \times Z_3$, respectively , and we get
$R/I \cong Z_2[T_1]/(T_1^2)$, Z_4 , Z_4 .
If $A \cong Z_4[T_1]/(2T_1, T_1^2 - 2) \times Z_2 \times Z_3$, then $R \cong A \times Z_2 \times Z_3$
If $I \cong (2) \times Z_2 \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1^2)$.
If $n = 2$, then $R \cong R_1 \times R_2$, s.t $|R_1| = p^n$and $|R_2| = 3$ since $|R_1| = 16 = p^t$, then
there are 21 rings satisfied this condition , where $R_1 \cong F_{16}$, Z_{16} , $Z_2[T_1]/(T_1^4)$, $Z_2[T_1,T_2]/(T_1^2, T_2^2)$, $Z_2[T_1,T_2]/(T_1^2, T_2^2 - T_1T_2)$, $Z_2[T_1,T_2]/(T_1^3, T_1T_2, T_2^2)$, $Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2$, $Z_4[T_1]/(T_1^2)$, $Z_4[T_1]/(T_1^2 + 2)$, $Z_4[T_1]/(T_1^2 + 1)$, $Z_4[T_1]/(T_1^2 + 3)$, $Z_4[T_1]/(T_1^2 + T_1 + 1)$, $Z_4[T_1]/(T_1^3, 2T_1)$, $Z_4[T_1]/(T_1^3 - 2, 2T_1)$, $Z_4[T_1,T_2]/(T_1^2, T_2^2, T_1T_2, 2T_1, 2T_2)$, $Z_4[T_1,T_2]/(T_1^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2)$, $Z_4[T_1,T_2]/(T_1^2, T_2^2, T_1T_2 - 2, 2T_1, 2T_2)$, $Z_4[T_1,T_2]/(T_1^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2)$, $Z_8[T_1]/(2T_1, T_1^2)$, $Z_8[T_1]/(2T_1, T_1^2 + 4)$.

1- $R \cong Z_{16} \times Z_3$, if $I \cong (4) \times Z_3$, then $R/I \cong Z_4$.
2- $R \cong F_{16} \times Z_3$ or $F_4[T_1]/(T_1^2)$, which is a contradiction .
3- $R \cong Z_2[T_1]/(T_1^4) \times Z_3$, if $I = (T_1^2) \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1^2)$.
4- $R \cong Z_2[T_1,T_2]/(T_1^2, T_2^2) \times Z_3$, if $I_1 = (T_1) \times Z_3$, then $R/I_1 \cong Z_2[T_2]/(T_2^2)$.
If $I_2 \cong (T_2) \times Z_3$, then $R/I_2 \cong Z_2[T_1]/(T_1^2)$, if $I_3 \cong (T_1 + T_2) \times Z_3$, then $R/I_3 \cong Z_2[T_1]/(T_1^2)$.
5- $R \cong Z_2[T_1,T_2]/(T_1^2, T_2^2 - T_1T_2) \times Z_3$,if $I_1 \cong (T_1) \times Z_3$, then $R/I_1 \cong Z_2[T_2]/(T_2^2)$, if $I_2 \cong (T_2) \times Z_3$, then $R/I_2 \cong Z_2[T_1]/(T_1^2)$,
If $I_3 \cong (T_1 + T_2) \times Z_3$, then $R/I_3 \cong Z_2[T_1]/(T_1^2)$.
6- $R \cong Z_2[T_1,T_2]/(T_1^3, T_1T_2, T_2^2) \times Z_3$, if $I_1 \cong (T_1) \times Z_3$, then $R/I_1 \cong Z_2[T_2]/(T_2^2)$, if $I_2 \cong (T_1 + T_2) \times Z_3$, then $R/I_2 \cong Z_2[T_2]/(T_2^2)$.
7- $R \cong Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2 \times Z_3$, if $I_1 \cong (T_1,T_2) \times Z_3$, then $R/I_1 \cong Z_2[T_3]/(T_3^2)$
If $I_2 = (T_1,T_3) \times Z_3$, then $R/I_2 \cong Z_2[T_2]/(T_2^2)$, if $I_3 \cong (T_2,T_3) \times Z_3$, then $R/I_3 \cong Z_2[T_1]/(T_1^2)$, if $I_4 \cong (T_1, T_2 + T_3) \times Z_3$, then $R/I_4 \cong Z_2[T_3]/(T_3^2)$, if $I_5 \cong (T_2, T_1 + T_3) \times Z_3$, then $R/I_5 \cong Z_2[T_3]/(T_3^2)$, if $I_6 \cong (T_3, T_1 + T_2) \times Z_3$, then $R/I_6 \cong Z_3[T_2]/(T_2^2)$, if $I_7 \cong (T_1 + T_2, T_{1+}T_3) \times Z_3$, then $R/I_7 \cong Z_2[T_1]/(T_1^2)$
8- $R \cong Z_4[T_1]/(T_1^2) \times Z_3$, if $I_1 \cong (2) \times Z_3$, then $R/I_1 \cong Z_2[T_1]/(T_1^2)$,
if $I_2 \cong (T_1) \times Z_3$, then $R/I_2 \cong Z_4$, if $I_3 \cong (2 + T_1) \times Z_3$, then $R/I_3 \cong Z_4$.
9- $R \cong Z_4[T_1]/(T_1^2 + 2) \times Z_3$, if $I \cong (2) \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1^2)$.
10- $R \cong Z_4[T_1]/(T_1^2 + 1) \times Z_3$, if $I \cong (2) \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1^2)$.
11- $R \cong Z_4[T_1]/(T_1^2 + 3) \times Z_3$, if $I \cong (2) \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1^2)$.
12- $R \cong Z_4[T_1]/(T_1^2 + T_1 + 1) \times Z_3$, if $I \cong (2) \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1^2)$ that is a contradiction .
13- $R \cong Z_4[T_1]/(T_1^3, 2T_1) \times Z_3$, if $I_1 \cong (T_1) \times Z_3$, then $R/I_1 \cong Z_4$,
If $I_2 \cong (2 + T_1) \times Z_3$, then $R/I_2 \cong Z_4$, $I_3 \cong (2, T_1) \times Z_3$, then $R/I_3 \cong Z_2$.
14- $R \cong Z_4[T_1]/(T_1^3 - 2, 2T_1) \times Z_3$, if $I \cong (T_1^2) \times Z_3$, then $R/I \cong Z_4[T_1]/(T_1^2 - 2, 2T_1)$ that is a contradiction .
15- $R \cong Z_4[T_1,T_2]/(T_1^2, T_2^2, T_1T_2, 2T_1, 2T_2) \times Z_3$, if $I_1 \cong (2, T_1) \times Z_3$, then $R/I_1 \cong Z_2[T_2]/(T_2^2)$, if $I_2 \cong (2, T_2) \times Z_3$, then $R/I_2 \cong Z_2[T_1]/(T_1^2)$, if $I_3 \cong (T_1, T_2) \times Z_3$, then $R/I_3 \cong Z_4$, if $I_4 \cong (2, T_1 + T_2) \times Z_3$, then $R/I_4 \cong Z_2[T_1]/(T_1^2)$, if $I_5 \cong (T_1, 2 + T_2) \times Z_3$,then $R/I_5 \cong Z_4$, if $I_6 \cong (T_2, 2 + T_1) \times Z_3$, then $R/I_6 \cong Z_4$, if $I_7 \cong (2 + T_1, 2 + T_2) \times Z_3$, then $R/I_7 \cong Z_4$.
16- $R \cong Z_4[T_1,T_2]/(T_1^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2) \times Z_3$, if $I_1 \cong (T_1) \times Z_3$, then $R/I_1 \cong Z_4[T_2]/(T_2^2, 2T_2)$ that is a contradiction , if $I_2 \cong (T_1 + T_2) \times Z_3$, then $R/I_2 \cong Z_4[T_2]/(T_2^2, 2T_2)$ that is a contradiction , if $I_3 \cong (2, T_2) \times Z_3$, then $R/I_3 \cong Z_2[T_1]/(T_1^2)$.
17- $R \cong Z_4[T_1,T_2]/(T_1^2, T_2^2, T_1T_2 - 2, 2T_1, 2T_2) \times Z_3$, if $I_1 \cong (T_1) \times Z_3$, then $R/I_1 \cong Z_4[T_2]/(T_2^2, 2T_2)$ that is a contradiction , if $I_2 \cong (T_2) \times Z_3$, then $R/I_2 \cong Z_4[T_2]/(T_1^2, 2T_1)$ that is a contradiction if $I_3 \cong (T_1 + T_2) \times Z_3$, then $R/I_3 \cong Z_4[T_2]/(T_2^2, 2T_2)$ that is a contradiction .
18- $R \cong Z_4[T_1,T_2]/(T_1^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2) \times Z_3$, if $I_1 \cong (T_1) \times Z_3$, then $R/I_1 \cong Z_4[T_2]/(T_2^2 - 2, 2T_2)$ that is a contradiction , if $I_2 \cong (T_2) \times Z_3$, then $R/I \cong Z_4[T_1]/(T_1^2 - 2, 2T_1)$ that is a contradiction , if $I_3 \cong$

$(T_1+T_2)\times Z_3$, then $R/I_3 \cong Z_4[T_2]/(T_2{}^2-2,2T_2)$.

19- $R \cong Z_8[T_1]/(2T_1,T_1{}^2)\times Z_3$, if $I_1 \cong (2)\times Z_3$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2)$,if $I_2 \cong (2+T_1)\times Z_3$, then $R/I_2 \cong Z_8$ that is a contradiction , if $I_3 \cong (4,T_1)\times Z_3$, then $R/I_3 \cong Z_4$.

20- $R \cong Z_8[T_1]/(2T_1,T_1{}^2+4)\times Z_3$, if $I_1 \cong (2)\times Z_3$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2)$

if $I_2 \cong (T_1)\times Z_3$, then $R/I_2 \cong Z_8$ that is a contradiction , if $I_3 \cong (2+T_1)\times Z_3$, then $R/I_3 \cong Z_8$ that is a contradiction .■

Theorem 2.3:

Let R be a non local ring satisfied $|\Gamma_I(R)| = 12$ with $|I| = 6$. Then R corresponding the following rings in Table 2.

Table 2- $|\Gamma_I(R)| = 12$ where $|I| = 6$

Ring	Ideal	Figure
$Z_2\times Z_9\times Z_3$ or $Z_2\times Z_3[T_1]/(T_1{}^2)\times Z_3$	$Z_2\times(0)\times Z_3$	K_{12}
$Z_2\times Z_{27}$	$Z_2\times(9)$	K_{12}
$Z_2\times Z_3[T_1]/(T_1{}^3)$	$Z_2\times(T_1{}^2)$	K_{12}
$Z_2\times Z_3[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$	$Z_2\times(T_1)$or$Z_2\times(T_2)$or$Z_2\times(T_1+T_2)$or$Z_2\times(2T_1+T_2)$	K_{12}
$Z_2\times Z_9[T_1]/(3T_1,T_1{}^2)$	$Z_2\times(3)$or$Z_2\times(T_1)$or$Z_2\times(3+T_1)$or$Z_2\times(3+2T_1)$	K_{12}
$Z_2\times Z_9[T_1]/(3T_1,T_1{}^2-3)$	$Z_2\times(3)$	K_{12}
$Z_2\times Z_9[T_1]/(3T_1,T_1{}^2-6)$	$Z_2\times(3)$	K_{12}
$Z_2\times Z_2\times Z_2\times Z_3$	$(0)\times(0)\times Z_2\times Z_3$	$K_{6,6}$
$Z_4\times Z_2\times Z_3$ or $Z_2[T_1]/(T_1{}^2)\times Z_2\times Z_3$	$(2)\times(0)\times Z_3$or$(T_1)\times(0)\times Z_3$	Fig 1

Proof:

Since $|\Gamma_I(R)| = 12$ and $|I| = 6$, then $|V(\Gamma(R/I))| = 2$, by [2] $R/I \cong Z_9$, $Z_3[T_1]/(T_1^2)$ or $Z_2\times Z_2$ and so,

Sub case(a) If $R/I \cong Z_9$ or $Z_3[T_1]/(T_1)$, then $|R/I| = 9$ so

$|R| = |I|.|R/I|$

$|R| = 54$. Therefor $R \cong R_1\times R_2\times \dots\times R_n$, $n = 2,3$ or 4 .

If $n = 4$, then $R \cong Z_2\times Z_3\times Z_3\times Z_3$

If $I \cong Z_2\times Z_3 \times 0\times 0$, then $R/I \cong Z_3\times Z_3 \ncong Z_9$ or $Z_3[T_1]/(T_1{}^2)$ that is a contradiction .

