Generalized Spline Approximation Method for Solving Ordinary and Partial Differential Equations

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Abstract
         The main theme of this paper is to approximate the solution of ordinary differential equations by using basis of generalized spline functions then using tensor product analysis to generalize the approximation for solving partial differential equations, (PDE's).

Keywords: Generalized splines, tensor product, ordinary and partial differential equations**.**

<div dir="rtl">

حول دوال السبلاين العامة وتصميمها الى فضاءات ثنائية الابعاد

الخلاصة

        الهدف الرئيسي هو تقريب الحل للمعادلات التفاضلية الاعتيادية بأستخدام دوال السبلاين العامة (generalized spline) ثم استخدام الضرب الممتد(tensor product) لتعميم التقريب في حل المعادلات التفاضلية الجزئية.

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1. Introduction

Polynomial splines have been extensively used in several areas of applied mathematics such as computer graphics and approximation theory. Therefore, one of the first generalizations in this direction are the so called generalized splines which were introduced in the 50's of the 20th century by Ahlberg, Nilson and Walsh, [8]. In 1964 Schoenberg and Greville suggested that a spline function associated with a general linear differential operator of finite order which is called generalized spline and exhibited smoothness and best approximation results, [1]. In 2008, Saleh M. studied the numerical solutions of ordinary fractional differential equations by G-spline functions to solve the linear and nonlinear differential equations, [9]. The use of spline approximations for PDE's, and their relationship to finite difference schemes is considered by Hoskins and Sakai in 1970. Hoskins considers the two dimensional Poisson equation and show that a cubic spline approximation leads to a nine-point difference formula. For the heat conduction problem in one space dimension, Sakai studied a cubic spline approximation for the space derivative is combined with a

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difference approximation for the time derivative, [6].

2. Generalized Spline Functions

Schultz and Varga have defined generalized spline function $S(x)$ to be a function constructed in a piecewise manner, where in each piece $S(x)$ is a solution of the differential equation $L^* LS=0$. The linear differential operator L of order n, defined by, [2]:

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1}.$$

where $a_j(x)$ $C^n[a,b]$, (class of all functions which are n continuously differentiable defined on [a,b]) ; j=0,1,…,n; and $a_n(x) \neq 0$ on [a,b] , D = , and associate with L its formal adjoint operator:

Definition (2.1), [7]:

Let $\Delta : a = x_0 < x_1 < … < x_N = b$, $N \in$ be a partition on [a,b]. A real valued function S, defined on [a,b] is said to be generalized spline with partition Δ if the following conditions holds simultaneously:

1- $S \in {}^{2n}[x_{i-1},x_i]$; i=1,2, …, N.

2- $L^* LS(x) = 0$; $x \in [x_{i-1},x_i]$; i=1,2, …, N.

3-

where ${}^{2n}[x_{i-1},x_i]$ class of all functions defined on $[x_{i-1},x_i]$ which possess an absolutely continuous $(2n-1)^{th}$ derivative on $[x_{i-1},x_i]$ whose $2n^{th}$ derivative is in $L^2(x_{i-1},x_i)$.

3. Solution of ODE's Using Generalized Spline

The objective of this section is to search for an approximate solution in the form of generalized spline function of order 2n (since $L^* LS$ of order 2n) and class $C^{2n-2}[a,b]$ for the first order initial value problems given by:

where is the initial condition. If then problem (1) has a unique solution by assuming that f(t,y) is continuous and differentiable in D and Lipschitzian, [4] satisfies

where M is a constant called the Lipschitz constant.

The construction of the approximate solution starts by letting:

where a_j's are the unknown parameters which must be determined and the $u_j(t)$ are the generalized spline basis, and . The first algebraic equation related to the resulting system of S(t) is determined by the initial condition given in eq.(1), i.e., .

