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A Meromorphic Function and Its Derivative That Share One Value or Small Function

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Abstract

The aim of this work is to present two results of a uniqueness theorem of meromorphic functions. The first result is an improvement and a generalization of [1, Theorem1] and the second result gives an improvement of [2, Theorem1].

Keywords: meromorphic functions, sharing, Nevanlinnan's theory, small function.

دالة الميرومورفكية ومشتقتها التي لها حصة قيمة واحدة أو دالة صغيرة

الخلاصة

هدفنا في هذا العمل هو تقديم نتيجتين من نظرية الوحدانية للدوال الميرومورفكية، النتيجة الأولى هي تحسين و تعميم من نظرية 1 في [1] والنتيجة الثانية هي تحسين من نظرية 1 في [2].

0. Preliminaries

In this section we give some definitions and theorems relating to our research as found in [3], [4]. We assume that the reader is familiar with the basic results of meromorphic functions as found in [5], [6].

Definition 0.1([3, P.3]). For $x \geq 0$, we have

$$\log^+ x = \log x \text{, if } x \geq 1$$
$$= 0, \quad \text{if } 0 \leq x < 1.$$

Let f be a meromorphic functionin in $|z| \leq R\,(0 < R < \infty)$. For $0 < r < R$, we introduce the following definitions and theorems (see [3], [4]).

Definition 0.2([3, P.4]). Let f be a meromorphic function in the complex plane. For a real variable $r > 0$, we define a real valued function $m(r, f)$ by

$$m(r, f) = \frac{1}{2p}\int_0^{2p} \log^+\left|f(re^{iq})\right| dq .$$

The function $m(r, f)$ is a sort of averaged magnitude of $\log|f|$ on arcs of $|z| = r$ where $|f|$ is large.

Definition 0.3([3, P.42]). Let f be a meromorphic function in the complex plane. For a real variable $r > 0$, we define a real valued function $N(r, f)$ by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f)\log r,$$

where $n(t, f)$ is the number of poles of f in $|z| \leq t$, multiple poles being counted

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multiply and $n(0, f)$ is the multiplicity (order) of pole of f at $z = 0$ (if $f(0) \neq \infty$, then $n(0, f) =$ 0). The function $N(r, f)$ is a counting function of the poles of f.

Definition 0.4([3, P.42]).

Let f be a meromorphic function in the complex plane. For a real variable $r > 0$, we define a real valued function $\overline{N}(r, f)$ by

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f)\log r$$

where $\overline{n}(t, f)$ is the number of distinct poles of f in $|z| \leq t$. The function $\overline{N}(r, f)$ is a counting function of the poles of f each poles is counted only one.

Definition 0.5([4, P.189]).

For a positive integer k, we write

$$N_{k)}(r, f) = \int_0^r \frac{n_{k)}(t, f) - n_{k)}(0, f)}{t} dt + n_{k)}(0, f)\log r$$

where $n_{k)}(t, f)$ is the number of poles of f with multiplicities less than or equal to k in $|z| \leq t$, multiple poles being counted multiply. The function $N_{k)}(r, f)$ is a counting function of poles of f with multiplicity $\leq k$.

Definition 0.6([4, P.189]).

For a positive integer k, we write

$$N_{(k+1}(r, f) = \int_0^r \frac{n_{(k+1}(t, f) - n_{(k+1}(0, f)}{t} dt + n_{(k+1}(0, f)\log r$$

where $n_{(k+1}(t, f)$ is the number of poles of f with multiplicities greater than k in $|z| \leq t$, multiple poles being counted multiply. The function $N_{(k+1}(r, f)$ is a counting function of poles of f with multiplicity $> k$.

In the same way, we can define $\overline{N}_{k)}(t, f)$ and $\overline{N}_{(k+1}(r, f)$ (see [4, P.89]).

Definition 0.7([3, P.4]).

