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Convergence of The Discrete Classical Optimal Control Problem To The Continuous Classical Optimal Control Problem Including A Nonlinear Hyperbolic P.D. Equation

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Abstract

Our focus in this paper is to study the behaviour in the limit of the discrete classical optimal control problem including partial differential equations of nonlinear hyperbolic type. We study that the discrete state and its discrete derivative are stable in Hilbert spaces $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. The discrete state equations containing discrete controls converge to the continuous state equations. The convergent of a subsequence of the sequence of discrete classical optimal for the discrete optimal control problem, to a continuous classical optimal control for the continuous optimal control problem is proved. Finally the necessary conditions for optimality of the discrete classical optimal control problem converge to the necessary conditions for optimality of the continuous optimal control problem, so as the minimum principle in blockwise form for optimality.

Keywords: Optimal Control, Nonlinear Systems, State Constrains, Classical Controls, Converges, Stability.

تقارب مسألة السيطرة الامثلية التقليدية من النمط المقسم لمسألة السيطرة الامثلية التقليدية من النمط المستمر لمعادلة تفاضلية جزئية غير خطية من النمط الزائدي

الخلاصة

هدفنا في البحث هو دراسة سلوكية مسألة السيطرة الامثلية من النمط المقسم (discrete) أي بعبارة أخرى هو دراسة تقارب مسألة السيطرة الامثلية التقليدية من النوع المقسم لمعادلة تفاضلية غير خطية من النمط الزائدي لمسألة السيطرة الامثلية التقليدية من النمط المستمر. في هذا البحث برهنا استقرارية الحالة من النوع المقسم (discrete state) و مشتقاتها المقسمة (discrete derivative) بوجود سيطرة تقليدية مقسمة (discrete controls) . برهنا إن الغاية لمتتابعة جزئية من متتابعة من السيطرات الامثلية المقسمة لمسألة السيطرة الامثلية التقليدية المقسمة هي سيطرة أمثلية مستمرة لمسألة السيطرة الامثلية التقليدية المستمرة. كذلك برهنا إن الغاية لمتتابعة جزئية من متتابعة من السيطرات الامثلية المقسمة المقبولة (admissible) والتي تحقق الشروط الضرورية المثلى لمسألة السيطرة الامثلية المقسمة هي سيطرة أمثلية مقبولة (admissible) و تحقق الشروط الضرورية المثلى لمسألة السيطرة الامثلية المستمرة. وكذلك برهنا التقارب لصيغة مبدأ الحزم الاصغري(Minimum principle blockwise form).

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Introduction

The behaviour of the discrete classical optimal control problem is very important. Since it tell us that the discrete form for the continuous classical optimal control is suitable to use this discrete form or not. This importance leaded many researchers have studied about the behaviour in the limit of discrete optimal control problem involving ordinary differential equations as in [1], [2], [3], and [4]. For these causes we deal with in this work the study of the behaviour for the discrete classical optimal control problem. i.e. we prove that the discrete classical optimal control problem of a nonlinear hyperbolic partial differential equations which is studied in [5] converges in the limit to the continuous classical optimal control problem of a nonlinear hyperbolic partial differential equations which is studied in [6]. Therefore we describe in the first two sections some forms, assumptions, and results which were obtained from the study of the continuous and the discrete classical optimal control problem of a nonlinear hyperbolic partial differential equations.

We study first the stability of discrete state in the Hilbert space $H_0^1(\Omega)$ and its discrete derivative in the Hilbert space $L^2(\Omega)$. Then we show that the discrete solution of the weak form state equations (in the discrete optimal control problem) converges in the limit to the solution of the weak form state equation (in the continuous optimal control problem). We prove that a subsequence of the sequence of discrete classical optimal control problem (for the discrete control problem) converges in the limit to a classical optimal control for the continuous control problem. Also we prove if a subsequence of the sequence of discrete admissible classical controls which satisfy the necessary conditions for optimality for the discrete optimal control problem then the limit of this subsequence is an admissible classical control which satisfies the necessary conditions for optimality for the continuous optimal control problem, so as the minimum principle in blockwise form for optimality.

1. Description of the Continuous Classical Optimal Control Problem:-

In this section, some forms, and results (which will need their in this paper) of the continuous classical optimal control problem of a nonlinear hyperbolic partial differential equations (CCOCP) are described which studied by [6]. We begin with the weak form of the continuous state equations of a nonlinear partial differential equations is

$$< y_{tt}, v > + a(t, y, v) = \left(f\ (t, y(t), u(t)), v\right) \quad ...(1)$$

$y(.,t) \in V$, $\forall v \in V$, a.e. on I

$$y(0) = y^0, \text{ in } \Omega , \quad \dots (2)$$

$$y_t(0) = y^1, \text{ in } \Omega \quad \dots (3)$$

where $\Omega \subset R^d$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial\Omega$, and let $I = (0,T)$, $0 < T < \infty$, $Q = \Omega \times I$.

The operator $a(t,.,.)$ is the usual bilinear form which is symmetric and satisfies the following elliptic conditions for some $a_1 \ge 0$, $a_2 \ge 0$ $\forall v, w \in V$ and $t \in \bar{I}$,

$a(t,v,v) \ge a_1 \|v\|_1^2$,

and

$|a(t,v,w)| \le a_2 \|v\|_1 \|w\|_1$,

and the function f is defined on $Q \times R \times U$, continuous w.r.t.(with respect to) x, t, measurable w.r.t. y & u, and it satisfies:-

$|f(x,t,y,u)| \le F(x,t) + b\,|y|$,

where $(x,t) \in Q$, $y, u \in R$, and $F \in L^2(Q)$

$|f(x,t,y_1,u) - f(x,t,y_2,u)|$
$\le L\,|y_1 - y_2|$,

where $(x,t) \in Q$, $y_1, y_2, u \in R$.