Now, S(t) should satisfy eq.(1) for t = h which gives the equation for the first subinterval [0,h], i.e. :

and repeat the same steps for the interval [h,2h], yields to:

and continuing in like manner, a spline function S(t) is obtained and satisfying the equation:

$$S'(ih) = f\big(ih, S(ih)\big),$$
$$\forall i = 1,2 …,2n - 1 \qquad (3)$$

and then the following linear system of algebraic equations is obtained:

$$\sum_{j=1}^{2n} a_j u_j'(ih) = f\left(ih, \sum_{j=1}^{2n} a_j u_j(ih) \right),$$
$$\forall i = 1,2,…,2n - 1 \qquad (4)$$

Which may be solved uniquely for $a_j's$

The following example illustrate the above method of solution and the accuracy of the results:

Example (3.1)

Suppose the generalized spline method is be used to approximate the solution of the initial value problem:

$$y' = y - t^2 + 1,$$
$$y(0) = 0.5, 0 \leq t \leq 1$$

with h = 0.2 ; $t_i = 0.2i$; i=1,2,3,4 ;
$y_0 = 0.5$

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Let $L^*Lw = D^4w - 13D^2w + 36w$, with basis functions:

$$w_1(t) = e^{3t}, \quad w_2(t) = e^{-3t},$$
$$w_3(t) = e^{2t}, \quad w_4(t) = e^{-2t}$$

which will give a non-polynomial generalized spline function as:

$$W(t) = a_0 e^{3t} + a_1 e^{-3t} + a_2 e^{2t} + a_3 e^{-2t} \qquad (5)$$

Then the algebraic system of equations is:

$$a_0 + a_1 + a_2 + a_3 = 0.5$$
$$3a_0 e^{3(0.2)} - 3a_1 e^{-3(0.2)} + 2a_2 e^{2(0.2)} - 2a_3 e^{-2(0.2)} =$$
$$a_0 e^{3(0.2)} + a_1 e^{-3(0.2)} + a_2 e^{2(0.2)} + a_3 e^{-2(0.2)} - (0.2)^2 + 1$$

$$3a_0 e^{3(0.4)} - 3a_1 e^{-3(0.4)} + 2a_2 e^{2(0.4)} - 2a_3 e^{-2(0.4)} =$$
$$a_0 e^{3(0.4)} + a_1 e^{-3(0.4)} + a_2 e^{2(0.4)} + a_3 e^{-2(0.4)} - (0.4)^2 + 1$$

$$3a_0 e^{3(0.6)} - 3a_1 e^{-3(0.6)} + 2a_2 e^{2(0.6)} - 2a_3 e^{-2(0.6)} =$$
$$a_0 e^{3(0.6)} + a_1 e^{-3(0.6)} + a_2 e^{2(0.6)} + a_3 e^{-2(0.6)} - (0.6)^2 + 1$$

and rewrite this system in the matrix form $Ax = b$ where, $|A| \neq 0$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2e^{0.6} & -4e^{-0.6} & e^{0.4} & -3e^{-0.4} \\ 2e^{1.2} & -4e^{-1.2} & e^{0.8} & -3e^{-0.8} \\ 2e^{1.8} & -4e^{-1.8} & e^{1.2} & -3e^{-1.2} \end{bmatrix},$$

$$x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0.5 \\ 0.96 \\ 0.84 \\ 0.64 \end{bmatrix} \qquad (6)$$

Then Gauss elimination method may be used to solve the above system to find a_0, a_1, a_2, a_3 which are found to be $a_0 = -0.192$, $a_1 = -0.198$, $a_2 = 0.86$, $a_3 = 0.029$ and hence:

$$W(t) = -0.192e^{3t} - 0.198e^{-3t}$$

$$+ 0.86e^{2t} + 0.029e^{-2t}$$

As a comparison, the exact solution of the considered problem is given by, [3]:

$$y(t) = (t+1)^2 - 0.5e^t$$

Figure (1), give the exact solution $y(t)$ and the approximation $W(t)$ for the considered example, in which one can see the accuracy of the obtained results and the applicability of the method.