Let f be a meromorphic function in the complex plane. For a real variable $r > 0$, we define a real valued function $T(r, f)$ by

$T(r, f) = N(r, f) + m(r, f)$.

The function $T(r, f)$ is called the characteristic function of f. It plays a cardinal role in the whole theory of meromorphic functions.

Definition 0.8([4, P.2]).

For any complex number a, we write

$$N(r, \frac{1}{f-a}) = \int_0^r \frac{n(t, \frac{1}{f-a}) - n(0, \frac{1}{f-a})}{t} dt + n(0, \frac{1}{f-a})\log r,$$

where $n(t, \frac{1}{f-a})$ is the number of roots of the equation $f(z) = a$ in $|z| \leq t$, multiple roots being counted multiply and $n(0, \frac{1}{f-a})$ is the multiplicity (order) of zero of $f - a$

at $z=0$ (if $f(0)-a\neq 0$, then $n(0,\frac{1}{f-a})=0$). The function $N(r,\frac{1}{f-a})$ is a counting function of the zero of $f-a$. Definitions of the $m(r,\frac{1}{f-a})$, $\overline{N}(r,\frac{1}{f-a})$, $N_{k)}(r,\frac{1}{f-a})$, $N_{(k+1}(r,\frac{1}{f-a})$, $\overline{N}_{k)}(r,\frac{1}{f-a})$, $\overline{N}_{(k+1}(r,\frac{1}{f-a})$ and $T(r,\frac{1}{f-a})$, can be similarly formulated (see [4]).

Definition 0.9([6, P.14]).

We write $f(z)=o(g(z))$ (with the understanding that z is near some point z_0, possibly ∞, that we are interested in) if $\lim_{z\to z_0}\frac{f(z)}{g(z)}=0$; and $f(z)=O(g(z))$ if $\left|\frac{f(z)}{g(z)}\right|$ is bounded in the neighborhood of z_0.

Definition 0.10([3, P.55]).

Let f be a non-constant meromorphic function in the complex plane. The error function, denoted by $S(r,f)$, is any function satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside a set E of r of finite linear measure. A meromorphic function a is called "small" with respect to f if $T(r,a)=S(r,f)$.

Definition 0.11([4, P.116]).

Two non-constant meromorphic functions f and g share a finite value or small function $a\,(\not\equiv\infty)$ CM (counting multiplicities), if $f-a$ and $g-a$ have the same zeros with the same multiplicities.

Remark 0.0. Let a function g be defined at all points x satisfying $|x|>K$, with some $K>0$. If $\lim_{x\to\infty} g(x)=l$, where l is a finite number, then we define $\overline{\lim}_{x\to\infty} g(x)$ = $\underline{\lim}_{x\to\infty} g(x)=l$. If $\lim_{x\to\infty} g(x)=\infty$ or $-\infty$, then in this case we define $\overline{\lim}_{x\to\infty} g(x)=$ $\underline{\lim}_{x\to\infty} g(x)=\infty$ or $\overline{\lim}_{x\to\infty} g(x)=$ $\underline{\lim}_{x\to\infty} g(x)=-\infty$ respectively. Now, suppose that $\lim_{x\to\infty} g(x)$ does not exists. It follows from theorem ($\lim_{x\to\infty} g(x)=l$ if and only if $\lim_{n\to\infty} g(x_n)=l$, for any sequence $\{x_n\}$ tending to ∞.) that there are two sequences $\{x_n\}$ and $\{x'_n\}$ tends to ∞, but $\{g(x_n)\}$ and $\{g(x'_n)\}$ converging to different limits (may be include ∞ or $-\infty$). We then define a set $L=\{\lim_{n\to\infty} g(x_n)\in\mathbb{R}^* \quad : x_n\to\infty\}$, where $\mathbb{R}^*$ is the extended real number system. If ∞ or $-\infty$ belongs to L, then we define $\overline{\lim}_{x\to\infty} g(x)=\infty$ or $\underline{\lim}_{x\to\infty} g(x)=-\infty$

respectively. Thus we consider only the case in which L is a bounded set. Let $b=\sup L$ and $n=\inf L$. Define $\overline{\lim_{x\to\infty}} g(x)=b$ and $\underline{\lim_{x\to\infty}} g(x)=n$ (see[7, P.1]).