We denote by $|.|$ the Euclidean norm in R^n, by $\|.\|_\infty$ the norm in $L^\infty(\Omega)$, by $(.,.)$ and $\|.\|_0$ the inner product and norm in $L^2(\Omega)$, by $(.,.)_1$ and $\|.\|_1$ the inner product and norm in Sobolev space $V = H_0^1(\Omega)$, by $<.,.>$ the duality bracket between V and its dual V^*, and by $\|.\|_Q$ the norm in $L^2(Q)$.

The set of *classical controls* is $u \in W$, $W \subset L^2(Q)$, where

$W = \{u \in L^2(Q) \,|\, u(x,t) \in U, \text{a.e.in} Q\}$, where U is a compact and convex subset of R^v (usually $v = 1$ or $v = 2$),

The continuous classical optimal control problem (CCOCP) is to find $u \in W$, such that

$G_0(u) = \min_{w \in W_A} G_0(w)$

where W_A, is the set of *admissible control* which is defined by:

$W_A =$
$\{u \in W^n \,|\, G_m(u) = 0, (\forall\ 1 \le m \le p)$
$, G_m(u) \le 0, (\forall\ p+1 \le m \le q)\}$

$G_0(u)$ is the *cost function* and $G_m(u)$ are the *constraints* on the state and control variables y and u which are defined by

$G_m(u) =$
$\int_Q g_m(x,t,y(x,t),u(x,t))\,dxdt$

for each $m = 0,1,...,q$

where $y = y_u$ is the solution of (1-3), for the control u. This solution proved exist and unique [6].

The existence of a continuous classical optimal control proved in [6], under the above assumption, with $W_A \ne f$, and the function g_m, $(m = 0,1,...,q)$ is defined on $Q \times R \times U$, measurable for fixed y and u, continuous for fixed (x,t) and satisfies

$|g_m(x,t,y,u)| \le G_m(x,t) + g_m y^2$,

$\forall (x,t,y,u) \in Q$, $G_m \in L^2(Q)$, $g_m \ge 0$

The adjoint-state $f = f_u$ (where $y = y_u$) equations for each $v \in V$, a.e. in I satisfy (with drooping the index m) :-

$$< f_{tt}, v > + a(t, v, f) = (f f_y(t, y(t), u(t)) + g_y(x, t, y(t), u(t)), v) \quad \dots (4)$$

$$f(x, T) = f_t(x, T) = 0 \quad \dots (5)$$

While the *Hamiltonian* H which is defined by

$$H(x, t, y, z, u) := f f(x, t, y, u) + g(x, t, y, u)$$

for $u, u' \in W$ the *directional derivative* of G is given by

$$DG(u, u' - u) = \int_Q H_u(y, z, u)(u' - u)\,dxdt$$

$$H_u(y, z, u) = H_u(x, t, y, z, u)$$

with y, z and u are functions of x and t.

The necessary conditions for optimality is obtained when $u \in W$ is an optimal classical control, i.e. there exist multipliers $l_m \in R$, $m = 1, 2, \dots, p$, $l_m \geq 0$, $m = p+1, \dots, q$, $l_0 \geq 0$, with $\sum_{m=0}^{q} |l_m| = 1$, such that

$$\sum_{m=0}^{q} l_m DG_m(u, u' - u) \geq 0, \quad \text{for each } u' \in W \quad \dots (6)$$

and

$$l_m G_m(u) = 0, \quad \dots (7)$$

for each $m = p+1, \dots, q$

which are equivalent to the (weak) pointwise minimum principle

$$[f f_u(x, t, y, u) + g_u(x, t, y, u)]u(x, t) = \min_{u' \in U}[f f_u(x, t, y, u) + g_u(x, t, y, u)]u'(x, t) \quad \dots (8)$$

2. Description of the Discrete Classical Optimal Control Problem: -

In this section some forms, assumptions, and results (which will need their in this paper) of the discrete classical optimal control problem of a nonlinear hyperbolic partial differential equations (DCOCP) are presented which studied by [5]. We begin with the operator $a(t, ., .)$ which is supposed independent on t. The region Q is divided into subregions $Q_{ij} := S_i^n \times I_j^n$, where $\{I_j^n\}_{j=0}^{N(n)-1}$ be a subdivision of the interval $\bar{I}$ into $N(n)$ intervals, where $I_j^n := [t_j^n, t_{j+1}^n]$ of equal lengths $\Delta t = T / N$, and $\{S_i^n\}_{i=1}^{M(n)}$ be an admissible regular triangulation of $\bar{\Omega}$, for every integer n.

Let $V_n \subset V = H_0^1(\Omega)$ be the space of continuous piecewise affine in Ω.

Let W^n be the set of discrete (blockwise constants) classical controls (piecewise constants classical controls), i.e.

$$W^n = \{w = w^n \in W \mid w(x, t) = w_{ij} \text{ in } Q_{ij}\}$$

The *discrete state* equations, for each $v \in V_n$ and $j = 0, 1, \dots, N-1$ is written in the form

$(z^n_{j+1} - z^n_j, v) + \Delta t\, a(y^n_{j+1}, v)$

$= \Delta t (f\,(t^n_j, y^n_{j+1}, u^n_j), v),$ ....(9)

$y^n_{j+1} - y^n_j = \Delta t\, z^n_{j+1},$ …(10)

$(y^n_0, v) = (y^0, v),$ …(11)

$(z^n_0, v) = (y^1, v),$ ….(12)

where $y^0 \in V$, $y^1 \in L^2(\Omega)$ are given, and $y^n_j, z^n_j \in V_n$, for $j = 0,1,..., N$, and the function f is defined on

$S^n_i \times I^n_j \times \quad \times W^n$ ($i = 1,2,..., M$)

continuous w.r.t. y^n_j , u^n_j and satisfies:-

$\left| f\,(x, t^n_j, y^n_{j+1}, u^n_j) \right| \le F_j(x) + b \left| y^n_{j+1} \right|,$

for $0 \le j \le N-1$, and $x \in \Omega$

and

$\left| f\,(x, t^n_j, y^n_{j+1}, u^n_j) - f\,(x, t^n_j, y^n_j, u^n_j) \right|$

$\le L \left| y^n_{j+1} - y^n_j \right|$

for $0 \le j \le N-1$, and $x \in \Omega$

where, $F_j(x) = F(x, t^n_j) \in L^2(\Omega)$, and L is the Lipschtiz constant for any j . *The discrete classical optimal control problem is* to find $u^n \in W^n_A$, such that