If $n = 3$, then $R \cong Z_2\times Z_9\times Z_3$ or $Z_2\times Z_3[T_1]/(T_1{}^2)\times Z_3$, Since $|I| = 6$.

If $I_1 \cong Z_2\times(0)\times Z_3$, then $R/I_1 \cong Z_9 \cong Z_9$ or $Z_3[T_1]/(T_1{}^2)$, if $I_2 \cong Z_2\times(3)\times(0)$, then $R/I_2 \cong 0\times Z_3\times Z_3 \ncong Z_9$ or $Z_3[T_1]/(T_1{}^2)$, that is a contradiction .

If $I_3 \cong Z_2\times(T_1)\times(0)$, then $R/I_3 \cong Z_3\times Z_3$ that is a contradiction .

If $n = 2$, then $R \cong R_1\times R_2$, s.t $|R_2| = p^3$ and $|R_1| = 2$ .since $|R_2| = 27 = p^n$, then there are 6 rings satisfied this condition . $R_2 \cong F_{27}, Z_{27}$, $Z_3[T_1]/(T_1{}^3)$, $Z_3[T_1,T_2]/(T_1{}^2,T_1T_2,T_2^2)$, $Z_9[T_1]/(3T_1,T_1{}^2)$, $Z_9[T_1]/(3T_1,T_1{}^2-3)$ Or $Z_9[T_1]/(3T_1,T_1{}^2-6)$.

1- $R \cong Z_2\times Z_{27}$, if $I \cong Z_2\times(9)$, then $R/I \cong Z_9$.

2- $R \cong Z_2\times Z_3[T_1]/(T_1{}^3)$, if $I \cong Z_2\times(T_1{}^2)$, then $R/I \cong Z_3[T_1]/(T_1{}^2)$.

3- $R \cong Z_2\times Z_3[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$, if $I_1 \cong Z_2\times(T_1)$, then $R/I_1 \cong Z_3[T_2]/(T_2{}^2)$, if $I_2 \cong Z_2\times(T_2)$, then $R/I_2 \cong Z_3[T_1]/(T_1{}^2)$, if $I_3 \cong Z_2\times(T_1+T_2)$, then $R/I_3 \cong Z_3[T_1]/(T_1{}^2)$, if $I_4 \cong Z_2\times(2T_1+T_2)$, then $R/I_1 \cong Z_3[T_1]/(T_1{}^2)$.

4- $R \cong Z_2\times Z_9[T_1]/(3T_1,T_1{}^2)$, if $I_1 \cong Z_2\times(3)$, then $R/I_1 \cong Z_3[T_1]/(T_1{}^2)$, if $I_2 \cong Z_2\times(T_1)$, then $R/I_2 \cong Z_9$, if $I_3 \cong Z_2\times(3+T_1)$, then $R/I_1 \cong Z_9$, if $I_4 \cong Z_2\times(3+2T_1)$, then $R/I_4 \cong Z_9$.

5- $R \cong Z_2\times Z_9[T_1]/(3T_1,T_1{}^2-3)$, if $I \cong Z_2\times(3)$, then $R/I \cong Z_3[T_1]/(T_1{}^2)$.

6- $R \cong Z_2\times Z_9[T_1]/(3T_1,T_1{}^2-6)$, if $I \cong Z_2\times(3)$, then $R/I \cong Z_3[T_1]/(T_1{}^2)$.

7- $R \cong Z_2\times F_{27}$, which is a contradiction .

Sub case(b) If $R/I \cong Z_2\times Z_2$, then $|R/I| = 4$, so

$|R| = |I|.|R/I|$

$|R| = 6.4 = 24$. Therefor $R \cong R_1\times R_2\times\dots\times R_n$, $n = 2,3$ or 4 .

If $n = 4$, then $R \cong Z_2\times Z_2\times Z_2\times Z_3$

If $I \cong (0)\times(0)\times Z_2\times Z_3$, then $R/I \cong Z_2\times Z_2$

If $n = 3$, then $R \cong R_1\times R_2\times R_3$, $\Rightarrow R \cong Z_4\times Z_2\times Z_3$ or $Z_2[T_1]/(T_1{}^2)\times Z_2\times Z_3$

If $I_1 \cong (0)\times Z_2\times Z_3$, then $R/I_1 \cong Z_4$ that is a contradiction .

If $I_2 \cong (2)\times(0)\times Z_3$, then $R/I_2 \cong Z_2\times Z_2$, thus $\Gamma_I(R) \cong$ Fig 1 .

If $I \cong (T_1)\times(0)\times Z_3 \subseteq Z_2[T_1]/(T_1{}^2)\times Z_2\times Z_3$, then $R/I \cong Z_2\times Z_2$, thus $\Gamma_I(R) \cong$ fig 1 .

If $n = 2$, then $R \cong R_1\times R_2$, s.t $|R_1| = p^3$ and $|R_2| = 3$ .since $|R_1| = 8 = p^t$, then there are 6 rings satisfied this condition .

$R_2 \cong Z_8$, F_8 , $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$, $Z_4[T_1]/(2T_1,T_1{}^2)$, $Z_4[T_1]/(2T_1,T_1{}^2-2)\times Z_3$.

1- $R \cong Z_8\times Z_3$, if $I = (4)\times Z_3$, then $R/I \cong Z_4$, that is a contradiction .

2- $R \cong F_8\times Z_3$, that is a contradiction .

3- $R \cong Z_2[T_1]/(T_1{}^3)\times Z_3$, if $I \cong (T_1^2)\times Z_3$, then $R/I \cong Z_2[T_1]/(T_1{}^2)$, that is a contradiction .

4- $R \cong Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)\times Z_3$, if $I_1 \cong (T_1)\times Z_3$, then $R/I_1 \cong Z_2[T_2]/(T_2{}^2)$, that is a

contradiction . if $I_2 \cong (T_2) \times Z_3$, then $R/I_2 \cong Z_2[T_1]/(T_1{}^2)$, , that is a contradiction .
if $I_3 \cong (T_1 + T_2) \times Z_3$, then $R/I_3 \cong Z_2[T_2]/(T_2{}^2)$, that is a contradiction .

5- $R \cong Z_4[T_1]/(2T_1, T_1{}^2) \times Z_3$, if $I_1 \cong (2) \times Z_3$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2)$, that is a contradiction .
if $I_2 \cong (T_1) \times Z_3$, then $R/I_2 \cong Z_4$, , that is a contradiction . if $I_3 \cong (2 + T_1) \times Z_3$, then $R/I_3 \cong Z_4$, that is a contradiction .

6- $R \cong Z_4[T_1]/(2T_1, T_1{}^2 - 2) \times Z_3$

If $I \cong (2) \times Z_3$, then $R/I \cong Z_2[T_1]/(T_1{}^2)$, that is a contradiction . ■

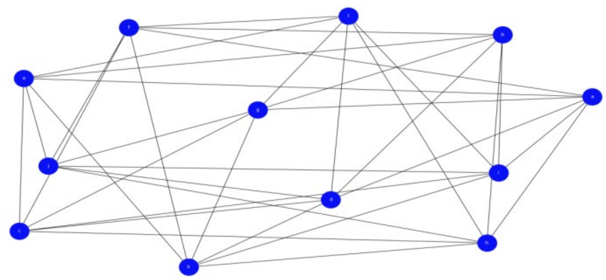

Fig 1. $\Gamma_I(R)$, where $I = (2) \times (0) \times Z_3, (T_1) \times (0) \times Z_3$ or $Z_2 \times (0) \times (0)$ and $R = Z_4 \times Z_2 \times Z_3,\ Z_2[T_1]/(T_1{}^2) \times Z_2 \times Z_3$ or $Z_2 \times F_4 \times F_4$.

Theorem 2.4:

Let R be a non-local ring satisfied $|\Gamma_I(R)| = 12$ with $|I| = 2$. Then R corresponding the following rings in Table 3.

Table 3. $|\Gamma_I(R)| = 12$ where $|I| = 2$

Ring	Ideal	Figure
$Z_2 \times Z_3 \times Z_5$	$Z_2 \times (0) \times (0)$	Fig 2
$Z_2 \times Z_2 \times Z_2 \times Z_2$	$Z_2 \times (0) \times (0) \times (0)$	Fig 3
$Z_2 \times F_4 \times F_4$	$Z_2 \times (0) \times (0)$	Fig 1
$Z_2 \times Z_{49}$ or $Z_2 \times Z_7[T_1]/(T_1{}^2)$	$Z_2 \times (0)$	K_{12}

Proof:

Since $|\Gamma_I(R)| = 12$ and $|I| = 2$, then $|V(\Gamma(R/I))| = 6$, by [14] $R/I \cong Z_{49}$ or $Z_7[T_1]/(T_1{}^2)$, $Z_2 \times Z_2 \times Z_2$, $Z_3 \times Z_5$, $F_4 \times F_4$, and so

Sub case(a) When $R/I \cong Z_3 \times Z_5$, then $|R| = 30$. Therefor $R \cong R_1 \times R_2 \times \dots \times R_n$, $n = 2$ or 3 .
If $n = 3$, then $R \cong Z_2 \times Z_3 \times Z_5$, if $I \cong Z_2 \times (0) \times (0)$, then $R/I \cong Z_3 \times Z_5$, thus $\Gamma_I(R) \cong$ figure 2 .
If $n = 2$, then $R \cong Z_2 \times Z_{15}$
If $I \cong Z_2 \times 0$, then $R/I \cong Z_{15}$ that is a contradiction.

Sub case(b) If $R/I \cong Z_2 \times Z_2 \times Z_2$, If $|R/I| = 8$, $|I| = 2$, then $|R| = 16$.Therefor $R \cong R_1 \times R_2 \times \dots \times R_n$, $n = 2,3$ or 4 .
If $n = 4$, then $R \cong Z_2 \times Z_2 \times Z_2 \times Z_2$
If $I = Z_2 \times (0) \times (0) \times (0)$, then $R/I \cong Z_2 \times Z_2 \times Z_2$, thus $\Gamma_I(R) \cong$ figure 3 .
If $n = 3$, then $R \cong Z_2 \times Z_4 \times Z_2$ or $R \cong Z_2 \times Z_2[T_1]/(T_1{}^2) \times Z_2$
If $I = Z_2 \times (0) \times (0)$, then $R/I \cong (0) \times Z_2 \times Z_4$ or $Z_2[T_1]/(T_1{}^2)$ that is contradiction .
If $n = 2$, then $R \cong R_1 \times R_2$, s.t $|R_2| = p^3$ and $|R_1| = 2$ .since $|R_2| = 8 = p^t$, then there are 6 rings $R_2 \cong Z_8$, F_8 , $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2 - 2) \times Z_3$ but they are all contradiction .

Sub case(c) If $R/I \cong F_4 \times F_4$, If $|R/I| = 16$, $|I| = 2$, then $|R| = 32$. Therefor $R \cong R_1 \times R_2 \times \dots \times R_n$, $n = 2,3,4$ or 5 .
If $n = 5$, then $R \cong Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$
If $I = Z_2 \times (0) \times (0) \times (0) \times (0)$, then $R/I \cong Z_2 \times Z_2 \times Z_2 \times Z_2$ that is a contradiction .
If $n = 4$, then $R \cong Z_2 \times Z_2 \times Z_2 \times Z_4$ or $Z_2 \times Z_2 \times Z_2 \times Z_2[T_1]/(T_1{}^2)$
If $I = Z_2 \times (0) \times (0) \times (0)$, then $R/I \cong Z_2 \times Z_2 \times Z_4$ or $Z_2 \times Z_2 \times Z_2[T_1]/(T_1{}^2)$ that is a contradiction.
If $n = 3$, then $R \cong Z_2 \times Z_2 \times Z_8$ or $Z_2 \times A_1 \times A_2$, when $A_1, A_1 \cong Z_4$, $Z_2[T_1]/(T_1{}^2)$ or F_4 .
$R \cong Z_2 \times Z_2 \times A$, where $A \cong Z_8$, F_8 , $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2)$ or $Z_4[T_1]/(2T_1, T_1{}^2 - 2) \times Z_3$.
If $I = Z_2 \times (0) \times (0)$, then $R/I \cong (0) \times Z_2 \times Z_8$, F_8 , $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2 - 2) \times Z_3$ that is a contradiction .
$R \cong Z_2 \times A_1 \times A_2$ when $A_1, A_1 \cong Z_4$, $Z_2[T_1]/(T_1{}^2)$, if $I = Z_2 \times (0) \times (0)$, then $R/I \cong (0) \times A_1 \times A_2$ that is contradiction .
$R \cong Z_2 \times F_4 \times F_4$, if $I = Z_2 \times (0) \times (0)$, then $R/I \cong F_4 \times F_4$, thus $\Gamma I(R) \cong$ figure 1 .