4. Solution of PDE's Using Generalized Spline and Tensor Product

Generalized spline method will be used here to approximate the solution of PDE's for example of the form:

$$u_t = f(x, t, u_x, u_{xx}),$$
$$0 \le x \le 1, \ t > 0 \qquad (7)$$

with boundary conditions:

$$u(0, t) = u(1, t) = 0, \ t > 0 \qquad (8)$$

and initial condition

$$u(x, 0) = g(x), \ 0 \le x \le 1 \qquad (9)$$

where $g(x)$ is given function.

Let:

$$z(x, t) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} c_{i,j} \psi_j(t) \phi_i(x) \qquad (10)$$

where $\phi_i(x)$, are the basis of the generalized spline function for the equation $L_1^* L_1 = 0$ and $\psi_j(t)$, are the basis of generalized spline function for the equation $L_2^* L_2 = 0$. From the initial condition given by eq.(9), one may get:

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} c_{i,j} \psi_j(0) \phi_i(x) = g(x) \qquad (11)$$

substituting the knot points for the x-axis to get an equation for each knot point, and from the boundary conditions given by eq.(8), we have:

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$$\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{i,j}\psi_j(t)\phi_i(0) - 0 \qquad (12)$$

and

$$\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{i,j}\psi_j(t)\phi_i(1) = 0 \qquad (13)$$

Similarly, substitute the knot points for the t-axis to get an equation for each knot point at $x = 0$ and at $x = 1$, and from the eq.(7), we have:

$$\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{i,j}\psi_j'(t)\phi_i(x) =$$

$$f\left(x, t, \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{ij}\psi_j(t)\phi_i'(x), \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} c_{ij}\psi_j(t)\phi_i''(x)\right) \qquad (14)$$

substitute the mesh points (x_i, t_j), for all i, j where, $i = 1,2,...,m_1; j = 1,2,...,m_2$

, to get an equation for each pair (i,j), then from eqs.(11)-(14), and a system with unknown coefficients c_{ij} must be determined to compute eq.(10).

The next example is an illustration to the above method:

Example (4.1)

Consider the partial differential equation:
$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0$$

with boundary and initial conditions:
$$u(0,t) = u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = \sin(\pi x), \qquad 0 \le x \le 1$$

Let Δ_1 be a partition for the x-axis, such that:
$$\Delta_1 : 0 = x_0 < x_1 < x_2 < x_3 = 1$$

where h=1/3, then
$$x_0 = 0, \; x_1 = \frac{1}{3}, \; x_2 = \frac{2}{3}, \; x_3 = 1,$$

are the mesh points with respect to the x-axis.

Let the differential operator for the generalized spline function be given as Lu=D^2u+Du–6u, then the fundamental solutions of L^*Lu=0 are
$$u_1(x) = e^{3x}, u_2(x) = e^{-3x},$$

$$u_3(x) = e^{2x}, u_4(x) = e^{-2x}.$$

Let Δ_2 be a partition for the t-axis such that:
$$\Delta_2 : 0 = t_0 < t_1 < t_2 < t_3 = 0.03$$

where k=0.01, then

$$t_0 = 0, \; t_1 = 0.01, \; t_2 = 0.02, \; t_3 = 0.03$$
, will be the mesh points with respect to t-axis. Assume the same differential operator with respect to t, was applied to get the following fundamental solutions
$$u_1(t) = e^{3t}, u_2(t) = e^{-3t},$$
$$u_3(t) = e^{2t}, u_4(t) = e^{-2t}$$

Now, from eq.(11), one may have:
$$(c_{00} + c_{01} + c_{02} + c_{03})e^{3x} +$$
$$(c_{10} + c_{11} + c_{12} + c_{13})e^{-3x} +$$
$$(c_{20} + c_{21} + c_{22} + c_{23})e^{2x} +$$
$$(c_{30} + c_{31} + c_{32} + c_{33})e^{-2x}$$
$$= \qquad \sin((22/7)x)$$
(15)

and substituting $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$ in eq.(15) to get the first four algebraic equations. From eq.(12), we have:

$$(c_{00}e^{3t} + c_{01}e^{-3t} + c_{02}e^{2t} + c_{03}e^{-2t}) +$$
$$(c_{10}e^{3t} + c_{11}e^{-3t} + c_{12}e^{2t} + c_{13}e^{-2t}) +$$
$$(c_{20}e^{3t} + c_{21}e^{-3t} + c_{22}e^{2t} + c_{23}e^{-2t}) +$$
$$(c_{30}e^{3t} + c_{31}e^{-3t} + c_{32}e^{2t} + c_{33}e^{-2t})$$
$$=0 \qquad (16)$$