$G^n_0(u^n) = \min_{w^n \in W^n_A} G^n_0(w^n)$

where W^n_A is the set of all *discrete admissible classical controls* for the discrete optimal problem given by

$W^n_A = \{u^n \in W^n : \left| G^n_m(u^n) \right| \le e^n_{1m}$,for $(1 \le m \le p)$,

$G^n_m(u^n) \le e^n_{2m}, (p+1 \le m \le q)\}$

where e^n_{1m} and e^n_{2m} are given numbers, tend to zero as n goes to infinity. $G^n_0(u^n)$ is *the discrete cost,* and $G^n_m(u^n)$ *is the discrete constraints* on the control $u^n \in W^n$, *and* the discrete state which are defined by

$G^n_m(u^n) =$

$\Delta t \sum_{j=0}^{N-1} \int_\Omega g_m(x, t^n_j, y^n_{j+1}, u^n_j)\,dx$

for each $m = 0,1,2,..., q$

The existence of a discrete classical optimal control is obtained under the above assumptions, with $W^n_A \ne f$, W^n is compact, and the function $g^n_m(x, t^n_j, y^n_j, u^n_j)$, for each ($m = 0,1,..., q$) is defined on $\Omega \times I^n_j \times S^n_i \times W^n$, ($\forall i = 1,..., M$ & $\forall j = 0,1,..., N$), continuous w.r.t. u^n_j and u^n_j for fixed x and j , measurable w.r.t. x for fixed y^n_j & u^n_j , and satisfies

$\left| g^n_m(x, t^n_j, y^n_j, u^n_j) \right| \le$

$G^n_{jm}(x) + g_{jm}(y^n_j)^2$

, $\forall x \in \Omega$, $j = 0,1..., N$

where, $g_{jm} \ge 0$, $j = 0,1..., N$ and $G^n_{jm}(x) = G^n_m(x, t^n_j) \in L^2(\Omega)$.

The *general discrete* classical *adjoint state* $f^n_{u^n} = f^n = (f^n_0, f^n_1, ..., f^n_{N-1})$, (with drooping the index m) is given by (for $j = N-1, N-2, ..., 0$):

$(y^n_{j+1} - y^n_j, v) + \Delta t\, a(f^n_j, v)$

$= \Delta t (f^n_j f_y (y^n_{j+1}, u^n_j)$

$$+g_y(y^n_{j+1},u^n_j),v), v \in V_n \quad \ldots(13)$$

$$f^n_{j+1} - f^n_j = \Delta t\, y^n_j, \quad \ldots.(14)$$

$$f^n_N = y^n_N = 0, \quad \ldots..(15)$$

where $f^n_j, y^n_j \in V_n$, for each $j = 0,1,\ldots,N$

The *directional derivative* of G is given by

$$DG^n(u^n, u'^n - u^n) = \Delta t \sum_{j=0}^{N-1} (H^n_u(t^n_j, y^n_{j+1}, f^n_j, u^n_j), du^n_j) \quad \ldots(16)$$

where $u^n, u'^n \in W^n$, $du^n_j = u'^n_j - u^n_j$, and the *discrete Hamiltonian* H^n is defined by

$$H^n(x, t^n_j, y^n_{j+1}, f^n_j, u^n_j) := f^n_j f(x, t^n_j, y^n_{j+1}, u^n_j) + g^n(x, t^n_j, y^n_j, u^n_j),\ j = 0,1,\ldots,N-1$$

The necessary conditions for optimality is satisfied if $u^n \in W^n$ is an optimal classical control of the considered problem, W^n is convex, then u^n (classical weakly) extremal, i.e. there exists multipliers $l^n_m \in R$, (for each $m = 0,1,\ldots,q$) with $l^n_0 \geq 0$, and $l^n_m \geq 0$ (for $m = p+1, p+2,\ldots,q$) satisfy $\sum_{m=0}^{q} |l^n_m| = 1$, such that

$$\sum_{m=0}^{q} l^n_m DG^n_m(u^n, u'^n - u^n) \geq 0, \quad ..(17)$$

$\forall u'^n_j \in W^n$

and

$$l^n_m [G^n_m(u^n) - e^n_m] = 0, \quad \ldots(18)$$

for each $m = p+1, p+2,\ldots,q$,

where

$f^n_j = \sum_{m=0}^{q} l^n_m f^n_{mj}$, and $g^n_u = \sum_{m=0}^{q} l^n_m g^n_{mu}$

in the definition of $H^n_u = \sum_{m=0}^{q} H^n_{mu}$

If W^n has the form $W^n = \{u' = u'^n_j : u'^n_j \in U,\ j = 0,1,\ldots,N-1\}$ with $U \subset R$, then the above relations are equivalent to the following minimum principle in blockwise form (for each $j = 0,1,\ldots,N-1$, and $i = 1,2,\ldots,M$):

$$(f^n_j f_u(y^n_{j+1}, u^n_j) + g^n_u(y^n_j, u^n_j), u^n_{ij})_{T_i} = \min_{u'^n \in U} (f^n_j f_u(y^n_{j+1}, u^n_j) + g^n_u(y^n_j, u^n_j), u'^n)_{T_i}, \quad \ldots.(19)$$

3. Stability:-

In this section we study the stability of the discrete state solutions and its discrete derivatives for the discrete state equations in weak form by the following lemma.

Lemma 3.1:-

For every discrete control $u^n \in W^n$, if Δt is sufficiently small, then

$\|y^n_j\|^2_1 \leq c$, and $\|z^n_j\|^2_0 \leq c$,

for each $j = 0,1,\ldots,N$

$$\sum_{j=0}^{N-1} \|y^n_{j+1} - y^n_j\|^2_1 \leq c$$

and

$$\sum_{j=0}^{N-1}\left\|z_{j+1}^{n}-z_{j}^{n}\right\|_0^2 \le c$$

where c denotes to the various constants.