Sub case(d) If $R/I \cong Z_{49}$ or $Z_7[T_1]/(T_1{}^2)$, If $|R/I| = 49$, $|I| = 2$, then $|R| = 98$. Therefor $R \cong R_1 \times R_2 \times \dots \times R_n$, $n = 2$ or 3 .
If $n = 3$, then $R \cong Z_2 \times Z_7 \times Z_7$, If $I = Z_2 \times (0) \times (0)$, then $R/I \cong Z_7 \times Z_7$ that is a contradiction .
If $n = 2$, then $R \cong Z_2 \times Z_{49}$ or $Z_2 \times Z_7[T_1]/(T_1{}^2)$
If $I = Z_2 \times (0)$, then $R/I \cong Z_{49}$ or $Z_7[T_1]/(T_1{}^2)$. ■

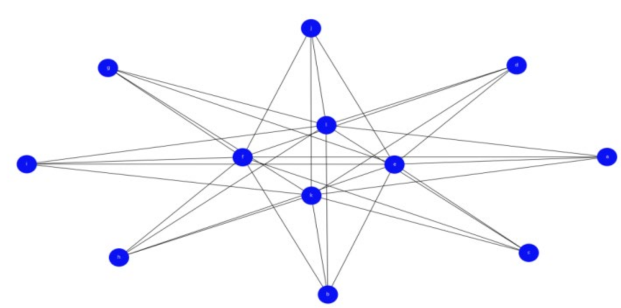

Fig 2. $\Gamma_I(R)$, where $I = Z_2 \times (0) \times (0), Z_2 \times Z_2 \times (0) \times (0), Z_4 \times (0) \times (0), (2) \times Z_2 \times (0)$ or $(T_1) \times Z_2 \times (0)$ and $R = Z_2 \times Z_3 \times Z_5, Z_4 \times Z_2 \times Z_3, Z_2[T_1]/(T_1{}^2) \times Z_2 \times Z_3$ or

$Z_2 \times Z_2 \times Z_2 \times Z_3$.

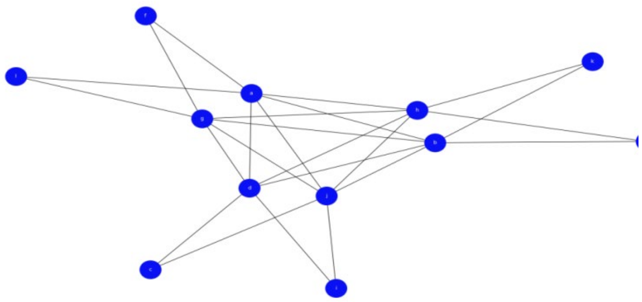

Fig 3. $\Gamma_I(R)$, where $I = Z_2 \times (0) \times (0) \times (0)$ and $R = Z_2 \times Z_2 \times Z_2 \times Z_2$

Theorem 2.5:

Let R be a non local ring satisfied $|\Gamma_I(R)| = 12$ with $|I| = 4$. Then R corresponding the following rings in Table 4.

Table 4. $|\Gamma_I(R)| = 12$ where $|I| = 4$

Ring	Ideal	Figure
$Z_2 \times Z_2 \times Z_2 \times Z_3$	$Z_2 \times Z_2 \times (0) \times (0)$	Fig 2
$Z_4 \times Z_2 \times Z_3$ or $Z_2[T_1]/(T_1^2) \times Z_2 \times Z_3$	$Z_4 \times (0) \times (0)$ or $(2) \times Z_2 \times (0)$ or $(T_1) \times Z_2 \times (0)$	Fig 2
$Z_3 \times Z_8$	$(0) \times (2)$	$K_{4,8}$
$Z_3 \times Z_2[T_1]/(T_1^3)$	$(0) \times (T_1)$	$K_{4,8}$
$Z_3 \times Z_2[T_1,T_2]/(T_1^2,T_1T_2,T_2^2)$	$(0) \times (T_1,T_2)$	$K_{4,8}$
$Z_3 \times Z_4[T_1]/(2T_1,T_1^2)$	$(0) \times (2,T_1)$	$K_{4,8}$
$Z_3 \times Z_4[T_1]/(2T_1,T_1^2-2)$	$(0) \times (T_1)$	$K_{4,8}$
$Z_2 \times Z_2 \times Z_8$	$Z_2 \times Z_2 \times (0)$	Fig 4
$Z_2 \times Z_2 \times Z_2[T_1]/(T_1^3)$	$Z_2 \times Z_2 \times (0)$	Fig 4
$Z_2 \times Z_2 \times Z_2[T_1,T_2]/(T_1^2,T_1T_2,T_2^2)$	$Z_2 \times Z_2 \times (0)$	K_{12}
$Z_2 \times Z_2 \times Z_4[T_1]/(2T_1,T_1^2)$	$Z_2 \times Z_2 \times (0)$	K_{12}
$Z_2 \times Z_2 \times Z_4[T_1]/(2T_1,T_1^2-2)$	$Z_2 \times Z_2 \times (0)$	Fig 4
$Z_2[T_1]/(T_1^4) \times Z_2$	$(T_1^3) \times Z_2$	Fig 4
$Z_2[T_1,T_2]/(T_1^2,T_2^2) \times Z_2$	$(T_1T_2) \times Z_2$	K_{12}
$Z_2[T_1,T_2]/(T_1^2,T_2^2-T_1T_2) \times Z_2$	$(T_1T_2) \times Z_2$	K_{12}
$Z_2[T_1,T_2]/(T_1^3,T_1T_2,T_2^2) \times Z_2$	$(T_1^2) \times Z_2$	K_{12}
$Z_2[T_1,T_2]/(T_1^3,T_1T_2,T_2^2) \times Z_2$	$(T_2) \times Z_2$	Fig 4
$Z_2[T_1,T_2]/(T_1^3,T_1T_2,T_2^2) \times Z_2$	$(T_2+T_1^2) \times Z_2$	Fig 4
$Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2 \times Z_2$	$(T_1) \times Z_2$ or $(T_2) \times Z_2$ or $(T_3) \times Z_2$ or $(T_1+T_2) \times Z_2$ or $(T_1+T_3) \times Z_2$ or $(T_2+T_3) \times Z_2$ or $(T_1+T_2+T_3) \times Z_2$	K_{12}
$Z_4[T_1]/(T_1^2) \times Z_2$	$(2T_1) \times Z_2$	K_{12}
$Z_4[T_1]/(T_1^2+2) \times Z_2$	$(2T_1) \times Z_2$	Fig 4
$Z_4[T_1]/(T_1^2+1) \times Z_2$	$(2+2T_1) \times Z_2$	Fig 4
$Z_4[T_1]/(T_1^2+3) \times Z_2$	$(2+2T_1) \times Z_2$	K_{12}
$Z_4[T_1]/(T_1^3,2T_1) \times Z_2$	$(T_1^2) \times Z_2$	K_{12}
$Z_4[T_1]/(T_1^3,2T_1) \times Z_2$	$(2) \times Z_2$	Fig 4
$Z_4[T_1]/(T_1^3,2T_1) \times Z_2$	$(2+T_1^2) \times Z_2$	Fig 4
$Z_4[T_1]/(T_1^3-2,2T_1) \times Z_2$	$(2) \times Z_2$	Fig 4
$Z_4[T_1,T_2]/(T_1^2,T_2^2,T_1T_2,2T_1,2T_2) \times Z_2$	$(2) \times Z_2$ or $(T_1) \times Z_2$ or $(T_2) \times Z_2$ or $(2+T_1) \times Z_2$ or $(2+T_2) \times Z_2$ or $(T_1+T_2) \times Z_2$ or $(2+T_1+T_2) \times Z_2$	K_{12}
$Z_4[T_1,T_2]/(T_1^2-2,T_2^2,T_1T_2,2T_1,2T_2) \times Z_2$	$(2) \times Z_2$	K_{12}
$Z_4[T_1,T_2]/(T_1^2-2,T_2^2,T_1T_2,2T_1,2T_2) \times Z_2$	$(T_2) \times Z_2$	Fig 4
$Z_4[T_1,T_2]/(T_1^2-2,T_2^2,T_1T_2,2T_1,2T_2) \times Z_2$	$(2+T_2) \times Z_2$	Fig 4
$Z_4[T_1,T_2]/(T_1^2,T_2^2,T_1T_2-2,2T_1,2T_2) \times Z_2$	$(2) \times Z_2$	K_{12}
$Z_4[T_1,T_2]/(T_1^2-2,T_2^2-2,T_1T_2,2T_1,2T_2) \times Z_2$	$(2) \times Z_2$	K_{12}
$Z_8[T_1]/(2T_1,T_1^2) \times Z_2$	$(4) \times Z_2$	K_{12}

$Z_8[T_1]/(2T_1,{T_1}^2)\times Z_2$	$(T_1)\times Z_2$	Fig 4
$Z_8[T_1]/(2T_1,{T_1}^2)\times Z_2$	$(4+T_1)\times Z_2$	Fig 4
$Z_8[T_1]/(2T_1,{T_1}^2+4)\times Z_2$	$(4)\times Z_2$	K_{12}
$Z_8\times Z_4$	$(0)\times Z_4$	Fig 4
$Z_2[T_1]/({T_1}^3)\times Z_4$	$(0)\times Z_4$	Fig 4
$Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)\times Z_4$	$(0)\times Z_4$	K_{12}
$Z_4\,[T_1]/(2T_1,{T_1}^2)\times Z_4$	$(0)\times Z_4$	K_{12}
$Z_4\,[T_1]/(2T_1,{T_1}^2-2)\times Z_4$	$(0)\times Z_4$	Fig 4
$Z_2\times Z_2\times F_4[T_1]/({T_1}^2)$	$Z_2\times Z_2\times(0)$	K_{12}
$Z_2\times Z_2\times Z_4[T_1]/({T_1}^2+T_1+1)$	$Z_2\times Z_2\times(0)$	K_{12}
$Z_4\times F_4[T_1]/({T_1}^2)$	$Z_4\times(0)$	K_{12}
$Z_4\times Z_4[T_1]/({T_1}^2+T_1+1)$	$Z_4\times(0)$	K_{12}

Proof:

Since $|\,\Gamma_I(R)\,| = 12$ and $|I| = 4$, then $|V(\Gamma(R/I))| = 3$, by [2] $R/I\cong Z_2\times Z_3$, Z_8 or $Z_2[T_1]/({T_1}^3)$, $Z_4[T_1]/(2T_1,{T_1}^2-2)$, $Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)$ or $F_4\,[T_1]/({T_1}^2)$, $Z_4[T_1]/(2T_1,{T_1}^2)$ or $Z_4[T_1]/({T_1}^2+T_1+1)$. and so

Sub case(a) When $R/I\cong Z_2\times Z_3$, then $|R| = 24$.Therefor $R\cong R_1\times R_2\times\ldots\times R_n$,$n = 2,3$or 4 .
If $n = 4$, then $R\cong Z_2\times Z_2\times Z_2\times Z_3$
If $I\cong Z_2\times Z_2\times(0)\times(0)$, then $R/I\cong Z_2\times Z_3$, thus $\Gamma_I(R)\cong$ figure 2 .
If $n = 3$, then $R\cong Z_4\times Z_2\times Z_3$ or $Z_2[T_1]/({T_1}^2)\times Z_2\times Z_3$
If $I_1\cong Z_4\times(0)\times(0)$, then $R/I_1\cong Z_2\times Z_3$, thus $\Gamma_I(R)\cong$ figure 2, if $I_2\cong(2)\times Z_2\times(0)$, then $R/I_2\cong Z_2\times Z_3$, thus $\Gamma_I(R)\cong$ figure 2, if $I_3\cong(T_1)\times Z_2\times(0)$, then $R/I_3\cong Z_2\times Z_3$, thus $\Gamma_I(R)\cong$ figure 2 .
If $n\cong 2$, then $R\cong R_1\times R_2\ \Rightarrow\ R\cong A\times Z_3$, when $A\cong F_8$, Z_8, $Z_2[T_1]/({T_1}^3)$, $Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)$, $Z_4[T_1]/(2T_1,{T_1}^2)$ or $Z_4[T_1]/(2T_1,{T_1}^2-2)$

1- $R\cong Z_3\times F_8$ that is a contradiction .
2- $R\cong Z_3\times Z_8$, if $I\cong(0)\times(2)$, then $R/I\cong Z_2\times Z_3$, thus $\Gamma_I(R)\cong K_{4,8}$.
3- $R\cong Z_3\times Z_2[T_1]/({T_1}^3)$, if $I\cong(0)\times(T_1)$, then $R/I\cong Z_3\times Z_2$, thus $\Gamma_I(R) = K_{4,8}$.
4- $R\cong Z_3\times Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)$, if $I\cong(0)\times(T_1,T_2)$, then $R/I\cong Z_3\times Z_2$, thus$\Gamma_I(R)\cong K_{4,8}$.
5- $R\cong Z_3\times Z_4\,[T_1]/(2T_1,{T_1}^2)$, if $I\cong(0)\times(2,T_1)$, then $R/I\cong Z_3\times Z_2$, thus $\Gamma_I(R)\cong K_{4,8}$.
6- $R\cong Z_3\times Z_4\,[T_1]/(2T_1,{T_1}^2-2)$, if $I\cong(0)\times(T_1)$, then $R/I\cong Z_3\times Z_2$, thus $\Gamma_I(R) = K_{4,8}$.