Definition 0.12([3, P.42]).

Let f be a meromorphic function in the complex plane. We define two numbers $d(0,f)$ and $\Theta(\infty,f)$ by

$$d(0,f)=1-\overline{\lim_{r\to\infty}}\frac{N(r,\frac{1}{f})}{T(r,f)}$$ and

$$\Theta(\infty,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}.$$

Remark 0.1. $0\le d(0,f)\le 1$ and $0\le\Theta(\infty,f)\le 1$.

Theorem0.1([3, P.5])(Nevanlinnan's first fundamental theorem). If a is any complex number then $T(r,\frac{1}{f-a})=T(r,f)+O(1)$.

Theorem0.2([4,P.15])(Nevanlinnan's second fundamental theorem). Let f be a non-constant meromorphic function in the complex plane and let $a_1,a_2,\ldots,a_q$ where $q\ge 2$ be a distinct complex numbers. Then

$$(q-1)T(r,f)\le\sum_{j=1}^{q}N(r,\frac{1}{f-a_j})+\overline{N}(r,f)-N(r,\frac{1}{f'})+S(r,f).$$

Theorem 0.3([3, P.55]). For a positive integer k, we have $m(r,\frac{f^{(k)}}{f})=S(r,f)$ and $T(r,f^{(k)})\le(k+1)T(r,f)+S(r,f)$.

1. Introduction

In [8] R. Brück proved the following theorem.

Theorem A. Let f be a non-constant entire function satisfying $N(r,\frac{1}{f'})=S(r,f)$. If f and f' share the value 1 CM, then

$$f-1=c(f'-1), \qquad (1.1)$$

for some nonzero constant c.

The author [1] improved Theorem A and proved the following theorem.

Theorem B. Let f be a non-constant meromorphic function satisfying $N(r,\frac{1}{f'})=S(r,f)$. If f and f' share the value 1 CM, then f satisfies the identity (1.1).

On the other hand L. Liu and Y. Gu [2] proved the following theorem.

Theorem C. Let f be a non-constant meromorphic function and let $a(z)$ $(\not\equiv 0,\infty)$ be a meromorphic small function of f. If $f-a(z)$ and $f^{(k)}-a(z)$ share the value 0 CM and $f^{(k)}$ and $a(z)$ do not have any common poles of same multiplicity and $2d(0,f)+4\Theta(\infty,f)>5$, then $f\equiv f^{(k)}$.

We introduce the following definition.

Definition 1.1. For a positive integer n, we write

$$d_n(0,f)=1-\overline{\lim_{r\to\infty}}\frac{N_n(r,\frac{1}{f})}{T(r,f)},$$

where in $N_n(r, \frac{1}{f})$ a zero of f with multiplicity p is counted with multiplicity $\min(n, p)$.

Remark 1.1. For every positive integer n, we have $0 \le d(0, f) \le d_n(0, f) \le 1$.

The purpose of this paper is to give an improvement and generalization of Theorem B and improvement of Theorem C. In other words, we shall prove the following theorems.

Theorem 1. Let f be a non-constant meromorphic function satisfying $\overline{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$. If f and $f^{(k)}$ $(k \ge 1)$ share the value 1 CM, then

$$f - 1 = c(f^{(k)} - 1), \qquad (1.2)$$

for some nonzero constant c.

It is obvious that Theorem 1 is an improvement and generalization of Theorem B.