Proof:-

Substituting $v = z_{j+1}^{n}$ in (9), rewriting the first term in the L.H.S. (left hand side) of the obtained equation by another way, i.e.

$$\left\|z_{j+1}^{n}\right\|_0^2-\left\|z_{j}^{n}\right\|_0^2+\left\|z_{j+1}^{n}-z_{j}^{n}\right\|_0^2$$
$$+\Delta t\, a(y_{j+1}^{n}, z_{j+1}^{n}) =$$
$$\Delta t\,(f\ (t_j^n, y_{j+1}^{n}, u_j^n), z_{j+1}^{n}) \qquad ..(20)$$

Since

$$a(y_{j+1}^{n}-y_{j}^{n}, y_{j+1}^{n}-y_{j}^{n}) =$$
$$(\Delta t)^2 a(z_{j+1}^{n}, z_{j+1}^{n})$$

and

$$a(y_{j+1}^{n}, y_{j+1}^{n}) - a(y_j^n, y_j^n) =$$
$$-(\Delta t)^2 a(z_{j+1}^{n}, z_{j+1}^{n}) + 2\Delta t a(y_{j+1}^{n}, z_{j+1}^{n})$$

then

$$\Delta t\, a(y_{j+1}^{n}, z_{j+1}^{n}) =$$
$$\tfrac{1}{2}[a(y_{j+1}^{n}, y_{j+1}^{n}) - a(y_j^n, y_j^n)$$
$$+a(y_{j+1}^{n}-y_{j}^{n}, y_{j+1}^{n}-y_{j}^{n})]$$

Now, substituting this equality in the L.H.S. of (20), then summing both sides of the obtained equation, for $j=0$, to $j=l-1$, using the assumptions on the operator $a(.,.)$, set $b=\min(1,\frac{a_2}{2})$ in the obtained equation, we get

$$b\left\|z_{l}^{n}\right\|_0^2+b\sum_{j=0}^{l-1}\left\|z_{j+1}^{n}-z_{j}^{n}\right\|_0^2+b\left\|y_{l}^{n}\right\|_1^2$$
$$+b\sum_{j=0}^{l-1}\left\|y_{j+1}^{n}-y_{j}^{n}\right\|_1^2 \le \left\|z_0^n\right\|_0^2+\tfrac{a_2}{2}\left\|y_0^n\right\|_1^2$$
$$+\sum_{j=0}^{l-1}\Delta t\,(f\ (t_j^n, y_{j+1}^{n}, u_j^n), z_{j+1}^{n}) \ \ldots (21)$$

From the assumptions on f, and by using the Cauchy-Schwarz inequality [7] we get

$$\left|(f\ (y_{j+1}^{n}, u_j^n), z_{j+1}^{n})\right| \le$$
$$\le \left\|F_j\right\|_0^2 + b\left\|y_{j+1}^{n}\right\|_1^2+\bar{b}\left\|z_{j+1}^{n}\right\|_0^2, \ ..(22)$$

where $\bar{b} = b+1$

But

$$\left\|y_{j+1}^{n}\right\|_1^2 = 2\left\|y_{j+1}^{n}-y_{j}^{n}\right\|_1^2+2\left\|y_{j}^{n}\right\|_1^2,$$

and

$$\left\|z_{j+1}^{n}\right\|_0^2 = 2\left\|z_{j+1}^{n}-z_{j}^{n}\right\|_0^2+2\left\|z_{j}^{n}\right\|_0^2$$

Substituting these equalities in (22), we have

$$\left|(f\ (y_{j+1}^{n}, u_j^n), z_{j+1}^{n})\right| \le$$
$$\left\|F_j\right\|_0^2+2b\left\|y_{j+1}^{n}-y_{j}^{n}\right\|_1^2+2b\left\|y_{j}^{n}\right\|_1^2$$
$$+2\bar{b}\left\|z_{j+1}^{n}-z_{j}^{n}\right\|_0^2+2\bar{b}\left\|z_{j}^{n}\right\|_0^2$$

Now, set $c=\max(b,\bar{b})$ substituting this inequality in the R.H.S. (right hand side) of (21), one obtains

$$b\left\|z_{l}^{n}\right\|_0^2+(b-c\Delta t)\sum_{j=0}^{l-1}\left\|z_{j+1}^{n}-z_{j}^{n}\right\|_0^2$$
$$+b\left\|y_{l}^{n}\right\|_1^2+(b-c\Delta t)\sum_{j=0}^{l-1}\left\|y_{j+1}^{n}-y_{j}^{n}\right\|_1^2$$

$$\leq \|z_0^n\|_0^2 + \tfrac{a_2}{2}\|y_0^n\|_1^2 + \|F\|_Q^2$$
$$+c\Delta t\sum_{j=0}^{l-1}\|y_j^n\|_1^2 + c\Delta t\sum_{j=0}^{l-1}\|z_j^n\|_0^2 \quad ..(23)$$

Now, with $\Delta t < b/c$, the 2nd and the 4th terms in the L.H.S. of (23) become positive, and on the other hand we have $\|z_0^n\|_0^2$, $\|y_0^n\|_1^2$, & $\|F\|_Q^2$ are bounded from the projection theorem and the assumptions on f , then using the discrete Gronwall's inequality [5], we get that

$$\|z_l^n\|_0^2 + \|y_l^n\|_1^2 \leq c,$$

where c denoting for various constants

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$\|y_l^n\|_1^2 \leq c$ and $\|z_l^n\|_0^2 \leq c$, for any arbitrary index l

Then

$\|y_j^n\|_1^2 \leq c$ and $\|z_j^n\|_0^2 \leq c$, for any $j = 0,1,\ldots,N$

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$$c\Delta t\sum_{j=0}^{N-1}\|y_j^n\|_1^2 + c\Delta t\sum_{j=0}^{N-1}\|z_j^n\|_0^2$$
$$\leq 2c\Delta t N = 2cT = \overline{c}$$

Now, by substituting $l = N$, in both sides of (23), using the last results, we obtain that all the terms in the R.H.S. of (23) are bounded, and with $\Delta t < b/c$, the 1st, 2nd, and the 3rd terms in the L.H.S. of (23), become positive, and we get

$$\sum_{j=0}^{N-1}\|y_{j+1}^n - y_j^n\|_1^2 \leq c \qquad \ldots.(24)$$

by the same above way, we can get also that

$$\sum_{j=0}^{N-1}\|z_{j+1}^n - z_j^n\|_0^2 \leq c \qquad \ldots(25)$$

4. Convergence:-

In this section we study the behavior of the discrete classical optimal control problem in the limit, i.e. we study the discrete classical optimal control problem and its main results which were considered in section 3 of this paper converge to the continuous classical optimal control problem and its main results were considered in section 2 of this paper. First we state the following control approximation lemma.

