Bernstein Polynomials Method For Solving Linear Volterra Integral Equation of The Second Kind

Dr. Haleema S. Ali*

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Abstract

In this paper, Bernstein polynomials method are used to find an approximate solution for linear Volterra integral equation of the second kind. These polynomials are incredibly useful mathematical tools, because they are simply defined. It has been shown that the polynomial has a fast convergences with only few steeps. Numerical example is prepared to illustrate the efficiency and accuracy of this method

طريقة متعددة حدود برنشتن لحل معادلة فولتيرة التكاملية الخطية من النوع الثاني

الخلاصة

في هذا البحث استعملت طريقة متعددة حدود برنشتن لإيجاد الحل التقريبي لمعادلة فرولتيرا التكاملية الخطية من النوع الثاني. وأن متعددات الحدود تستعمل بشكل كبير في الطرق الرياضية وذلك لبساطة تعريفها، والحل بهذه الطريقة يتقارب بسرعة وبخطوات قليلة. والمثال العددي أعد ليوضح كفاءة ودقة هذه الطريقة.

1. Introduction

New methods are always needed to solve integral equations because no single method works well for all such equations.

There are considerable interest in solving differential and integral equations using techniques which involve Bernstein polynomials method.

The integral equation is an equation in which the unknown function y(x) appears under the integral sign.

A linear integral equation is an integral equation which involves a linear expression of the unknown function.

The general form of integral equation is given by [1],[5][6],[7]

$$h(x)y(x)-\int_{\Omega}k(x,t)y(t)dt=f(x) \qquad \ldots(1)$$

where $h(x)$, $f(x)$ and the kernel $k(x,t)$ are known functions; $y(x)$ is the function to be determined, and Ω is a finite interval $[a,x]\subseteq R$.

If the upper limit of the integral in equation (1) is variable then equation (1) is called Volterra integral equation.

Now we can distinguish between two types of Volterra integral equations which are:

1. Volterra Integral equation of the first kind when $h(x)=0$ in equation (1).

$$f(x)=-\int_{a}^{x}k(x,t)y(t)dt \qquad \ldots(2)$$

2. Volterra Integral equation of the second kind when $h(x)\neq 0$ (for

*Engineering College, University of Al-Mast./ Baghdad

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2412-0758/University of Technology-Iraq, Baghdad, Iraq

simple of, $h(x)=1)$ in equation (1).

$$y(x)= f(x)+\int_a^x k(x,t)y(t)dt \qquad \dots(3)$$

In this paper, an approximate method is introduced to solve the following linear Volterra integral equation of the second kind by using Bernstein polynomials.

Barghi et al. (2001) studied fluidization regimes in liquid-solid and gas-liquid-solid fluidized beds. The liquid velocities at which regime transition occurs in liquid-solid and gas-liquid-solid systems were

2. Bernstein polynomials method

Polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions.

The Bernstein polynomials of degree n are defined by [3], [4].

$$B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i} \quad \text{for} \quad i=0,1,2,\dots,n \qquad \dots(4)$$

where

$\binom{n}{i}=\frac{n!}{i!(n-i)!}$, (n) is the degree of polynomials, (i) is the index of polynomials and (t) is the variable.

The exponents on the (t) term increase by one as (i) increases, and the exponents on the (1-t) term decrease by one as (i) increases.

The Bernstein polynomial of degree (n) can be defined by blending together two Bernstein polynomials of degree (n-1). That is, the nth degree Bernstein polynomial can be written as, [4].

$$B_k^n(t)=(1-t)B_k^{n-1}(t)+tB_{k-1}^{n-1}(t) \qquad \dots(5)$$

Bernstein polynomials of degree (n) can be written in terms of the power basis. This can be directly calculated using the equation (4) and the binomial theorem as follows, [4].

$$B_k^n(t)=\binom{n}{k}t^k(1-t)^{n-k}=\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{i}\binom{i}{k}t^i$$

Where the binomial theorem is used to Expand $(1-t)^{n-k}$.

3. A Matrix Representation for Bernstein Polynomials

In many applications, a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if only looking at a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis functions [3].

$$B(t)=c_0B_0^n(t)+c_1B_1^n(t)+c_2B_2^n(t)+\dots+c_nB_n^n(t) \qquad \dots(6)$$

It is easy to write this as a dot product of two vectors

$$B(t)=\begin{bmatrix}B_0^n(t) & B_1^n(t) & B_2^n(t) & \cdots & B_n^n(t)\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ \vdots\\ c_n\end{bmatrix} \qquad \dots(7)$$

which can be converted to the following form:

$$B(t)=\begin{bmatrix}1 & t & t^2 & \cdots & t^n\end{bmatrix}\begin{bmatrix}b_{00} & 0 & 0 & \cdots & 0\\ b_{10} & b_{11} & 0 & \cdots & 0\\ b_{20} & b_{21} & b_{22} & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix} \qquad \dots(8)$$

where b_{nn} are the coefficients of the power basis that are used to determine the respective Bernstein polynomials, we note that the matrix in this case lower triangular.

4. Solution of Voltarra integral equation with Bernstein polynomials

In this section, Bernstein polynomials are used to find the approximate solution for Voltarra integral equation, as follows.