Theorem 2. Let f be a non-constant meromorphic function and let $a(z)$ $(\not\equiv 0, \infty)$ be a meromorphic small function of f. If $f - a(z)$ and $f^{(k)} - a(z)$ share the value 0 CM and if $d_2(0, f) + d_{k+1}(0, f) +$

$$3\Theta(\infty, f) > 4 \qquad (1.3)$$

then $f \equiv f^{(k)}$.

It can be seen that Theorem 2 is an improvement of Theorem C.

2. Some Lemmas

For the proof of our results we need the following lemmas.

Lemma 1 [9]. Let f be a non-constant meromorphic function and let $a(\not\equiv 0, \infty)$ be a meromorphic small function of f. If f and $f^{(k)}$ share a CM, and if $\overline{N}(r, f) + \overline{N}(r, \frac{1}{f^{(k)}}) < (\mathbf{l} + o(1))T(r, f^{(k)})$, for some real constant $\mathbf{l} \in (0, \frac{1}{k+1})$, then

$$f - a = (1 - \frac{p_{k-1}}{a})(f^{(k)} - a), \quad (2.1)$$

where p_{k-1} is a polynomial of degree at most $k-1$ and $1 - \frac{p_{k-1}}{a} \not\equiv 0$.

Lemma 2[1]. Let k be a positive integer, and let f be a meromorphic function such that $f^{(k)}$ is not constant. Then either $(f^{(k+1)})^{k+1} = c(f^{(k)} - l)^{k+2}$, for some nonzero constant c, or $kN_{1)}(r, f) \le \overline{N}_{(2}(r, f) + N_{1)}(r, \frac{1}{f^{(k)} - l}) + \overline{N}(r, \frac{1}{f^{(k+1)}}) + S(r, f)$, where l is a constant.

Lemma 3([4, P.75]). Let f_j $(j = 1, 2, \mathbf{K}, n)$ be n linearly independent meromorphic functions, if $\sum_{j=1}^{n} f_j \equiv 1$, then, for $1 \le j \le n$

$$T(r, f_j) \le \sum_{j=1}^{n} N(r, \frac{1}{f_j}) + N(r, f_j) + N(r, D) - \sum_{j=1}^{n} N(r, f_j) - N(r, \frac{1}{D}) + S(r),$$

where D is the Wronskian

determinant $W(f_1, f_2, \mathbf{K}, f_n)$, $S(r) = o(T(r))$ as $r \to \infty$, $r \notin E$ and $T(r) = \max_{1 \le j \le n} \{T(r, f_j)\}$.

Lemma 4([3, P.47]). Let f be a non-constant meromorphic function. a_1, a_2 and a_3 are distinct small functions of f, then

$$T(r, f) \le \sum_{j=1}^{3} \overline{N}(r, \frac{1}{f - a_j}) + S(r, f)$$

3. The proofs

3.1. Proof of Theorem 1

By Lemma 1, there are two cases that we need to observe separately.

Case I. If (2.1) is not true, then

$$T(r, f^{(k)}) \le (k+1)(\overline{N}(r, f) + \overline{N}(r, \frac{1}{f^{(k)}})) + S(r, f). \qquad (3.1)$$

It is easy to see that

$$N(r, f^{(k)}) = N(r, f) + k\overline{N}(r, f). \qquad (3.2)$$

Since $\overline{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$, we see from (3.1), Definition 0.7 and (3.2) that $N_{(2}(r, f) = S(r, f)$ and

$$m(r, f^{(k)}) = S(r, f). \qquad (3.3)$$

Applying Lemma 2 for $l = 0$, which divided into two cases.