Lemma 4.1:-

For every control $u^n \in W^n$, there exists a sequence of $\{u^n\}$ in W^n, such that $u^n \to u$ strongly in $L^2(Q)$ (for prove see[8]).

Now, before indulging in the details, it is necessary to define the following functions almost every where on $\overline{I}$, as

$y_-^n(t) := y_j^n, t \in I_j^n$, $\forall j = 0,1,\ldots,N$,

$y_+^n(t) := y_{j+1}^n, t \in I_j^n$,

$\forall j = 0,1,\ldots,N-1$,

$z_-^n(t) := z_j^n$, $t \in I_j^n$, $\forall j = 0,1,\ldots,N$,

$z_+^n(t) := z_{j+1}^n, t \in I_j^n$,

$\forall j = 0,1,\ldots,N-1$,

$y_\wedge^n(t) :=$ the functions which is affine on each I_j^n, such that

$y_\wedge^n(t_j^n) := y_j^n$, for each $j = 0,1,\ldots,N$.

and

$z_{\wedge}^{n}(t) :=$ the functions which is affine on each I_j^n, such that

$z_{\wedge}^{n}(t_j^n) := z_j^n$, for each $j = 0,1,...,N$.

Theorem 4.1:-

If $u^n \to u$ strongly in $L^2(Q)$, then the corresponding discrete state y_{-}^{n}, y_{+}^{n}, $y_{\wedge}^{n}$ converges strongly in $L^2(Q)$, as $n \to \infty$.

Proof:-

From lemma 3.1, we got for any $j = 0,1,...,N$, that

$\|y_j^n\|_1^2 \le c$ and $\|z_j^n\|_0^2 \le c$

which give that $\|y_{-}^{n}\|_{L^2(I,V)}^2$, $\|y_{+}^{n}\|_{L^2(I,V)}^2$, $\|y_{\wedge}^{n}\|_{L^2(I,V)}^2$, $\|z_{-}^{n}\|_{L^2(Q)}^2$, $\|z_{+}^{n}\|_{L^2(Q)}^2$, and $\|z_{\wedge}^{n}\|_{L^2(Q)}^2$ are bounded.

From the inequalities (24) and (25), when $\Delta t \to 0$, we get that

$y_{+}^{n} - y_{-}^{n} \to 0$, strongly in $L^2(I,V)$, then in $L^2(Q)$,

and

$z_{+}^{n} - z_{-}^{n} \to 0$, strongly in $L^2(Q)$

which give clearly that

$y_{+}^{n} - y_{\wedge}^{n} \to 0$, strongly in $L^2(Q)$, and $z_{+}^{n} - z_{\wedge}^{n} \to 0$, strongly in $L^2(Q)$.

Therefore by using Alaoglu theorem [9] there exist a subsequences of $\{y_{-}^{n}\}$, $\{y_{+}^{n}\}$, $\{y_{\wedge}^{n}\}$ say again $\{y_{-}^{n}\}$, $\{y_{+}^{n}\}$, $\{y_{\wedge}^{n}\}$ (same notation) which converge weakly to some y in $L^2(I,V)$, and there exist a subsequences of $\{z_{-}^{n}\}$, $\{z_{+}^{n}\}$, $\{z_{\wedge}^{n}\}$ say again $\{z_{-}^{n}\}$, $\{z_{+}^{n}\}$, $\{z_{\wedge}^{n}\}$ (same notation) which converge weakly to some z in $L^2(Q)$, i.e.

$y_{-}^{n} \to y$, $y_{+}^{n} \to y$, $y_{\wedge}^{n} \to y$ weakly in $L^2(I,V)$

and $z_{-}^{n} \to z$, $z_{+}^{n} \to z$, $z_{\wedge}^{n} \to z$ weakly in $L^2(Q)$

Now, by using the Aubin compactness theorem [10, P.271], there exists a subsequence of $\{y_{\wedge}^{n}\}$ say again $\{y_{\wedge}^{n}\}$ (also same notation) which converges strongly to the same y in $L^2(Q)$, i.e. $y_{\wedge}^{n} \to y$ strongly in $L^2(Q)$, and then

$y_{+}^{n} \to y$ strongly in $L^2(Q)$, and $y_{-}^{n} \to y$ strongly in $L^2(Q)$

Now, let V_n (for each n) be the set of continuous and piecewise affine functions in Ω. By using the approximation of Galerkin [9], let $\{V_n\}_{n=1}^{\infty}$ be a sequence of subspaces of V, such that for each $v \in V$, there exists a sequence $\{v_n\}$, with $v_n \in V_n$, $\forall n$, and $v_n \to v$ strongly in V (by), hence $v_n \to v$ strongly in $L^2(\Omega)$ (this sequence is the continuous piecewise affine interpolation of v with respect to the subregions S_i^n). Let $z(t) \in C^2[0,T]$, such that $z(T) = z'(T) = 0$, $z(0) \ne 0$, and $z'(0) \ne 0$, and let $z^n(t)$ be the continuous piecewise interpolation of $z(t)$ w.r.t. I_j^n. Set $w = vz(t)$, and $w^n = v_n z^n(t)$, with