Sub case(b) If $R/I\cong Z_8$ or $Z_2[T_1]/({T_1}^3)$, $Z_4[T_1]/(2T_1,{T_1}^2)$, then $|R/I| = 8$ so , $|R| = |I|.|R/I|$
$|R| = 32$. Therefor $R\cong R_1\times R_2\times\ldots\times R_n$, $n = 2,3$ or 5 .
If $n = 5$, then $R\cong Z_2\times Z_2\times Z_2\times Z_2\times Z_2$
If $I\cong Z_2\times Z_2\times(0)\times(0)\times(0)$, then $R/I\cong Z_2\times Z_2\times Z_2$ that is a contradiction .
If $n = 4$, then $R\cong Z_4\times Z_2\times Z_2\times Z_2$ or $Z_2[T_1]/({T_1}^2)\times Z_2\times Z_2\times Z_2$
If $I_1 = (0)\times Z_2\times Z_2\times(0)$, then $R/I_1\cong Z_4\times Z_2$ that is a contradiction .
If $I_2 = (2)\times Z_2\times(0)\times(0)$, then $R/I_2\cong Z_2\times 0\times Z_2\times Z_2$ that is a contradiction , if $I_3 = Z_4\times(0)\times(0)\times(0)$, then $R/I_3\cong Z_2\times Z_2\times Z_2$that is a contradiction .
If $n = 3$, then

1- $R\cong B_1\times B_2\times Z_2$, where $B_1,B_2\cong Z_4$ or $Z_2[T_1]/({T_1}^2)$.

If $I_1\cong B_1\times(0)\times(0)$, then $R/I_1\cong B_2\times Z_2$ that is a contradiction , if $I_2\cong(0)\times(2)\times Z_2$, $R/I_2\cong B_1\times Z_2$ that is a contradiction , if $I_3\cong(0)\times(T_1)\times Z_2$, $R/I_3\cong B_1\times Z_2$, that is a contradiction , if $I_4\cong(2)\times(2)\times(0)$, then $R/I_4\cong Z_2\times Z_2$ that is a contradiction , if $I_5\cong(T_1)\times(T_1)\times(0)$, then $R/I_5\cong Z_2\times Z_2$ that is a contradiction .

2- $R\cong Z_2\times Z_2\times A$ where $A\cong Z_8$, F_8, $Z_2[T_1]/({T_1}^3)$, $Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)$, $Z_4[T_1]/(2T_1,{T_1}^2)$ or $Z_4[T_1]/(2T_1,{T_1}^2-2)$, there are 6 rings satisfied this condition .

1- $R\cong Z_2\times Z_2\times F_8$ that is a contradiction .
2- $R\cong Z_2\times Z_2\times Z_8$, if $I_1\cong Z_2\times Z_2\times(0)$, then $R/I_1\cong Z_8$, thus $\Gamma_I(R)\cong$ figure 4 , if $I_2\cong Z_2\times(0)\times(4)$, then $R/I_2\cong Z_2\times Z_4$ that is a contradiction , if $I_3\cong Z_2\times(0)\times({T_1}^2)$, then $R/I_3\cong Z_2\times Z_4$ that is a contradiction
3- $R\cong Z_2\times Z_2\times Z_2[T_1]/({T_1}^3)$, if $I_1\cong Z_2\times Z_2\times(0)$, then $R/I_1\cong Z_2[T_1]/({T_1}^3)$, thus $\Gamma_I(R)\cong$ figure 4, if $I_2\cong Z_2\times(0)\times({T_1}^2)$, then $R/I_2\cong Z_2\times Z_2[T_1]/({T_1}^3)$ that is a contradiction .
4- $R\cong Z_2\times Z_2\times Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)$, if $I_1\cong Z_2\times Z_2\times(0)$, then $R/I_1\cong Z_2[T_1,T_2]/({T_1}^2,T_1T_2,{T_2}^2)$, if $I_2\cong(0)\times Z_2\times(T_1)$, then $R/I_2\cong Z_2\times Z_2[T_2]/({T_2}^2)$, that is a contradiction , if $I_3\cong(0)\times Z_2\times(T_2)$, then $R/I_3\cong Z_2\times Z_2[T_1]/({T_1}^2)$, that is a contradiction , if $I_4\cong(0)\times Z_2\times(T_1+T_2)$, then $R/I_4\cong Z_2\times Z_2[T_2]/({T_2}^2)$, that is a contradiction .
5- $R\cong Z_2\times Z_2\times Z_4\,[T_1]/(2T_1,{T_1}^2)$, if $I_1\cong Z_2\times Z_2\times(0)$, then $R/I_1\cong Z_4\,[T_1]/(2T_1,{T_1}^2)$, if $I_2\cong(0)\times Z_2\times(2)$, then $R/I_2\cong Z_2\times Z_2[T_1]/({T_1}^2)$, that is a contradiction , if $I_3\cong(0)\times Z_2\times(T_1)$, then $R/I_3\cong Z_2\times Z_4$, that is a contradiction , if $I_4\cong(0)\times Z_2\times(2+T_1)$, then $R/I_4\cong Z_2\times Z_4$, that is a contradiction .
6- $R\cong Z_2\times Z_2\times Z_4\,[T_1]/(2T_1,{T_1}^2-2)$, if $I_1\cong Z_2\times Z_2\times(0)$, then $R/I_1\cong Z_4\,[T_1]/(2T_1,{T_1}^2-2)$, thus $\Gamma_I(R)\cong$ figure 4, if $I_2\cong(0)\times Z_2\times(2)$, then

$R/I_2 \cong Z_2 \times Z_2[T_1]/(T_1{}^2)$ that is a contradiction .

If $n = 2$, then

(a) $R \cong R_1 \times R_2$, s.t $|R_1| = p^n$ and $|R_2| = 2$ since $|R_1| = 16 = p^t$, then there are 21 rings satisfied this condition , where $R_1 \cong F_{16}$, Z_{16} , $Z_2[T_1]/(T_1{}^4)$, $Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2)$, $Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2 - T_1T_2)$, $Z_2[T_1,T_2]/(T_1{}^3,T_1T_2,T_2{}^2)$, $Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2 \times Z_3$, $Z_4[T_1]/(T_1{}^2) \times Z_3$, $Z_4[T_1]/(T_1{}^2 + 2)$, $Z_4[T_1]/(T_1{}^2 + 1)$, $Z_4[T_1]/(T_1{}^2 + 3)$, $Z_4[T_1]/(T_1{}^2 + T_1 + 1)$, $Z_4[T_1]/(T_1{}^3, 2T_1)$, $Z_4[T_1]/(T_1{}^3 - 2,2T_1)$, $Z_4[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1{}^2 - 2,T_2^2,T_1T_2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1{}^2,T_2^2,T_1T_2 - 2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1{}^2 - 2,T_2^2 - 2,T_1T_2,2T_1,2T_2)$, $Z_8[T_1]/(2T_1,T_1{}^2)$, $Z_8[T_1]/(2T_1,T_1{}^2 + 4)$.

1- $R \cong Z_{16} \times Z_2$, if $\mathrm{I} = (4) \times (0)$, then $R/\mathrm{I} \cong Z_4 \times Z_2$, which is a contradiction .

2- $R \cong F_{16} \times Z_2$ or $F_4[T_1]/(T_1{}^2) \times Z_2$, which is a contradiction .

3- $R \cong Z_2[T_1]/(T_1{}^4) \times Z_2$, if $\mathrm{I} = (T_1^3) \times Z_2$, then $R/\mathrm{I} \cong Z_2[T_1]/(T_1{}^3)$, thus $\Gamma_I(R) \cong$ figure 4 .

4- $R \cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2) \times Z_2$, if $I = (T_1T_2) \times Z_2$, then $R/I \cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2)$.

5- $R \cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2 - T_1T_2) \times Z_2$, if $I \cong (T_1T_2) \times Z_2$, then $R/I \cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2)$

6- $R \cong Z_2[T_1,T_2]/(T_1{}^3,T_1T_2,T_2{}^2) \times Z_2$, if $I_1 \cong (T_1^2) \times Z_2$, then $R/I_1 \cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2)$, if $I_2 \cong (T_2) \times Z_2$, then $R/I_2 \cong Z_2[T_1]/(T_1{}^3)$, thus $\Gamma_I(R) \cong$ figure 4, if $I_2 \cong (T_2 + T_1^2) \times Z_2$, then $R/I_2 \cong Z_2[T_1]/(T_1{}^3)$, thus $\Gamma_I(R) \cong$ figure 4 .

7- $R \cong Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2 \times Z_2$,if $I_1 = (T_1) \times Z_2$, then $R/I_1 \cong Z_2[T_1,T_2]/(T_1,T_2)^2$

If $I_2 = (T_2) \times Z_2$, then $R/I_2 \cong Z_2[T_1,T_3]/(T_1,T_3)^2$, if $I_3 = (T_3) \times Z_2$, then $R/I_3 \cong Z_2[T_1,T_2]/(T_1,T_2)^2$, if $I_4 = (T_1 + T_2) \times Z_2$, then $R/I_4 \cong Z_2[T_1,T_3]/(T_1,T_3)^2$, if $I_5 = (T_1 + T_3) \times Z_2$, then $R/I_5 \cong Z_2[T_1,T_2]/(T_1,T_2)^2$, if $I_6 = (T_2 + T_3) \times Z_2$, then $R/I_6 \cong Z_2[T_1,T_2]/(T_1,T_2)^2$, if $I_7 = (T_1 + T_2 + T_3) \times Z_2$, then $R/I_7 \cong Z_2[T_2,T_3]/(T_2,T_3)^2$.

8- $R \cong Z_4[T_1]/(T_1{}^2) \times Z_2$, if $I_1 \cong (2T_1) \times Z_2$, then $R/I_1 \cong Z_4[T_1]/(T_1{}^2, 2T_1)$.

9- $R \cong Z_4[T_1]/(T_1{}^2 + 2) \times Z_2$, if $\mathrm{I} \cong (2T_1) \times Z_2$, then $R/\mathrm{I} \cong Z_4[T_1]/(2T_1,T_1{}^2 + 2)$, thus $\Gamma_I(R) \cong$ figure 4 .

10- $R \cong Z_4[T_1]/(T_1{}^2 + 1) \times Z_2$, if $\mathrm{I} \cong (2 + 2T_1) \times Z_2$, then $R/\mathrm{I} \cong Z_2[T_1]/(T_1{}^2)$, thus $\Gamma_I(R) \cong$ figure 4 .

11- $R \cong Z_4[T_1]/(T_1{}^2 + 3) \times Z_2$, if $\mathrm{I} \cong (2 + 2T_1) \times Z_2$, then $R/\mathrm{I} \cong Z_2[T_1]/(T_1{}^2)$.

12- $R \cong Z_4[T_1]/(T_1{}^2 + T_1 + 1) \times Z_2$ which is a contradiction .

13- $R \cong Z_4[T_1]/(T_1{}^3, 2T_1) \times Z_2$, if $I_1 \cong (2) \times Z_2$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^3)$, thus $\Gamma_I(R) \cong$ figure 4,

If $I_2 \cong (T_1^2) \times Z_2$, then $R/I_2 \cong Z_4[T_1]/(2T_1,T_1{}^2)$, $I_3 \cong (2 + T_1^2) \times Z_2$, then $R/I_3 \cong Z_4[T_1]/(2T_1,T_1{}^2 - 2)$, thus $\Gamma_I(R) \cong$ figure 4 .