Case I.1.

$$kN_{1)}(r, f) \le \overline{N}_{(2}(r, f) + N_{1)}(r, \frac{1}{f^{(k)}}) + \overline{N}(r, \frac{1}{f^{(k+1)}}) + S(r, f)$$

.From this, $\overline{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$ and (3.3) are satisfied, therefore

$$kN_{1)}(r, f) \le \overline{N}(r, \frac{1}{f^{(k+1)}}) + S(r, f). \qquad (3.4)$$

It follows from Theorem 0.2, Theorem 0.1 and (3.3) that

$$N(r, \frac{1}{f^{(k+1)}}) + m(r, \frac{1}{f^{(k)} - 1}) \le N(r, \frac{1}{f^{(k)}}) + N_{1)}(r, f) + S(r, f)$$

Combining this with (3.4) we get

$$N(r, \frac{1}{f^{(k+1)}}) + m(r, \frac{1}{f^{(k)} - 1}) \le N(r, \frac{1}{f^{(k)}}) + \frac{1}{k}\overline{N}(r, \frac{1}{f^{(k+1)}}) + S(r, f)$$

This implies that

$$N^*(r, \frac{1}{f^{(k+1)}}) + m(r, \frac{1}{f^{(k)} - 1}) \le \overline{N}(r, \frac{1}{f^{(k)}}) + \frac{1}{k}\overline{N}_{(2}(r, \frac{1}{f^{(k)}}) + \frac{1}{k}\overline{N^*}(r, \frac{1}{f^{(k+1)}}) + S(r, f), \qquad (3.5)$$

where $N^*(r, \frac{1}{f^{(k+1)}})$ denotes the counting function corresponding to the zeros of $f^{(k+1)}$ that are not zeros of $f^{(k)}$ with the multiple zeros are counted multiplicity times and $\overline{N^*}(r, \frac{1}{f^{(k+1)}})$ denotes that case the multiple zeros are only counted one time. From $\overline{N}(r, \frac{1}{f^{(k)}}) = S(r, f)$ and (3.5) we have

$$N^*(r, \frac{1}{f^{(k+1)}}) + m(r, \frac{1}{f^{(k)} - 1}) \le \frac{1}{k}\overline{N^*}(r, \frac{1}{f^{(k+1)}}) + S(r, f)$$

This inequality reduces to $k=1$, $N^{*}{}_{(2}(r,\frac{1}{f''})=S(r,f)$ and

$$m(r,\frac{1}{f'-1})=S(r,f), \qquad (3.6)$$

It can be obtained from (3.3), (3.2), (3.4), $\overline{N}(r,\frac{1}{f'})=S(r,f)$ and (3.6) that

$$T(r,f')=m(r,f')+N(r,f')$$
$$=2N_{1)}(r,f)+S(r,f)$$
$$\leq 2N^{*}{}_{1)}(r,\frac{1}{f''})+S(r,f). \qquad (3.7)$$

By using exactly the same argument as in [1, P.137-140], we get

$$N^{*}{}_{1)}(r,\frac{1}{f''})=S(r,f). \qquad (3.8)$$

Thus we deduce from Theorem 0.1, f and f' share the value 1 CM, Theorem 0.3, (3.7) and (3.8) that

$$T(r,f)=N(r,\frac{1}{f-1})+m(r,\frac{1}{f-1})$$
$$\leq N(r,\frac{1}{f'-1})+m(r,\frac{f'}{f-1})+$$
$$m(r,\frac{1}{f'})\leq T(r,f')+S(r,f)+$$

$T(r,f')=S(r,f)$, which is a contradiction.

Case I.2. $(f^{(k+1)})^{k+1}=c(f^{(k)})^{k+2}$.

If $f^{(k)}\equiv 0$, then f is a polynomial. So f and $f^{(k)}$ can not share the value 1 CM which contradicts the condition of Theorem 1. Therefore $f^{(k)}\not\equiv 0$ and we rewrite the above equation in the form