$w_-^n := v_n z_-^n(t), t \in I_j^n$,

$j = 0,1,...,N-1,\ v_n \in V_n$

$w_+^n := v_n z_+^n(t), t \in I_j^n$,

$j = 0,1,...,N-1,\ v_n \in V_n$

$w_\wedge^n := v_n z^n(t),\ t \in I,\ v_n \in V_n$

Now, by substituting $v = w_{j+1}^n$ in (9), summing both sides of the obtained equation for $j = 0$ to $j = N-1$, we get

$$\Delta t \sum_{j=0}^{N-1} \left(\frac{z_{j+1}^n - z_j^n}{\Delta t}, w_{j+1}^n\right)$$

$$+\Delta t \sum_{j=0}^{N-1} a(y_{j+1}^n, w_{j+1}^n)$$

$$= \Delta t \sum_{j=0}^{N-1} (f(t_j^n, y_{j+1}^n, u_j^n), w_{j+1}^n)$$

which can be written in the form

$$\int_0^T ((z_\wedge^n)', w_+^n)dt + \int_0^T a(y_+^n, w_+^n)dt$$

$$= \int_0^T (f(t_-^n, y_+^n, u^n), w_+^n)dt$$

Bu using the discrete integration by part formula on the 1st term in the L.H.S. of this equation, it becomes

$$-\int_0^T (z_-^n, (w_\wedge^n)')dt + \int_0^T a(y_+^n, w_+^n)dt$$

$$= \int_0^T (f(t_-^n, y_+^n, u^n), w_+^n)dt +$$

$$+(z_0^n, v_n)z(0) \quad \text{.... (26)}$$

From (10), we have

$$\left(\frac{y_+^n - y_-^n}{\Delta t}\right) = z_+^n$$

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$$(y_\wedge^n)' = z_+^n$$

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$$((y_\wedge^n)', v)z' = (z_+^n, v)z'$$

Integrating both sides of this equality from $t = 0$, to $t = T$, then using integrating by parts for the term in the L.H.S. of the obtained equation, we get

$$-\int_0^T (y_\wedge^n, v)z''(t)dt$$

$$= \int_0^T (z_+^n, v)z'(t)dt$$

$$+(y_0^n, v)z'(0) \quad \dots(27)$$

Since

$$\left.\begin{array}{l} z^n(t) \to z(t) \text{ in } C(I) \subset L^2(I) \\ v_n \to v \text{ strongly in } L^2(I,V) \\ \& \\ v_n \to v \text{ strongly in } L^2(\Omega) \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l} w_+^n = v_n z_+^n \to vz = w \text{ strongly in } L^2(I,V) \\ w_+^n = v_n z_+^n \to vz = w \text{ strongly in } L^2(Q) \\ v_n z^n(0) \to v z(0) \text{ strongly in } L^2(Q) \\ (w_\wedge^n)' = v_n z'^n \to vz' = w' \text{ strongly in } L^2(I,V) \end{array}\right.$$

On the other hand, we have $t_-^n \to t$ strongly in $L^\infty(I)$, $z_+^n, z_-^n \to z$ weakly in $L^2(Q)$, $y_+^n, y_-^n, y_\wedge^n \to y$ strongly in $L^2(Q)$, $y_+^n, y_-^n \to y$ weakly in $L^2(I,V)$, $y_0^n \to y^0$ strongly in V (from the projection theorem), $z_0^n \to y^1$ strongly in $L^2(\Omega)$ (from the projection theorem), and $u^n \to u$ strongly in $L^2(Q)$.

From the above convergences, and the assumptions on the function f, we can passage to the limit in (26) & in (27), to get

$$-\int_0^T (z,v)z'dt+\int_0^T a(y,v)z\,dt$$
$$=\int_0^T (f(t,y,u),v)z\,dt$$
$$+(y^1,v)z(0) \qquad ..(28)$$

and

$$-\int_0^T (y,v)z''(t)dt =$$
$$\int_0^T (z,v)z'(t)dt+(y^0,v)z'(0)$$
..(29)

Now, we have the following cases:-

Case I:- Choose $z\in C^2[0,T]$, such that $z(0)=z'(0)=z(T)=z'(T)=0$

Substituting $z'(0)=0$ in (29), integrating by parts the obtained equation, we get

$$\int_0^T (y_t,v)z'(t)dt=\int_0^T (z,v)z'(t)dt$$
$$\Rightarrow\ y_t=z$$

Substituting $z=y_t$ in (28), using $z(0)=0$, integrating by parts the obtained equation, we get

$$\int_0^T (y_{tt},v)z\,dt+\int_0^T a(y,v)z\,dt$$
$$=\int_0^T (f(t,y,u),v)z\,dt \qquad ..(30)$$

$\Rightarrow$
$(y_{tt},v)+a(y,v)=(f(t,y,u),v)$, $v\in V$, a.e. in I.
i.e. u is a solution of the state equation.

Case II:- Let $z(t)\in D[0,T]$, such that $z(0)\neq 0$, and $z(T)=0$.

Integrating by parts the 1st term in L.H.S. of (30), we have

$$-\int_0^T (y_t,v)z'dt+\int_0^T a(y,v)z\,dt$$
$$=\int_0^T (f(t,y,u),v)z\,dt$$
$$+(y_t(0),v)z(0) \qquad ..(31)$$

Substituting $z=y_t$ in (28), subtracting the obtained equation from (31), we get

$(y_t(0),v)z(0)=(y^1,v)z(0)$
$\Rightarrow$
$(y_t(0),v)=(y^1,v)$, $\forall v$
$\Rightarrow$
$y_t(0)=y^1(0)$.

Case III:- Let $z(t)\in D[0,T]$, such that $z(0)=0$, $z'(0)\neq 0$, and $z(T)=z'(T)=0$.