14- $R \cong Z_4[T_1]/(T_1{}^3 - 2,2T_1) \times Z_2$, if $I \cong (2) \times Z_2$, then $R/\mathrm{I} \cong Z_2[T_1]/(T_1{}^3)$, thus $\Gamma_I(R) \cong$ figure 4 .

15- $R \cong Z_4[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2,2T_1,2T_2) \times Z_2$, if $I_1 \cong (2) \times Z_2$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2,T_2{}^2,T_1T_2)$, if $I_2 \cong (T_1) \times Z_2$, then $R/I_2 \cong Z_4[T_2]/(2T_2,T_2{}^2)$, if $I_3 \cong (T_2) \times Z_2$, then $R/I_3 \cong Z_4[T_1]/(2T_1,T_1{}^2)$, if $I_4 \cong (2 + T_1) \times Z_2$, then $R/I_4 \cong Z_4[T_2]/(2T_2,T_2{}^2)$, if $I_5 \cong (2 + T_2) \times Z_2$,then $R/I_5 \cong Z_4[T_1]/(2T_1,T_1{}^2)$, if $I_6 \cong (T_1 + T_2) \times Z_2$, then $R/I_6 \cong Z_4[T_2]/(2T_2,T_2{}^2)$, if $I_7 \cong (2 + T_1 + T_2) \times Z_2$, then $R/I_7 \cong Z_4[T_1]/(2T_1,T_1{}^2)$.

16- $R \cong Z_4[T_1,T_2]/(T_1{}^2 - 2,T_2^2,T_1T_2,2T_1,2T_2) \times Z_2$, if $I_1 \cong (2) \times Z_2$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2,T_2{}^2,T_1T_2)$, if $I_2 \cong (T_2) \times Z_2$, then $R/I_2 \cong Z_4[T_1]/(T_1{}^2 - 2,2T_1)$, thus $\Gamma I(R\ ) \cong$ figure 4, if $I_3 \cong (2 + T_2) \times Z_2$, then $R/I_3 \cong Z_4[T_1]/(T_1{}^2 - 2,2T_1)$, thus $\Gamma_I(R) \cong$ figure 4 .

17- $R \cong Z_4[T_1,T_2]/(T_1{}^2,T_2^2,T_1T_2 - 2,2T_1,2T_2) \times Z_2$, if $I_1 \cong (2) \times Z_2$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2,T_2{}^2,T_1T_2)$.

18- $R \cong Z_4[T_1,T_2]/(T_1{}^2 - 2,T_2^2 - 2,T_1T_2,2T_1,2T_2) \times Z_2$, if $I_1 \cong (2) \times Z_2$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^2,T_2{}^2,T_1T_2)$.

19- $R \cong Z_8[T_1]/(2T_1,T_1{}^2) \times Z_2$, if $I_1 \cong (4) \times Z_2$, then $R/I_1 \cong Z_4[T_1]/(2T_1,T_1{}^2)$, if $I_2 \cong (T_1) \times Z_2$, then $R/I_2 \cong Z_8$, thus $\Gamma_I(R) \cong$ figure 4, if $I_3 \cong (4 + T_1) \times Z_2$, then $R/I_3 \cong Z_8$, thus $\Gamma_I(R) \cong$ figure 4 .

20- $R \cong Z_8[T_1]/(2T_1,T_1{}^2 + 4) \times Z_2$, if $I_1 \cong (4) \times Z_2$, then $R/I_1 \cong Z_4[T_1]/(2T_1,T_1{}^2)$.

(b) $R \cong R_1 \times R_2$, s.t $|R_1| = P^n$ and $|R_2| = 4$ since $|R_1| = 8 = P^t$, then there are 6 rings satisfied this condition .

1- $R \cong Z_8 \times Z_4$, if $I_1 \cong (0) \times Z_4$, then $R/I_1 \cong Z_8$, thus $\Gamma I(R\ ) =$ figure 4, if $I_2 \cong (2) \times (0)$, then $R/I_2 \cong Z_2 \times Z_4$, that is contradiction , if $I_3 \cong (4) \times (2)$, then $R/I_3 \cong Z_4 \times Z_2$ that is contradiction .

2- $R \cong F_8 \times Z_4$, if $\mathrm{I} \cong (0) \times Z_4$, then $R/I \cong F_8$ that is contradiction .

3- $R \cong Z_2[T_1]/(T_1{}^3) \times Z_4$, if $I_1 \cong (0) \times Z_4$, then $R/I_1 \cong Z_2[T_1]/(T_1{}^3)$, thus$\Gamma_I(R) \cong$ figure 4, if $I_2 \cong (T_1) \times (0)$, then $R/I_2 \cong Z_2 \times Z_4$, that is contradiction , if $I_3 \cong (T_1{}^2) \times (2)$, then $R/I_3 \cong Z_2 \times Z_2$ that is contradiction .

4- $R \cong Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2) \times Z_4$, if $I_1 \cong (0) \times Z_4$, then $R/I_1 \cong Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$, if $I_2 \cong (T_1,T_2) \times (0)$, then $R/I_2 \cong Z_2 \times Z_4$ that is contradiction , if $I_3 \cong (T_1) \times (2)$, then $R/I_3 \cong Z_2[T_2]/(T_2{}^2) \times Z_2$, that is contradiction , if $I_4 \cong (T_2) \times (2)$, then $R/I_4 \cong Z_2[T_1]/(T_1{}^2) \times Z_2$, that is contradiction if $I_5 \cong (T_1 + T_2) \times (2)$, then $R/I_5 \cong Z_2[T_2]/(T_2{}^2) \times Z_2$ that is contradiction .

5- $R \cong Z_4 [T_1]/(2T_1, T_1{}^2) \times Z_4$, if $I_1 \cong (0) \times Z_4$, then $R/I_1 \cong Z_4 [T_1]/(2T_1, T_1{}^2)$, if $I_2 \cong (2, T_1) \times (0)$, then $R/I_2 \cong Z_2 \times Z_4$ that is contradiction , if $I_3 \cong (T_1) \times (2)$, then $R/I_3 \cong Z_4 \times Z_2$ that is contradiction , if $I_4 \cong (2) \times (2)$, then $R/I_4 \cong Z_2[T_1]/(T_1{}^2) \times Z_2$ that is contradiction , if $I_5 \cong (2 + T_1) \times (2)$, then $R/I_5 \cong Z_4 \times Z_2$ that is contradiction .

6- $R \cong Z_4 [T_1]/(2T_1, T_1{}^2 - 2) \times Z_4$, if $I_1 \cong (0) \times Z_4$, then $R/I_1 \cong Z_4 [T_1]/(2T_1, T_1{}^2 - 2)$, thus $\Gamma_I(R) \cong$ figure 4, if $I_2 \cong (T_1) \times (0)$, then $R/I_2 \cong Z_4 \times Z_4$ that is contradiction , , if $I_3 \cong (2) \times (2)$, then $R/I_3 \cong Z_2[T_1]/(T_1{}^2) \times Z_2$ that is contradiction .

Sub case(c)

If $R/I \cong F_4 [T_1]/(T_1{}^2)$ or $Z_4[T_1]/(T_1{}^2 + T_1 + 1)$, then $|R/I| = 16$, so $|R| = |I|.|R/I|$

$|R| = 64$. Therefor $R \cong R_1 \times R_2 \times \ldots \times R_n$, $n = 2,3,4,5$ or 6

If $n = 6$, then $R \cong Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$,

If $I = Z_2 \times Z_2 \times (0) \times (0) \times (0) \times (0)$, then $R/I \cong Z_2 \times Z_2$ that is contradiction .

If $n = 5$, then $R \cong Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_4$ or $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2[T_1]/(T_1{}^2)$.

If $I = Z_2 \times Z_2 \times (0) \times (0) \times (0)$, then $R/I \cong Z_2 \times Z_2 \times Z_4$ or $Z_2[T_1]/(T_1{}^2)$ that is contradiction.

If $n = 4$, then $R \cong Z_2 \times Z_2 \times Z_2 \times A$, where $A \cong Z_8$, F_8 , $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2)$ or $Z_4[T_1]/(2T_1, T_1{}^2 - 2)$.

If $I = Z_2 \times Z_2 \times (0) \times (0)$, then $R/I \cong Z_2 \times A$ that is a contradiction .

If $n = 3$, then $R \cong Z_2 \times Z_2 \times Z_{16}$ or $R \cong Z_2 \times Z_4 \times Z_8$.

If $R \cong Z_2 \times Z_2 \times A$, when $A \cong Z_{16}$, F_{16} , $Z_2[T_1]/(T_1{}^4)$, $Z_2[T_1, T_2]/(T_1{}^2, T_2{}^2)$, $Z_2[T_1, T_2]/(T_1{}^2, T_2{}^2 - T_1T_2)$, $Z_2[T_1, T_2]/(T_1{}^3, T_1T_2, T_2{}^2)$, $Z_2[T_1, T_2, T_3]/(T_1, T_2, T_3)^2$, $Z_4[T_1]/(T_1{}^2)$, $F_4[T_1]/(T_1{}^2)$, $Z_4[T_1]/(T_1{}^2 + 2)$, $Z_4[T_1]/(T_1{}^2 + 1)$, $Z_4[T_1]/(T_1{}^2 + 3)$, $Z_4[T_1]/(T_1{}^2 + T_1 + 1)$, $Z_4[T_1]/(T_1{}^3, 2T_1)$, $Z_4[T_1]/(T_1{}^3 - 2, 2T_1)$, $Z_4[T_1]/(T_1{}^2, T_2{}^2, T_1T_2, 2T_1, 2T_2)$, $Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2)$, $Z_4[T_1, T_2]/(T_1{}^2, T_2^2, T_1T_2 - 2, 2T_1, 2T_2)$, $Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2)$, $Z_8[T_1]/(2T_1, T_1{}^2)$, $Z_8[T_1]/(2T_1, T_1{}^2 + 4)$.

If $I_1 = Z_2 \times Z_2 \times (0)$, then $R/I_1 \cong F_4[T_1]/(T_1{}^2)$, If $I_2 = Z_2 \times Z_2 \times (0)$, then $R/I_2 \cong Z_4[T_1]/(T_1{}^2 + T_1 + 1)$. Otherwise it is a contradiction .

If $n = 2$, then $R \cong Z_4 \times Z_{16}$ or $R \cong Z_2 \times Z_{32}$ or $R \cong Z_8 \times Z_8$.

If $R \cong Z_4 \times A$, when $A \cong Z_{16}$, F_{16} , $Z_2[T_1]/(T_1{}^4)$, $Z_2[T_1, T_2]/(T_1{}^2, T_2{}^2)$, $Z_2[T_1, T_2]/(T_1{}^2, T_2{}^2 - T_1T_2)$, $Z_2[T_1, T_2]/(T_1{}^3, T_1T_2, T_2{}^2)$, $Z_2[T_1, T_2, T_3]/(T_1, T_2, T_3)^2$, $Z_4[T_1]/(T_1{}^2)$, $Z_4[T_1]/(T_1{}^2 + 2)$, $Z_4[T_1]/(T_1{}^2 + 1)$, $Z_4[T_1]/(T_1{}^2 + 3)$, $Z_4[T_1]/(T_1{}^2 + T_1 + 1)$, $Z_4[T_1]/(T_1{}^3, 2T_1)$, $Z_4[T_1]/(T_1{}^3 - 2, 2T_1)$, $Z_4[T_1]/(T_1{}^2, T_2{}^2, T_1T_2, 2T_1, 2T_2)$, $Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2)$, $Z_4[T_1, T_2]/(T_1{}^2, T_2^2, T_1T_2 - 2, 2T_1, 2T_2)$, $Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2)$, $Z_8[T_1]/(2T_1, T_1{}^2)$, $Z_8[T_1]/(2T_1, T_1{}^2 + 4)$.

If $I_1 = Z_4 \times (0)$, then $R/I_1 \cong Z_4[T_1]/(T_1{}^2)$, If $I_2 = Z_4 \times (0)$, then $R/I_2 \cong Z_4[T_1]/(T_1{}^2 + T_1 + 1)$, Otherwise it is a contradiction . ■

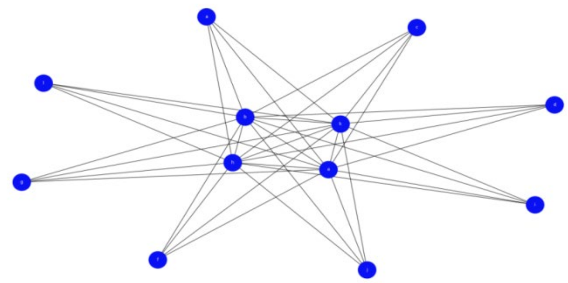

Fig 4. $\Gamma_I(R)$, where $I = Z_2 \times Z_2 \times (0), \cdots$ and $R = Z_2 \times Z_2 \times Z_8, \cdots$.