$$(\frac{f^{(k+1)}}{f^{(k)}})^{k+1}=cf^{(k)}. \qquad (3.9)$$

By differentiating once,

$$(k+1)(\frac{f^{(k+1)}}{f^{(k)}})^{k}(\frac{f^{(k+1)}}{f^{(k)}})'=cf^{(k+1)}.$$

Combining this with (3.9) we obtain

$$(\frac{f^{(k+1)}}{f^{(k)}})^{-2}(\frac{f^{(k+1)}}{f^{(k)}})'=\frac{1}{k+1}.$$

By integrating once and then using (3.9), we get $f^{(k)}(z)=$

$$\frac{1}{c}[\frac{-(k+1)}{z+c_1(k+1)}]^{k+1}, \qquad (3.10)$$

where $c(\neq 0)$ and c_1 are constants. By integrating k times we deduce that

$$f(z)=\frac{-(k+1)^{k+1}}{ck!(z+c_1(k+1))}+p_{k-1}(z),$$

,where p_{k-1} is a polynomial of degree at most $k-1$. Hence $f(z)-1$ has at most k zeros. But from (3.10) $f^{(k)}-1$ has exactly $k+1$ zeros. This contradicts with the fact f and $f^{(k)}$ share the value 1 CM.

Case II. If (2.1) is true, then $f-1=(1-p_{k-1})(f^{(k)}-1)$, from this we conclude that $N(r,f)=0$. Since f and $f^{(k)}$ share the value 1 CM, $1-p_{k-1}$ should be a constant. Therefore (1.2) holds. The proof of Theorem 1 is complete.

3.2. Proof of Theorem 2

We assume that $f\not\equiv f^{(k)}$. Consider the following function

$$h=\frac{f^{(k)}-a}{f-a}. \qquad (3.11)$$

If $h\equiv c(\neq 1)$ is a constant, then we deduce from (3.11) that

$$\overline{N}(r,f)+\overline{N}_{(k+1}(r,\frac{1}{f})=S(r,f). \quad (3.12)$$

Since $f-a$ and $f^{(k)}-a$ share 0 CM, it follows from (3.12) that

$$\overline{N}(r,\frac{1}{f-a})\le N(r,\frac{1}{\frac{f^{(k)}}{f}-1})\le$$

$$N(r,\frac{f^{(k)}}{f})+S(r,f)\le N_{k)}(r,\frac{1}{f})$$

$$+k\overline{N}_{(k+1}(r,\frac{1}{f})+k\overline{N}(r,f)+S(r,f)$$

$$=N_{k)}(r,\frac{1}{f})+S(r,f).$$

Thus, we get from this, (3.12) and Lemma 4 that

$$T(r,f)\le\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-a})+$$

$$\overline{N}(r,f)+S(r,f)\le\overline{N}_{k)}(r,\frac{1}{f})+$$

$$\overline{N}_{(k+1}(r,\frac{1}{f})+N_{k)}(r,\frac{1}{f})+\overline{N}(r,f)+$$

$$S(r,f)\le N_2(r,\frac{1}{f})+N_{k+1}(r,\frac{1}{f})+$$

$S(r,f)$, from which we get $d_2(0,f)+d_{k+1}(0,f)+3\Theta(\infty,f)$ $\le 1+3=4$. This contradicts (1.3).

In the following, we assume that h is not constant. Writing (3.11) as

$$\frac{f^{(k)}}{a}-\frac{hf}{a}+h=1.$$

Set

$$f_1=\frac{f^{(k)}}{a},\quad f_2=\frac{-hf}{a},\quad f_3=h.$$

Then $\sum_{i=1}^{3}f_i\equiv 1$. We distinguish the following two cases.