Integrating by parts twice the 1st term in L.H.S. of (30), we get

$$\int_0^T (y,v)z''dt+\int_0^T a(y,v)z\,dt$$
$$=\int_0^T (f(t,y,u),v)z\,dt$$
$$-(y(0),v)z'(0) \qquad \ldots(32)$$

From (29), we have

$$-\int_0^T (z,v)z'(t)dt$$
$$=\int_0^T (y,v)z''(t)dt$$
$$+(y^0,v)z'(0) \qquad ..(33)$$

Substituting $z(0)=0$ in (28), then substituting (33) in the 1st term in the L.H.S. of the obtained equation, then subtracting the last obtained equation from (32), we get

$$(y(0),v)z'(0)=(y^0,v)z'(0)$$
$$\Rightarrow$$
$$(y(0),v)=(y^0,v),\ \forall v$$
$$\Rightarrow$$
$$y(0)=y^0(0)$$

i.e. the limit point y is a solution to the weak form state equations in the continuous control problem.

Lemma 4.2:-

If $u^n \to u$ strongly in $L^2(Q)$, then $G_m^n(u^n) \to G_m(u)$, $\forall m=0,1,...,q$.

Proof:-

Since the operator $u^n \mapsto y^n = y_{u^n}^n$ is continuous (from lemma3.1 in ref. [5]), and since $G_m^n(u^n)$,(for each $m=0,1,...,q$) is continuous w.r.t. y and u (from lemma 3.2 in ref. [5]), and since $y^n = y_{u^n}^n \to y = y_u$, strongly in $L^2(Q)$ if $u^n \to u$ strongly in $L^2(Q)$ (from theorem 4.1), then $G_m^n(u^n) \to G_m(u)$, for each $m=0,1,...,q$.

Lemma 4.3:-

If $u^n \to u$ strongly in $L^2(Q)$, then $G_m'^n(u^n) \to G_m'(u)$, $\forall m=0,1,...,q$.

Proof:-

Form the assumptions on $g_m^n(x,t,y^n,u^n)$, for each $0 \le m \le q$, the Frechét derivatives for $G_m^n(u^n)$ (for each $0 \le m \le q$) exists, and from lemma 4.2 the convergence follows.

Corollary 4.1:-

If $u^n \to u$ strongly in $L^2(Q)$, then the corresponding adjoint-state (in the discrete form) f_-^n, f_+^n, $f_\wedge^n$ converge strongly in $L^2(Q)$ to $f = f_u$.

Proof:-

The convergence follows by using lemma 4.3, and by the same way which we used to prove theorem 4.1.

Theorem 4.2:-

For every n, let $\{u^n\}$ be a sequence of classical optimal controls for the discrete control problem. Then the limit of any strongly convergent subsequence of the sequence $\{u^n\}$ is classical optimal control for the continuous control problem.

Proof:-

Let $\{\bar{u}^n\}$ be a subsequence of the sequence $\{u^n\}$ of a classical optimal controls for the discrete control problem, such that

$\bar{u}^n \to \bar{u}$ strongly in $L^2(Q)$,

$\left|G_m^n(\bar{u}^n)\right| \le e_{1m}^n$, $e_{1m}^n \to 0$, as $n \to \infty$, for each $m=1,2,...,p$

and $G_m^n(\bar{u}^n) \le e_{2m}^n$, $e_{2m}^n \to 0$, as $n \to \infty$, for each $m=p+1,p+2,...,q$

From theorem 3.1 in ref. [6], the continuous classical optimal control problem has a classical optimal control say it $\tilde{u}$, and from lemma 4.1, there exists a sequence of discrete classical controls $\{\tilde{u}^n\}$, such that $\tilde{u}^n \to \tilde{u}$ strongly in $L^2(Q)$, then from lemma 4.2, we get that $G_m^n(\tilde{u}^n) \to G_m(\tilde{u})$, $\forall\ m = 0,1,\ldots,p$, or in other word

$G_m^n(\tilde{u}^n) \to G_m(\tilde{u}) = 0$, $\forall\ m = 0,1,\ldots,p$

and

$G_m^n(\tilde{u}^n) \to G_m(\tilde{u}) = 0$

$\forall m = p+1, p+2,\ldots,q$

Hence we can choose e_{1m}^n and e_{2m}^n, such that

$\left|G_m^n(\tilde{u}^n)\right| \le e_{1m}^n$, $e_{1m}^n \to 0$, as $n \to \infty$, for each $m = 1,2,\ldots,p$

$G_m^n(\tilde{u}^n) \le e_{2m}^n$, $e_{2m}^n \to 0$, as $n \to \infty$, for each $m = p+1, p+2,\ldots,q$

Which means the sequence $\{\tilde{u}^n\}$ is admissible for the discrete control problem.

But, in the other hand, we have that

$\left|G_m(\bar{u})\right| = \lim_{n\to\infty}\left|G_m^n(\bar{u}^n)\right| \le \lim_{n\to\infty} e_{1m}^n = 0$

, for each $m = 1,2,\ldots,p$

$$G_m(\bar{u}) = \lim_{n\to\infty} G_m^n(\bar{u}^n)$$

$$\le \lim_{n\to\infty} e_{2m}^n = 0$$

, for each $m = p+1, p+2,\ldots,q$

which implies that

$\left|G_m(\bar{u})\right| \le 0 \Rightarrow G_m(\bar{u}) \le 0$, for each $m = 1,2,\ldots,p$

and $G_m(\bar{u}) \le 0$, for each $m = p+1, p+2,\ldots,q$

and

$G_0^n(\bar{u}^n) \le G_0^n(\tilde{u}^n)$, for each n,

Then from lemma 4.2 we get

$$G_0(\bar{u}) = \lim_{n\to\infty} G_0^n(\bar{u}^n) \le \lim_{n\to\infty} G_0^n(\tilde{u}^n)$$

$$= G_0(\tilde{u})$$

i.e. $\bar{u}$ is a classical optimal control for the continuous problem.

Theorem 4.3:-

Let $\{u^n\}$ be a sequence of admissible classical controls and satisfies the necessary conditions for optimality for the discrete control problem (the K-T-L conditions). Then the limit of any strongly convergent subsequence of $\{u^n\}$ is admissible and satisfies the necessary conditions for optimality for the continuous classical optimal control problem.