Theorem 2.6:

Let R be a non local ring satisfied $|\Gamma_I(R)| = 12$ with $|I| = 3$. Then R corresponding the following rings in Table 5.

Table 5. $|\Gamma_I(R)| = 12$ where $|I| = 3$

Ring	Ideal	Figure
$Z_3 \times Z_2 \times F_4$	$Z_3 \times (0) \times (0)$	Fig 5
$Z_3 \times Z_3 \times Z_3$	$(0) \times (0) \times Z_3$	$K_{6,6}$
$Z_3 \times Z_9$ or $Z_3 \times Z_3[T_1]/(T_1{}^2)$	$(0) \times (3)$ or $(0) \times (T_1)$	$K_{6,6}$
$Z_3 \times Z_{25}$ or $Z_3 \times Z_5[T_1]/(T_1{}^2)$	$Z_3 \times (0)$	K_{12}

Proof:

Since $|\Gamma_I(R)| = 12$ and $|I| = 3$, then $|V(\Gamma(R/I))| = 4$, by [2] $R/I \cong Z_2 \times F_4$, $Z_3 \times Z_3$, Z_{25} or $Z_5[T_1]/(T_1{}^2)$, and so

Sub case(a) When $R/I \cong Z_2 \times F_4$, then $|R| = 24$. Therefor $R \cong R_1 \times R_2 \times \ldots \times R_n$, $n = 2,3$ or 4 .

We take $n = 4$, then $R \cong Z_2 \times Z_2 \times Z_2 \times Z_3$

The only ideal of order 3 is $(0) \times (0) \times (0) \times Z_3$, so $R/I \cong Z_2 \times Z_2 \times Z_2$ that is a contradiction.

Second when $n = 3$, then $R \cong Z_3 \times Z_2 \times A$, where $A \cong Z_4$, $Z_2[T_1]/(T_1{}^2)$ or F_4

So that the ideal I, $|I| = 3$ is $I \cong Z_3 \times (0) \times (0)$, that is leads $R/I \cong Z_2 \times A$,where $A \cong Z_4$, $Z_2[T_1]/(T_1{}^2)$ or F_4. If we take $A \cong F_4$ then $R/I \cong Z_2 \times F_4$, thus $\Gamma_I(R)$ = figure 5 .

If $n = 2$, then $R \cong Z_3 \times A$ where $A \cong Z_8$, F_8 , $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$, $Z_4[T_1]/(2T_1, T_1{}^2)$ or $Z_4[T_1]/(2T_1, T_1{}^2 - 2)$.

If we take ideal I in R of order 3, then $I \cong Z_3 \times (0)$.

So $R/\mathrm{I} \cong A$ that is a contradiction.

Sub case(b) when $R/\mathrm{I} \cong Z_3 \times Z_3$, then $|R| = 27$. Therefor $R \cong R_1 \times R_2 \times \ldots \times R_n$, $n = 2$ or 3.

If $n = 3$, then $R \cong Z_3 \times Z_3 \times Z_3$, and an ideal I, satisfieying $|I| = 3$ must be isomorphic with $(0) \times (0) \times Z_3$. Hence $R/\mathrm{I} \cong Z_3 \times Z_3$, thus $\Gamma_I(R) = K_{6,6}$

If $n = 2$, then $R \cong R_1 \times R_2$, where $R_1 \cong Z_3$ and $R_2 \cong Z_9$ or $Z_3[T_1]/(T_1{}^2)$. The ideals I of order 3 have a form $I = (0) \times \mathrm{J}$ where $J = (3)$ or $J = (T_1)$ the only ideals $I = (0) \times \mathrm{J}$ satisfied $R/I \cong Z_3 \times Z_3$, thus $\Gamma_I(R) = K_{6,6}$

Sub case(c) If $R/\mathrm{I} \cong Z_{25}$ or $Z_5[T_1]/(T^2)$, then $|R| = 75$ and we have $R \subseteq R_1 \times R_2 \times R_3$ or $R \subseteq R_1 \times R_2$

If $R \cong R_1 \times R_2 \times R_3$, then $R \cong Z_5 \times Z_5 \times Z_3$ that is lead a contradiction fact $R/I \cong Z_{25}$ or $Z_3[T_1]/(T_1{}^2)$.

So that $R \cong R_1 \times R_2$, where $R_1 \cong Z_3$, $R_1 \cong Z_{25}$ or $Z_5[T_1]/(T_1{}^2)$.

If I ideal in R such that $|I| = 3$, then $I \cong Z_3 \times (0)$.

So $R/\mathrm{I} \cong Z_{25}$ or $Z_5[T_1]/(T_1{}^2)$. ■

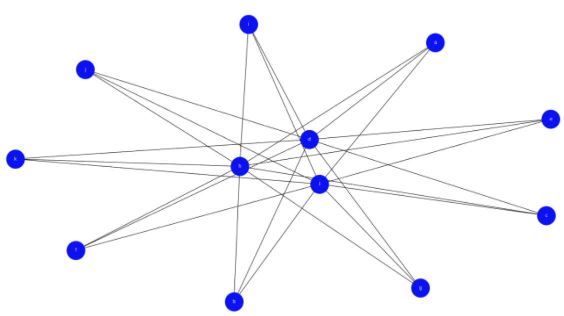

Fig 5. $\Gamma_I(R)$, where $I = Z_3 \times (0) \times (0)$ and $R = Z_3 \times Z_2 \times F_4$.

1. Ring with $|V(\Gamma_I(R))| = 13$:

In this section we give all possible ring with $|V(\Gamma_\mathrm{I}(\mathrm{R}))| = 13$ .First we give this observing.

observing 3.1:

We consider when a non-trivial $\Gamma_I(R)$ is the graph on 13 vertices since $|V(\Gamma_I(R)| = |V(\Gamma(R/I)|.|I|$ and $|I| \geq 2$ we get: -

1- $|I| = 13$ and $|V(\Gamma(R/I)| = 1$.

Theorem 3.2:

Let R be a non-local ring satisfied $|\Gamma_I(R)| = 13$ with $|I| = 13$. Then R corresponding the following rings in Table 6.

Table 6. $|\Gamma_\mathrm{I}(\mathrm{R})| = 13$ where $|I| = 13$

Ring	Ideal	Figure
$Z_4 \times Z_{13}$	$(0) \times Z_{13}$	K_{13}
$Z_2[T_1]/(T_1{}^2) \times Z_{13}$	$(0) \times Z_{13}$	K_{13}

Proof:

Since $|\Gamma_I(R)| = 13$ and $|I| = 13$, then $|V(\Gamma(R/I))| = 1$, by [12], $R/\mathrm{I} \cong Z_4$ or $Z_2[T_1]/(T_1{}^2)$ and so $|R/\mathrm{I}| = 4$

Since $|R| = |I||R/\mathrm{I}|$, which implies $|R| = 13 \times 4 = 52$

We note that R direct product of R

So $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $n = 2$ or 3

If $n = 3$, then $R \cong Z_2 \times Z_2 \times Z_{13}$, if $I = (0) \times (0) \times Z_{13}$, then $R/\mathrm{I} \cong Z_2 \times Z_2$, that is a contradiction .

If $n = 2$, then $R \cong R_1 \times R_2$ where $R_1 \cong Z_4$ or $Z_2[T_1]/(T_1{}^2)$ and $R_2 \cong Z_{13}$

If $\mathrm{I} \cong (0) \times Z_{13}$, then $R/\mathrm{I} \cong Z_4$. ■

2. Ring with $|V(\Gamma_I(R))| = 14$:

In this section we give all possible ring with $|V(\Gamma_\mathrm{I}(\mathrm{R}))| = 14$ .First we give this observing.

observing 4.1:

We consider when a non-trivial $\Gamma_I(R)$ is the graph on 14 vertices since $|V(\Gamma_I(R)| = |V(\Gamma(R/I)|.|I|$ *and* $|I| \geq 2$ we get three possibilities :-

1- $|I| = 14$ and $|V(\Gamma(R/I)| = 1$.

2- $|I| = 7$ and $|V(\Gamma(R/I)| = 2$.

3- $|I| = 2$ and $|V(\Gamma(R/I)| = 7$.

Theorem 4.2:

Let R be a non-local ring satisfied $|\Gamma_I(R)| = 14$ with $|I| = 14$. Then R corresponding the following rings in Table 7.

Table 7. $|\Gamma_\mathrm{I}(\mathrm{R})| = 14$ where $|I| = 14$

Ring	Ideal	Figure
$Z_7 \times Z_2 \times Z_4$	$Z_7 \times Z_2 \times (0)$	K_{14}
$Z_7 \times Z_2 \times Z_2[T_1]/(T_1{}^2)$	$Z_7 \times Z_2 \times (0)$	K_{14}
$Z_7 \times Z_8$	$Z_7 \times (4)$	K_{14}
$Z_7 \times Z_2[T_1]/(T_1{}^3)$	$Z_7 \times (T_1{}^2)$	K_{14}
$Z_7 \times Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2)$	$Z_7 \times (T_1)$or$Z_7 \times (T_2)$or $Z_7 \times (T_1 + T_2)$	K_{14}
$Z_7 \times Z_4\,[T_1]/(2T_1, T_1{}^2)$	$Z_7 \times (2)$or$Z_7 \times (T_1)$or $Z_7 \times (2 + T_1)$	K_{14}
$Z_7 \times Z_4\,[T_1]/(2T_1, T_1{}^2 - 2)$	$Z_7 \times (2)$	K_{14}

Proof :

Since $|\Gamma_I(R)| = 14$ and $|I| = 14$, then $|V(\Gamma(R/I))| = 1$, by [12], $R/\mathrm{I} \cong Z_4$ or $Z_2[T_1]/(T_1{}^2)$, F_4 we note that if

$R/\mathrm{I} \cong F_4$ we have a contradiction for all case and so $|R/\mathrm{I}| = 4$

Since $|R| = |I||R/\mathrm{I}|$, which implies $|R| = 14 \times 4 = 56$

We note that R direct product of local ring R_i, where $2 \leq i \leq 4$

So If $R \cong R_1 \times R_2 \times R_3 \times R_4$, then $R \cong Z_7 \times Z_2 \times Z_2 \times Z_2$

Therefore $I \cong Z_7 \times Z_2 \times (0) \times (0)$, and we get $R/\mathrm{I} \cong Z_2 \times Z_2$ that is a contradiction.

Addition if $R \cong Z_7 \times Z_2 \times Z_4$ or $Z_2[T_1]/(T_1^2)$, then $I \cong Z_7 \times Z_2 \times (0)$, then we have $R/\mathrm{I} \cong Z_4$ or $Z_2[T_1]/(T_1^2)$.

Similarly If $R \cong Z_7 \times A$, whenever $A \cong F_8$, Z_8, $Z_2[T_1]/(T_1^3)$, $Z_2[T_1,T_2]/(T_1^2,T_1T_2,T_2^2)$, $Z_4[T_1]/(2T_1,T_1^2)$ or $Z_4[T_1]/(2T_1,T_1^2-2)$.

Clearly if $A \cong F_8$ we have a contradiction. Otherwise all rings satisfied all conditions corresponding, $|I| = 14$ are $Z_7 \times Z_8$, $I \cong Z_7 \times (4)$

1- $R \cong Z_7 \times F_8$ that is a contradiction.
2- $R \cong Z_7 \times Z_8$, if $\mathrm{I} = Z_7 \times (4)$, then $R/\mathrm{I} \cong Z_4$
3- $R \cong Z_7 \times Z_2[T_1]/(T_1^3)$, if $\mathrm{I} = Z_7 \times (T_1^2)$, then $R/\mathrm{I} \cong Z_2[T_1]/(T_1^2)$
4- $R \cong Z_7 \times Z_2[T_1,T_2]/(T_1^2,T_1T_2,T_2^2)$, if $I_1 = Z_7 \times (T_1)$, then $R/I_1 \cong Z_2[T_2]/(T_2^2)$, if $I_2 = Z_7 \times (T_2)$, then $R/I_2 \cong Z_2[T_1]/(T_1^2)$, if $I_3 = Z_7 \times (T_1+T_2)$, then $R/I_3 \cong Z_2[T_1]/(T_1^2)$.
5- $R \cong Z_7 \times Z_4[T_1]/(2T_1,T_1^2)$, if $I_1 = Z_7 \times (2)$, then $R/I_1 \cong Z_2[T_1]/(T_1^2)$, if $I_2 = Z_7 \times (T_1)$, then $R/I_2 \cong Z_4$, if $I_3 = Z_7 \times (2+T_1)$, then $R/I_3 \cong Z_4$.
6- $R \cong Z_7 \times Z_4[T_1]/(2T_1,T_1^2-2)$, if $I = Z_7 \times (2)$, then $R/I \cong Z_2[T_1]/(T_1^2)$. ■

Theorem 4.3:

Let R be a non-local ring satisfied $|\Gamma_I(R)| = 14$ with $|I| = 7$. Then R corresponding the following rings in Table 8.