Also substituting:
$$t_1 = 0.01, t_2 = 0.02, t_3 = 0.03 \text{ in}$$
eq.(16) to get the second three algebraic equations. From eq.(13), we have:

$$(c_{00}e^{3t} + c_{01}e^{-3t} + c_{02}e^{2t} + c_{03}e^{-2t})e^3 +$$
$$(c_{10}e^{3t} + c_{11}e^{-3t} + c_{12}e^{2t} + c_{13}e^{-2t})e^{-3} +$$
$$(c_{20}e^{3t} + c_{21}e^{-3t} + c_{22}e^{2t} + c_{23}e^{-2t})e^2 +$$
$$(c_{30}e^{3t} + c_{31}e^{-3t} + c_{32}e^{2t} + c_{33}e^{-2t})e^{-2}$$
$$=0 \qquad (17)$$

Substituting:

$t_1 = 0.01,\ t_2 = 0.02,\ t_3 = 0.03$ in eq.(17) to get the third three equations. From eq.(14), we have:

$$(6c_{00}e^{3t} + 12c_{01}e^{-3t} + 7c_{02}e^{2t}$$
$$+11c_{03}e^{-2t})e^{3x} +$$
$$(6c_{10}e^{3t} + 12c_{11}e^{-3t} + 7c_{12}e^{2t}$$
$$+11c_{13}e^{-2t})e^{-3x} +$$
$$(c_{20}e^{3t} + 7c_{21}e^{-3t} + 2c_{22}e^{2t}$$
$$+6c_{23}e^{2x})e^{2x} +$$
$$(c_{30}e^{3t} + 7c_{31}e^{-3t} + 2c_{32}e^{2t}$$
$$6c_{33}e^{-2t})e^{-2x} = 0 \quad (18)$$

Substitute

$$\left(\tfrac{1}{3}, 0.01\right), \left(\tfrac{2}{3}, 0.01\right), \left(\tfrac{1}{3}, 0.02\right),$$
$$\left(\tfrac{2}{3}, 0.02\right), \left(\tfrac{1}{3}, 0.03\right), \left(\tfrac{2}{3}, 0.03\right) \quad \text{in}$$

eq.(18) to get six equations, then the resulting system of sixteen equations with sixteen algebraic unknown coefficients are obtained.

Gauss elimination method may be used to solve the algebraic equations (15)-(18), to get the approximate solution of the PDE given in the following function:

$$z(x,t) = (-2.593\,e^{3t} + 1.173\,e^{-3t} +$$
$$3.81\,e^{2t} - 2.59\,e^{-2t})\,e^{3x} + (8.2\,e^{3t} -$$
$$52.947\,e^{-3t} - 17.78\,e^{2t} +$$
$$58.493e^{-2t})e^{-3x} + (0.338e^{3t} +$$
$$3.92\,e^{-3t} - 0.142\,e^{2t} - 3.612e^{-2t})e^{2x} +$$
$$(-5.962\,e^{3t} + 47.855\,e^{-3t} + 14.112e^{2t} -$$
$$52.29e^{-2t})e^{-2x}$$

The exact solution is, [3]:

$$u(x,t) = \exp(-\pi^2 t)\sin(\pi x)\,,$$

$$t > 0,\ \ 0 \leq x \leq 1$$

The approximate solution $z(x,t)$ and the exact solution $u(x,t)$ are illustrated in table (1), figure (2), and figure (3).

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[10]-"Tensor Product Spline Surfaces",http://   folk.uio.no/in322.