Proof:-

Let $\{u^n\}$ be a sequence of admissible controls for the discrete optimal control problem and satisfies the necessary conditions for optimality (which studied in [5], theorem 4.1), i.e. for each $u_j'^n \in W^n$, and $u_j^n \in W^n$,

$$\Delta t \sum_{j=0}^{N-1} \int_\Omega [\sum_{m=0}^{q} l_m^n f_u(y_{j+1}^n, u_j^n) f_{mj}^n +$$

$$l_m^n g_{mu}^n(y_j^n, u_j^n)](u_j'^n - u_j^n)dx \ge 0,$$

(34)

and

$l_m^n[G_m^n(u^n) - e_{2m}^n] = 0$, for each $m = p+1, p+2,\ldots,q$

$\Rightarrow \forall u'^n \in W^n$

$$\int_Q \sum_{m=0}^{q} [(l_m^n f_u(y^n, u^n) f_m^n$$

$$+ l_m^n g_{mu}^n(y^n, u^n)](u'^n - u^n)dxdt \ge 0, \quad (34a)$$

and

$$l_m^n[G_m^n(u^n) - e_{2m}^n] = 0, \quad (34b)$$

for each $m = p+1, p+2,\ldots,q$

Now, let $\{u^n\}$ be a subsequence of $\{u^n\}$ (same notation), and assume that u be the limit of this subsequence, i.e.

$u^n \to u$ strongly in $L^2(Q)$.

Then $G_m^n(u^n) \to G_m(u)$, and $G_m'^n(u^n) \to G_m'(u)$, $\forall m = 0,1,...,q$ (from lemmas 4.2 and 4.3), and then form theorem 4.2, we get that the limit u is admissible for the continuous classical optimal control problem.

Now, since for fixed m ($m = 1,2,...,q$) the sequence of numbers $\{l_m^n\}$ belongs compact sphere with radius 1, then $l_m^n \to l_m$, as $n \to \infty$, for each m ($m = 1,2,...,q$).

On the other hand, since $u^n \to u$ strongly in $L^2(Q)$, then

$y^n = y_{u^n}^n \to y = y_u$ strongly in $L^2(Q)$, (theorem 5.1), and

$f^n = f_{u^n}^n \to f = f_u$, strongly in $L^2(Q)$, (corollary 4.1)

Hence by taking the limit as $n \to \infty$, for both sides of (34a) and (34b), we get that

$$\int_Q \sum_{m=0}^{q} [(l_m f_u(y,u) f_m + l_m g_{mu}(y,u)](u'-u)dxdt \geq 0,$$

$\forall u' \in W$

$\Rightarrow$

$$\sum_{m=0}^{q} l_m D_m(u'-u,u)\Delta u \geq 0, \qquad (35)$$

where $\Delta u = u' - u$, $u, u' \in W$

and

$l_m G_m(u) = 0$,

$\forall m = p+1, p+2,...,q$

i.e. the limit u of a subsequence of the sequence $\{u^n\}$ satisfies the necessary conditions for optimality for the continuous optimal control problem.

$$|G_m(u)| = \lim_{n\to\infty} |G_m^n(u^n)| \leq \lim_{n\to\infty} e_{1m}^n = 0$$

$\Rightarrow |G_m(u)| \leq 0$

$\Rightarrow$

$G_m(u) \leq 0$, $\forall m = 1,2,..., p$

and

$$G_m(u) = \lim_{n\to\infty} G_m^n(u^n) \leq \lim_{n\to\infty} e_{2m}^n = 0$$

$\Rightarrow$

$G_m(u) \leq 0$, $\forall$ $m = p+1, p+2,...,q$

Corollary 4.2:-

Let $\{u^n\}$ be a sequence of admissible classical controls and satisfies the minimum principle in blockwise form for optimality for the discrete control problem. Then the limit of any strongly convergent subsequence of $\{u^n\}$ is admissible and satisfies the pointwise minimum principle form for optimality for the continuous classical optimal control problem.

Proof:-

Let $\{u^n\}$ be a subsequence (of $\{u^n\}$, same notation) of admissible classical controls and satisfies the minimum principle in blockwise form (19), for optimality (which studied in [5], theorem 4.2) , i.e. for each $u_j'^n \in W^n$, and $u_j^n \in W^n$,

$$(f_j^n f_u(y_{j+1}^n, u_j^n) + g_u^n(y_j^n, u_j^n), u_{ij}^n)_{T_i}$$
$$= \min_{u'^n \in U} (f_j^n f_u(y_{j+1}^n, u_j^n)$$

$$+ g_u^n(y_j^n, u_j^n), u'^n)_{T_i}$$ for

each $j = 0,1,..., N-1$, and $i = 1,2,..., M$,

From theorem 4.1 in ref.[5], we got that this minimum principle is equivalent to the necessary conditions for optimality (34a). From theorem 5.3 above we got that the limit u of the subsequence $\{u^n\}$(which satisfy the necessary conditions for optimality (34a)), satisfies the necessary conditions for optimality (35) for the continuous problem. But the necessary conditions for optimality (35) is equivalent to the following pointwise minimum principle (8) for the continuous optimal control (theorem 4.1 in ref. [6]), i.e.

$$[ff_u(x,t,y,u) + g_u(x,t,y,u)]u(x,t) = \min_{u'\in U}[ff_u(x,t,y,u) + g_u(x,t,y,u)]u'(x,t).$$

Conclusions

In this work we concluded that the behaviour of the discrete classical optimal control problem in the limit is stable and converges to the continuous classical optimal control. In other words we got on the following results:-

The discrete state and its discrete derivative are stable in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. The solution of the discrete state equations in weak form with discrete controls converges in the limit to the solution of the state equations in weak form (for the continuous problem), so as the adjoint-state equations. The limit of a subsequence of a sequence of discrete classical optimal control for the discrete optimal control problem is a classical optimal control for the continuous optimal control problem. Finally, the limit of a subsequence of the sequence of admissible discrete classical controls which satisfy the necessary conditions for optimality for the discrete optimal control problem is an admissible classical control and satisfies the necessary conditions for optimality for the continuous optimal control problem. The same result is obtained for the minimum principle in blockwise form for the optimality.

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