Table 8. $|\Gamma_I(R)| = 14$ where $|I| = 7$

Ring	Ideal	Figure
$Z_7 \times Z_9$ or $Z_7 \times Z_3[T_1]/(T_1^2)$	$Z_7 \times (0)$	K_{14}
$Z_7 \times Z_2 \times Z_2$	$Z_7 \times (0) \times (0)$	$K_{7,7}$

Proof :

Since $|\Gamma_I(R)| = 14$ and $|I| = 7$, then $|V(\Gamma(R/I))| = 2$, by [2] $R/\mathrm{I} \cong Z_9$, $Z_3[T_1]/(T_1^2)$ or $Z_2 \times Z_2$.

Sub case(a) If $R/\mathrm{I} \cong Z_9$ or $Z_3[T_1]/(T_1^2)$, then $|R/\mathrm{I}| = 9$

Since $|R| = |I||R/\mathrm{I}|$, which implies $|R| = 7 \times 9 = 63$ we note that R direct product of ring

So $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $n = 2$ or 3.

If $R \cong R_1 \times R_2 \times R_3$ then $R \cong Z_7 \times Z_3 \times Z_3$ that is a contradiction

If $R \cong R_1 \times R_2$ then $R_1 \cong Z_7$, $R_2 \cong Z_9$ or $Z_3[T_1]/(T_1^2)$, and we have $I = Z_7 \times (0)$, therefore $R/\mathrm{I} \cong Z_9$ or $Z_3[T_1]/(T_1^2)$.

Sub case(b) If $R/\mathrm{I} \cong Z_2 \times Z_2$, then $|R/\mathrm{I}| = 4$,

Since $|R| = |I||R/\mathrm{I}|$, which implies $|R| = 7 \times 4 = 28$ we note that R direct product of ring

So $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $n = 2$ or 3.

If $n = 3$, then $R \cong Z_7 \times Z_2 \times Z_2$

If $I \cong Z_7 \times (0) \times (0)$, then $R/\mathrm{I} \cong Z_2 \times Z_2$.

If $n = 2$, then $R \cong Z_7 \times Z_4$ or $Z_2[T_1]/(T_1^2)$,

If $I \cong Z_7 \times (0)$, then $R/\mathrm{I} \cong Z_4$ or $Z_2[T_1]/(T_1^2)$ that is a contradiction. ■

Theorem 4.4:

Let R be a non local ring satisfied $|\Gamma_I(R)| = 14$ with $|I| = 2$. Then R corresponding the following rings in Table 9.

Table 9. $|\Gamma_I(R)| = 14$ where $|I| = 2$

Ring	Ideal	Figure
$Z_{16} \times Z_2$ or $Z_2[T_1]/(T_1^4) \times Z_2$	$(0) \times Z_2$	Fig 6
$Z_2[T_1,T_2]/(T_1^2,T_2^2) \times Z_2$	$(0) \times Z_2$	Fig 7
$Z_2[T_1,T_2]/(T_1^2,T_2^2-T_1T_2) \times Z_2$	$(0) \times Z_2$	Fig 8
$Z_2[T_1,T_2]/(T_1^3,T_1T_2,T_2^2) \times Z_2$	$(0) \times Z_2$	Fig 9
$Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2 \times Z_2$	$(0) \times Z_2$	K_{14}
$Z_4[T_1]/(T_1^2) \times Z_2$	$(0) \times Z_2$	Fig 10
$Z_4[T_1]/(T_1^2+2) \times Z_2$	$(0) \times Z_2$	Fig 6
$Z_4[T_1]/(T_1^3,2T_1) \times Z_2$	$(0) \times Z_2$	Fig 9
$Z_4[T_1]/(T_1^3-2,2T_1) \times Z_2$	$(0) \times Z_2$	Fig 6

$Z_4[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2,2T_1,2T_2)\times Z_2$	$(0)\times Z_2$	K_{14}
$Z_4[T_1,T_2]/(T_1{}^2-2,T_2^2,T_1T_2,2T_1,2T_2)\times Z_2$	$(0)\times Z_2$	Fig 11
$Z_4[T_1,T_2]/(T_1{}^2,T_2^2,T_1T_2-2,2T_1,2T_2)\times Z_2$	$(0)\times Z_2$	Fig 10
$Z_4[T_1,T_2]/(T_1{}^2-2,T_2^2-2,T_1T_2,2T_1,2T_2)\times Z_2$	$(0)\times Z_2$	Fig 12
$Z_8[T_1]/(2T_1,T_1{}^2)\times Z_2$	$(0)\times Z_2$	Fig 9
$Z_8[T_1]/(2T_1,T_1{}^2+4)\times Z_2$	$(0)\times Z_2$	Fig 8

Proof:

Since $|\Gamma_I(R)| = 14$ and $|I|=2$, then $|V(\Gamma(R/I))|=7$, by [13] $R/\mathrm{I}\cong Z_{16}$, $Z_2[T_1]/(T_1^4)$, $Z_4[T_1]/(T_1^2+2)$, $Z_4[T_1]/(T_1^2+2T_1+2)$, $Z_4[T_1]/(T_1^3-2,2T_1^2,2T_1)$, $Z_2[T_1,T_2]/(T_1^3,T_1T_2,T_2^2)$, $Z_8[T_1]/(2T_1,T_1^2)$, $Z_4[T_1]/(T_1^3,2T_1^2,2T_1)$, $Z_4[T_1,T_2]/(T_1^2-2,T_1T_2,T_2^2,2T_1,2T_2)$, $Z_4[T_1]/(T_1^2+2T_1)$, $Z_8[T_1]/(2T_1,T_1^2+4)$, $Z_2[T_1,T_2]/(T_1^2,T_2^2-T_1T_2)$, $Z_4[T_1,T_2]/(T_1^2,T_2^2-T_1T_2,T_1T_2-2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1^2,T_2^2,T_1T_2-2,2T_1,2T_2)$, $Z_2[T_1,T_2]/(T_1^2,T_2^2)$, $Z_4[T_1]/(T_1^2)$, $Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2$, $Z_4[T_1,T_2]/(T_1^2,T_2^2,T_1T_2,2T_1,2T_2)$, $F_8[T_1]/(T_1^2)$ or $Z_4[T_1]/(T_1^3+T_1+1)$ and so $|R/\mathrm{I}|=16$

Since $|R|=|I||R/\mathrm{I}|$, which implies $|R| = 2\times 16 = 32$

We note that R direct product of local rings.

So $R\cong R_1\times R_2\times\ldots\times R_n$, where $n=2,3,4$ or 5.

If $n=5$, then $R\cong Z_2\times Z_2\times Z_2\times Z_2\times Z_2$

If $I=Z_2\times(0)\times(0)\times(0)\times(0)$, then $R/\mathrm{I}\cong Z_2\times Z_2\times Z_2\times Z_2$ that is a contradiction.

If $n=4$, then $R\cong Z_2\times Z_2\times Z_2\times Z_4$ or $Z_2[T_1]/(T_1{}^2)$

If $I=Z_2\times(0)\times(0)\times(0)$, then $R/\mathrm{I}\cong Z_2\times Z_2\times Z_4$, that is a contradiction.

If $n=3$, then $R\cong Z_2\times Z_2\times A$, when $A\cong F_8$, Z_8, $Z_2[T_1]/(T_1{}^3)$, $Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$, $Z_4[T_1]/(2T_1,T_1{}^2)$ or $Z_4[T_1]/(2T_1,T_1{}^2-2)$.

1- $R\cong Z_2\times Z_2\times F_8$ that is a contradiction.
2- $R\cong Z_2\times Z_2\times Z_8$, if $\mathrm{I}=Z_2\times(0)\times(0)$, then $R/\mathrm{I}\cong Z_2\times Z_8$ that is a contradiction.
3- $R\cong Z_2\times Z_2\times Z_2[T_1]/(T_1{}^3)$, if $\mathrm{I}=Z_2\times(0)\times(0)$, then $R/\mathrm{I}\cong Z_2\times Z_2[T_1]/(T_1{}^3)$, that is a contradiction.
4- $R\cong Z_2\times Z_2\times Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$, if $\mathrm{I}=Z_2\times(0)\times(0)$, then $R/I\cong Z_2\times Z_2[T_1,T_2]/(T_1{}^2,T_1T_2,T_2{}^2)$ that is a contradiction.
5- $R\cong Z_2\times Z_2\times Z_4[T_1]/(2T_1,T_1{}^2)$, if $I_1=Z_2\times(0)\times(0)$, then $R/I_1\cong Z_2\times Z_4[T_1]/(2T_1,T_1{}^2)$ that is a contradiction.
6- $R\cong Z_2\times Z_2\times Z_4[T_1]/(2T_1,T_1{}^2-2)$, if $I=Z_2\times(0)\times(0)$, then $R/I\cong Z_2\times Z_4[T_1]/(2T_1,T_1{}^2-2)$ that is a contradiction.

If $n=2$, then $R\cong Z_2\times Z_{16}$ or $Z_4\times Z_8$.

(a) $R_1\times R_2$, s.t $|R_1|=p^n$ and $|R_2|=2$ since $|R_1|=16=p^t$, then

there are 21 rings satisfied this condition, where $R_1\cong F_{16}$, Z_{16}, $Z_2[T_1]/(T_1{}^4)$, $Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2)$, $Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2-T_1T_2)$, $Z_2[T_1,T_2]/(T_1{}^3,T_1T_2,T_2{}^2)$, $Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2\times Z_3$, $Z_4[T_1]/(T_1{}^2)\times Z_3$, $Z_4[T_1]/(T_1{}^2+2)$, $Z_4[T_1]/(T_1{}^2+1)$, $Z_4[T_1]/(T_1{}^2+3)$, $Z_4[T_1]/(T_1{}^2+T_1+1)$, $Z_4[T_1]/(T_1{}^3,2T_1)$, $Z_4[T_1]/(T_1{}^3-2,2T_1)$, $Z_4[T_1,T_2]/(T_1{}^2,T_2{}^2,T_1T_2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1{}^2-2,T_2^2,T_1T_2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1{}^2,T_2^2,T_1T_2-2,2T_1,2T_2)$, $Z_4[T_1,T_2]/(T_1{}^2-2,T_2^2-2,T_1T_2,2T_1,2T_2)$, $Z_8[T_1]/(2T_1,T_1{}^2)$, $Z_8[T_1]/(2T_1,T_1{}^2+4)$.

1- $R\cong Z_{16}\times Z_2$, if $\mathrm{I}=(0)\times Z_2$, then $R/\mathrm{I}\cong Z_{16}$, thus $\Gamma_I(R)\cong$ Fig 6.
2- $R\cong F_{16}\times Z_2$ or $F_4[T_1]/(T_1{}^2)\times Z_2$, which is a contradiction.
3- $R\cong Z_2[T_1]/(T_1{}^4)\times Z_2$, if $\mathrm{I}=(0)\times Z_2$, then $R/\mathrm{I}\cong Z_2[T_1]/(T_1{}^4)$, thus $\Gamma_I(R)=$ Fig 6.
4- $R\cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2)\times Z_2$, if $I=(0)\times Z_2$, then $R/I\cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2)$, thus $\Gamma_I(R)\cong$ Fig 7.
5- $R\cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2-T_1T_2)\times Z_2$, if $I\cong(0)\times Z_2$, then $R/I\cong Z_2[T_1,T_2]/(T_1{}^2,T_2{}^2-T_1T_2)$, thus $\Gamma_I(R)\cong$ Fig 8.
6- $R\cong Z_2[T_1,T_2]/(T_1{}^3,T_1T_2,T_2{}^2)\times Z_2$, if $I\cong(0)\times Z_2$, then $R/I\cong Z_2[T_1,T_2]/(T_1{}^3,T_1T_2,T_2{}^2)$, thus $\Gamma_I(R)\cong$ Fig 9.
7- $R\cong Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2\times Z_2$, if $I=(0)\times Z_2$, then $R/I\cong Z_2[T_1,T_2,T_3]/(T_1,T_2,T_3)^2$.
8- $R\cong Z_4[T_1]/(T_1{}^2)\times Z_2$, if $I_1\cong(0)\times Z_2$, then $R/I_1\cong Z_4[T_1]/(T_1{}^2)$, thus $\Gamma_I(R)=$ figure 10.
9- $R\cong Z_4[T_1]/(T_1{}^2+2)\times Z_2$, if $\mathrm{I}\cong(0)\times Z_2$, then $R/\mathrm{I}\cong Z_4[T_1]/(T_1{}^2+2)$, thus $\Gamma_I(R)\cong$ figure 6.
10- $R\cong Z_4[T_1]/(T_1{}^2+1)\times Z_2$, if $\mathrm{I}\cong(0)\times Z_2$, then $R/\mathrm{I}\cong Z_4[T_1]/(T_1{}^2+1)$ that is a contradiction. .
11- $R\cong Z_4[T_1]/(T_1{}^2+3)\times Z_2$, if $\mathrm{I}\cong(0)\times Z_2$, then $R/\mathrm{I}\cong Z_4[T_1]/(T_1{}^2+3)$ that is a contradiction.