Table (1) The approximate and the exact solutions of example (4.1):

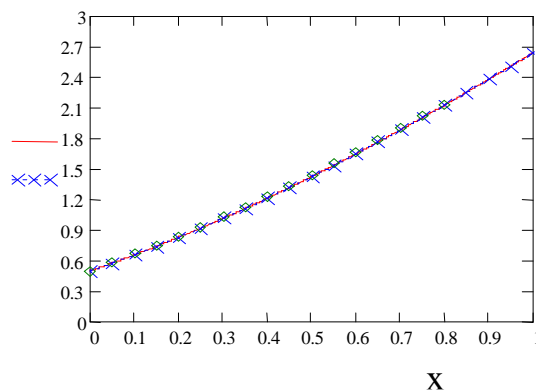| t	x	z(x,t)	u(x,t)	$\lvert u(x,t)-z(x,t)\rvert$
0	0	0.0001	0	0.0001
0.01	0	0.0009	0	0.0009
0.02	0	0.0009	0	0.0009
0.03	0	0.0009	0	0.0009
0	1/3	0.867	0.866	0.001
0.01	1/3	0.803	0.785	0.018
0.02	1/3	0.744	0.711	0.033
0.03	1/3	0.689	0.644	0.045
0	2/3	0.87	0.866	0.004
0.01	2/3	0.805	0.785	0.020
0.02	2/3	0.746	0.711	0.035
0.03	2/3	0.691	0.644	0.047
0	1	0.0096	0	0.0096
0.01	1	0.008	0	0.008
0.02	1	0.008	0	0.008
0.03	1	0.0081	0	0.0081

Figure (1): Exact and approximate solution of example
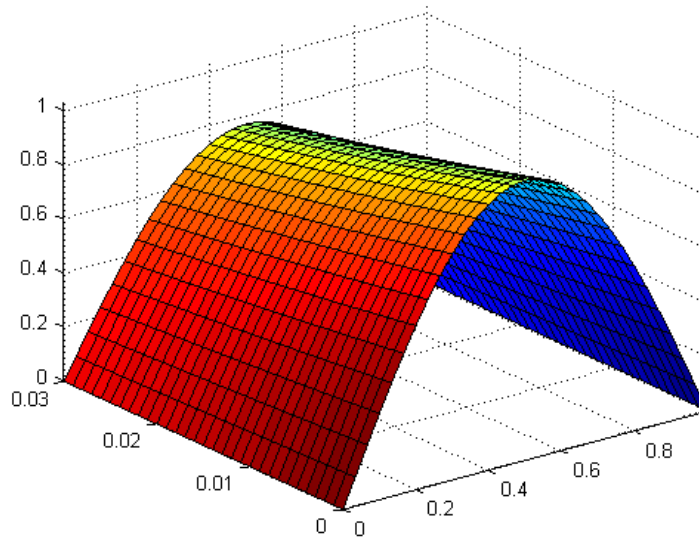
(3.1).
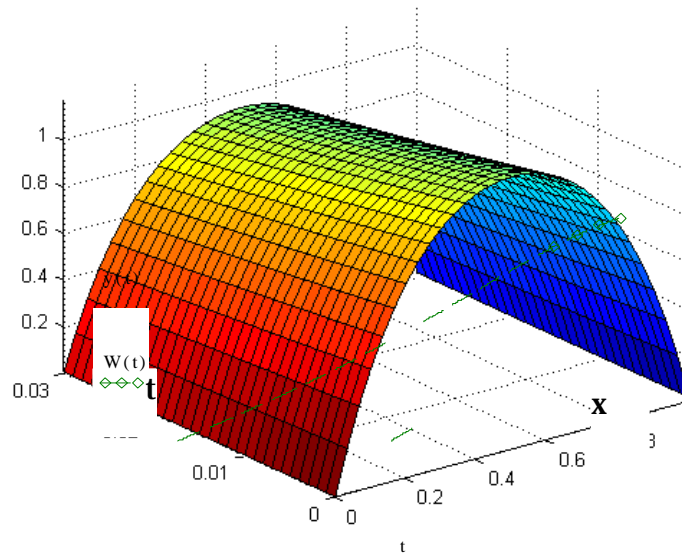
Figure (2):Approximate solution of example (4.1).

Figure (3):Exact solution of example (4.1).