Case 1. f_1, f_2, f_3 are three linearly independent meromorphic functions, then by Lemma 3 and Theorem 0.3 we have

$$T(r,f^{(k)})\le N(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{hf})$$

$$+N(r,\frac{1}{h})-N(r,hf)-N(r,h)+$$

$$N(r,D)-N(r,\frac{1}{D})+S(r,f), \quad (3.13)$$

where $D=$

$$\begin{vmatrix}\frac{f^{(k)}}{a} & \frac{-hf}{a} & h\\ (\frac{f^{(k)}}{a})' & (\frac{-hf}{a})' & h'\\ (\frac{f^{(k)}}{a})'' & (\frac{-hf}{a})'' & h''\end{vmatrix}=\begin{vmatrix}1 & \frac{-hf}{a} & h\\ 0 & (\frac{-hf}{a})' & h'\\ 0 & (\frac{-hf}{a})'' & h''\end{vmatrix}$$

$$=(\frac{hf}{a})''h'-(\frac{hf}{a})'h''. \quad (3.14)$$

The poles of D can only occur at the poles of f and h, or zeros of a. Since $f-a$ and $f^{(k)}-a$ share 0 CM, from (3.11) the poles of h can only occur at the poles of f. Furthermore, if z_∞ is a pole of f with multiplicity p and $a(z_\infty)\ne$ $0,\infty$, then z_∞ is a pole with multiplicity k of h and a pole with multiplicity at most $p+2k+3$ of D. Thus (3.13) imply

$$T(r,f^{(k)})\le N(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{f})$$

$$-2k\overline{N}(r,f)-N(r,f)+N(r,f)+ 2k\overline{N}(r,f)+3\overline{N}(r,f)-N(r,\frac{1}{D})+ S(r,f)=N(r,\frac{1}{f^{(k)}})+N(r,\frac{1}{f})+ 3\overline{N}(r,f)-N(r,\frac{1}{D})+S(r,f).$$

From this and Theorem 0.1 we find that

$$m(r,\frac{1}{f^{(k)}})\le N(r,\frac{1}{f})+3\overline{N}(r,f)- N(r,\frac{1}{D})+S(r,f). \qquad ..(3.15)$$

From Theorem 0.3 we have,

$$m(r,\frac{1}{f})=m(r,\frac{f^{(k)}}{f}.\frac{1}{f^{(k)}})\le m(r,\frac{f^{(k)}}{f})+m(r,\frac{1}{f^{(k)}})= m(r,\frac{1}{f^{(k)}})+S(r,f),$$

together with (3.15) we get

$$m(r,\frac{1}{f})\le N(r,\frac{1}{f})+3\overline{N}(r,f)- N(r,\frac{1}{D})+S(r,f).$$ Hence,

$$T(r,f)\le 2N(r,\frac{1}{f})+3\overline{N}(r,f)- N(r,\frac{1}{D})+S(r,f). \qquad ..(3.16)$$

On the other hand, differentiating (3.11) to obtain

$$h'=\frac{f(f^{(k+1)}-a')-a(f^{(k+1)}-f')}{(f-a)^2} -\frac{f^{(k)}(f'-a')}{(f-a)^2}.$$

It can be conclude from this and (3.14) that if z_0 is a zero of f with multiplicity $p\ge k+1$ and $a(z_0)\ne 0,\infty$, then z_0 may be a zero of D with multiplicity at least $2p-k-3$.

Also from (3.14) any zero of f with multiplicity $3\le p\le k$, which is not a zero of a, is a zero of D with multiplicity at least $p-2$. Thus

$$2N(r,\frac{1}{f})-N(r,\frac{1}{D})\le N_2(r,\frac{1}{f})+ N_{k+1}(r,\frac{1}{f})+S(r,f). \qquad (3.17)$$

Combining (3.16) and (3.17) we get

$$T(r,f)\le N_2(r,\frac{1}{f})+N_{k+1}(r,\frac{1}{f})+ 3\overline{N}(r,f)+S(r,f).$$ Hence,

$d_2(0,f)+d_{k+1}(0,f)+3\Theta(\infty,f)\le 4.$ This contradicts with (1.3).

Case 2. f_1, f_2, f_3 are three linearly dependent meromorphic functions. Using an argument similar to that in the proof of [2, Theorem 1], we can arrive at a contradiction. This completes the proof of Theorem 2.

Remark 3.1. We can use Lemma 2 in [10] for another proof of Theorem 2.

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