12- $R \cong Z_4[T_1]/(T_1{}^2 + T_1 + 1) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1]/(T_1{}^2 + T_1 + 1)$ that is a contradiction.

13- $R \cong Z_4[T_1]/(T_1{}^3, 2T_1) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1]/(T_1{}^3, 2T_1)$, thus $\Gamma_I(R) \cong$ figure 9 .

14- $R \cong Z_4[T_1]/(T_1{}^3 - 2, 2T_1) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1]/(T_1{}^3 - 2, 2T_1)$, thus $\Gamma_I(R) \cong$ figure 6 .

15- $R \cong Z_4[T_1, T_2]/(T_1{}^2, T_2{}^2, T_1T_2, 2T_1, 2T_2) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1, T_2]/(T_1{}^2, T_2{}^2, T_1T_2, 2T_1, 2T_2)$.

16- $R \cong Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2)$, thus $\Gamma_I(R) \cong$ figure 11 .

17- $R \cong Z_4[T_1, T_2]/(T_1{}^2, T_2^2, T_1T_2 - 2, 2T_1, 2T_2) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1, T_2]/(T_1{}^2, T_2^2, T_1T_2 - 2, 2T_1, 2T_2)$, thus $\Gamma_I(R) \cong$ figure 10 .

18- $R \cong Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_4[T_1, T_2]/(T_1{}^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2)$, thus $\Gamma_I(R) \cong$ figure 12 .

19- $R \cong Z_8[T_1]/(2T_1, T_1{}^2) \times Z_2$, if $I \cong (0) \times Z_2$, then $R/I \cong Z_8[T_1]/(2T_1, T_1{}^2)$, thus $\Gamma_I(R) =$ Fig 9 .

20- $R \cong Z_8[T_1]/(2T_1, T_1{}^2 + 4) \times Z_2$, if $I \cong (0) \times Z_3$, then $R/I \cong Z_8[T_1]/(2T_1, T_1{}^2 + 4)$, thus $\Gamma_I(R) \cong$ Fig 8 .

(b) $R \cong R_1 \times R_2$, s.t $|R_1| = p^n$ and $|R_2| = 4$ since $|R_1| = 8 = p^t$, then

there are 6 rings satisfied this condition .

1- $Z_8 \times Z_4$, if $I \cong (0) \times (2)$, then $R/I \cong Z_8 \times Z_2$ that is a contradiction .

2- $R \cong F_8 \times Z_4$, if $I \cong (0) \times (2)$, then $R/I \cong F_8 \times Z_2$ that is a contradiction .

3- $R \cong Z_2[T_1]/(T_1{}^3) \times Z_4$, if $I \cong (0) \times (2)$, then $R/I \cong Z_2[T_1]/(T_1{}^3) \times Z_2$ that is a contradiction .

4- $R \cong Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2) \times Z_4$, if $I \cong (0) \times (2)$ then $R/I \cong Z_2[T_1, T_2]/(T_1{}^2, T_1T_2, T_2{}^2) \times Z_2$ that is a contradiction .

5- $R \cong Z_4\, [T_1]/(2T_1, T_1{}^2) \times Z_4$, if $I \cong (0) \times (2)$, then $R/I \cong Z_4\, [T_1]/(2T_1, T_1{}^2) \times Z_2$ that is a contradiction .

6- $R \cong Z_4\, [T_1]/(2T_1, T_1{}^2 - 2) \times Z_4$, if $I \cong (0) \times (2)$, then $R/I \cong Z_4\, [T_1]/(2T_1, T_1{}^2 - 2) \times Z_2$ that is a contradiction . ■

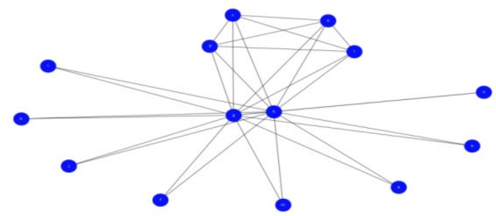

Fig 6. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_{16} \times Z_2$, $Z_2[T_1]/(T_1{}^4) \times Z_2$, $Z_4[T_1]/(T_1{}^2 + 2) \times Z_2$ or $Z_4[T_1]/(T_1{}^3 - 2, 2T_1) \times Z_2$

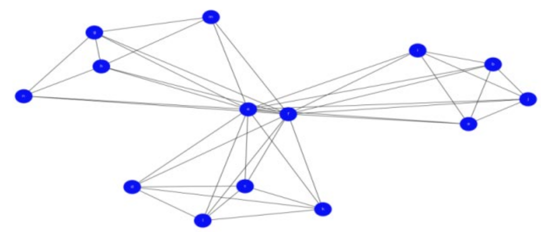

Fig 7. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_2[T_1, T_2]/(T_1{}^2, T_2{}^2) \times Z_2$.

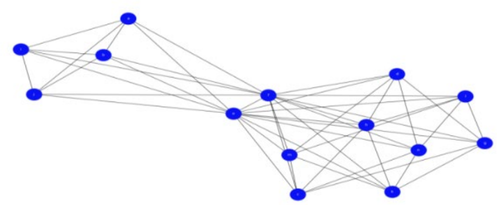

Fig 8. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_2[T_1, T_2]/(T_1{}^2, T_2{}^2 - T_1T_2) \times Z_2$ or $Z_8[T_1]/(2T_1, T_1{}^2 + 4) \times Z_2$.

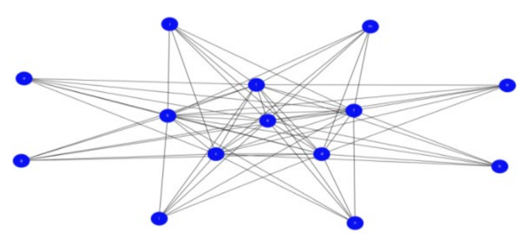

Fig 9. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_2[T_1, T_2]/(T_1{}^3, T_1T_2, T_2{}^2) \times Z_2$, $Z_4[T_1]/(T_1{}^3, 2T_1) \times Z_2$ or $Z_8[T_1]/(2T_1, T_1{}^2) \times Z_2$

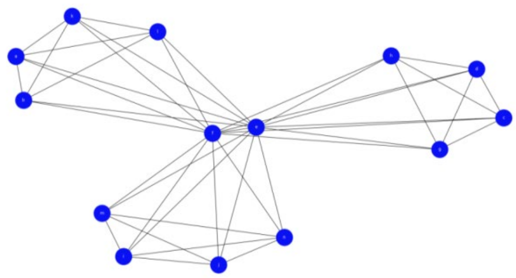

Fig 10. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_4[T_1]/({T_1}^2) \times Z_2$.

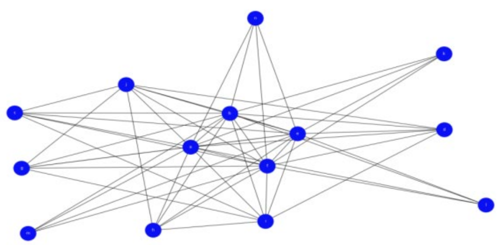

Fig 11. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_4[T_1, T_2]/({T_1}^2 - 2, T_2^2, T_1T_2, 2T_1, 2T_2) \times Z_2$

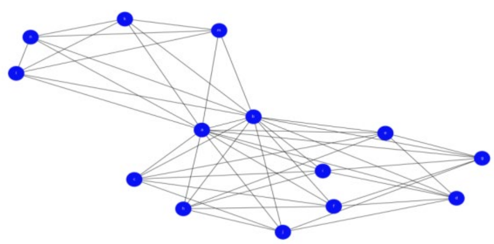

Fig 12. $\Gamma_I(R)$, where $I = (0) \times Z_2$ and $R = Z_4[T_1, T_2]/({T_1}^2 - 2, T_2^2 - 2, T_1T_2, 2T_1, 2T_2) \times Z_2$.

II. Conclusion

In this work, we introduce the notion if $|V(\Gamma_I(R))| = 12$, then there are eight graphs (Fig(1) , Fig(2), Fig(3), Fig(4), Fig(5), K_{12}, $K_{6,6}$, $K_{4,8}$) realized a ring R with respect ideal I, if $|V(\Gamma_I(R))| = 13$, then there are one graphs (K_{13}) realized a ring R with respect ideal I and if $|V(\Gamma_I(R))| = 14$, then there are nine graphs (K_{14}, $K_{7,7}$, Fig (6), Fig(7), Fig(8), Fig(9), Fig(10), Fig(11), Fig(12)) realized a ring R with respect ideal I.

Acknowledgement

The authors would like to express their gratitude to the College of Computer Science and Mathematics at the University of Mosul for its support of this report.

References

[1] S. Redmond, "An Ideal-Based Zero-Divisor Graph of a Commutative Ring," COMMUNICATIONS IN ALGEBRA® Vol. 31, vol. No. 9, no. 1, pp. 4425–4443, Jan. 2003.

[2] D. F. Anderson and P. S. Livingston, "The Zero-Divisor Graph of a Commutative Ring," Journal of Algebra, vol. 217, no. 2, pp. 434–447, Jul. 1999.

[3] S. E. Atani and A. Y. Darani, "Zero-divisor graphs with respect to primal and weakly primal ideals," Journal of the Korean Mathematical Society, vol. 46, no. 2, pp. 313–325, Mar. 2009.

[4] H. Ansari-Toroghy, F. Farshadifar, and F. Mahboobi-Abkenar, "On the ideal-based zero-divisor graphs," International Electronic Journal of Algebra, pp. 115–130, Jan. 2018.

[5] S. S. Essa and H. Q. Mohammad, "On the structure of a k-annihilating ideal hypergraph of commutative rings," vol. 38, no. 1, pp. 55–67, Jan. 2023.

[6] H. Maimani, M. R. Pournaki, and S. Yassemi, "Zero-Divisor Graph with Respect to an Ideal," Communications in Algebra - COMMUN ALGEBRA, vol. 34, pp. 923–929, Feb. 2006.

[7] M. Behboodi and R. Beyranvand, "On the structure of commutative rings with $P_1^{k_1} \dots P_n^{k_n}$ $(1 \le k_i \le 7)$ zero-divisors," European Journal of Pure and Applied Mathematics (elektronik), vol. 3, no. 2, Art. no. 2, 2010.

[8] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, by Addison-Wesley, Reading, 1969.

[9] D. M. Burton, Introduction to Modern Abstract Algebra, First Edition. Addison-Wesley, 1967.

[10] J. Smith, "Properties of Ideal-Based Zero-Divisor Graphs of Commutative Rings," Doctoral Dissertations, May 2014, [Online]. Available: https://trace.tennessee.edu/utk_graddiss/2729

[11] A. alstar Y. Sahbaa, "Annihilating - Ideals Graph of Commutative Rings," University of mosul, Iraq, 2019.

[12] B. Corbas and G. D. Williams, "Rings of Order p5 Part I. Nonlocal Rings," Journal of Algebra, vol. 231, no. 2, pp. 677–690, Sep. 2000.

[13] B. Corbas and G. D. Williams, "Rings of Order p5 Part II. Local Rings," Journal of Algebra, vol. 231, no. 2, pp. 691–704, Sep. 2000.

[14] J. T. Wang, "Zero Divisor of Commutative Rings." M.Sc. Thesis at the University of National Chung Cheng , Taiwan., 2005.