Recall Voltarra integral equation of the second kind.

$$y(x) = f(x) + \int_a^x k(x,t)y(t)dt \qquad x \in [a, x] \qquad \ldots (9)$$

Let

$$y(t) = B(t) = \left[B_0^n(t) \;\; B_1^n(t) \;\; B_2^n(t) \ldots B_n^n(t) \right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

by using equation (7), applying the Bernstein polynomials method for equation (9), we get the following formula.

$$\left[B_0^n(t) \;\; B_1^n(t) \ldots B_n^n(t) \right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = f(x) + \int_a^x k(x,t)[B_0^n(t) \;\; B_1^n(t) \ldots B_n^n(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt \qquad \ldots (10)$$

by using equation (8), which can be converted to the following form:

$$\left[1 \; t \ldots t^n \right] \begin{bmatrix} b_{00} & 0 & \ldots & 0 \\ b_{10} & b_{11} & \ldots & 0 \\ b_{20} & b_{21} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = f(x) + \int_a^x k(x,t) \left[1 \; t \ldots t^n \right] \begin{bmatrix} b_{00} & 0 & \ldots & 0 \\ b_{10} & b_{11} & \ldots & 0 \\ b_{20} & b_{21} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt \qquad \ldots (11)$$

now to find all integration in equation(11).

Then in order to determine $c_0, c_1, \ldots, c_n$, we need n equations;

Now Choice $x_i, i = 1,2,3,\ldots n$ in the interval [a, b], which give (n) equations.

Solve the (n) equations by Gauss elimination to find the values $c_0, c_1, \ldots, c_n$.

The following algorithm summarizes the steps for finding the approximate solution for the second kind of linear Voltarra integral equation.

5.Algorithm(BP-IE)(Bernstein polynomials in linear Volterra integral equation)

Input: $(f(t), k(t,s), y(s), a, t),$

Output: polynomials of degree n

Step1:

Choice n the degree of Bernstein polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

for $i = 0,1,2,\ldots,n$

Step2:

Put the Bernstein polynomials in linear Volterra integral equation of second kind.

$$\left[B_0^n(t) \;\; B_1^n(t) \ldots B_n^n(t) \right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = f(x) + \int_a^x k(x,t)[B_0^n(t) \;\; B_1^n(t) \ldots B_n^n(t)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} dt$$

Step3:

Compute

$$\int_a^x k(x,t) \left[1 \; t \ldots t^n \right] \begin{bmatrix} b_{00} & 0 & 0 & \ldots & 0 \\ b_{10} & b_{11} & 0 & \ldots & 0 \\ b_{20} & b_{21} & b_{22} & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n0} & b_{n1} & b_{n2} & & b_{nn} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} dt$$

Step4:

Compute $c_0, c_1, \ldots, c_n$, where $x_i, i = 1,2,3,\ldots,n$, $x_i \in [a,b]$

End:

6. Numerical Examples:

Example(1)

Consider the following linear Volterra integral equation of the second kind:

$$y(x) = 1 - \int_0^x 2t\, y(t)dt$$

with the exact solution $y(x) = e^{-x^2}$

Now to derive the solution by using the Bernstein polynomials method, we can use the following scheme:

When Bernstein polynomials algorithm is applied. And choice the degree of Bernstein polynomials n=2, we get:

$$c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2$$
$$= 1 - \int_0^x 2t\left[c_0(1-t)^2 + 2c_1t(1-t) + c_2t^2\right]dt$$

Next

$$c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2$$
$$= 1 - \left(2c_0\int_0^x t(1-t)^2dt + 4c_1\int_0^x t^2(1-t)dt + 2c_2\int_0^x t^3dt\right)$$

And after performing the integration.

$$c_0(1-x)^2 + 2c_1x(1-x) + c_2x^2$$
$$= 1 - c_0[x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4] - c_1[\frac{4}{3}x^3 - x^4] - c_2[\frac{1}{2}x^4]$$

Then in order to determine c_0, c_1 *and* c_2, we need three equations;

Now Choice $x_i, i = 1,2,3$ in the interval [0, 1], which give three equations.

$$c_0 = 1$$
$$\frac{1}{6}c_0 + \frac{1}{3}c_1 + \frac{3}{4}c_2 = 1$$
$$\frac{35}{96}c_0 + \frac{29}{48}c_1 + \frac{9}{32}c_2 = 1$$

Solve the three equation by Gauss elimination to find the values c_0, c_1 *and* c_2 as follows

$$c_0 = 1,$$
$$c_1 = 0.8846,$$
$$c_2 = 0.3590$$

Then the solution of linear Volterra integral equation of the second kind is:

$$y(x) = (c_0 - 2c_1 + c_2)x^2 - 2((c_0 - c_1)x + c_0$$
$$y(x) = -0.4102x^2 - 0.2308x + 1$$

Approximated solution for some values of (x) by using Bernstein polynomials method and exact values $y(x) = e^{-x^2}$ of the example1, depending on the least square error (L.S.E) is presented in Table(1) and figure(1).

Example(2)

Consider the following linear Volterra integral equation of the second kind:

$$y(x) = x + \int_0^x (t - x)\, y(t)dt$$

with the exact solution $y(x) = \sin(x)$

Approximated solution for some values of (x) by using Bernstein polynomials method and exact values $y(x) = \sin(x)$ of the example(2), depending on the least square error (L.S.E) is presented in Table(2) and figure(2).

7. Conclusions

This paper presents the use of the Bernstein polynomials method, for solving linear Voltarra integral equation of the second kind. From solving some numerical examples the following points have been identified:

1. This method can be used to solve all kinds of linear Voltarra integral equation.
2. It is clear that using the Bernstein polynomial basis function to approximate when the n^th degree of Bernstein polynomial increases the error is decreases.

3. We can see also from Figure(1) and Figure(2) that the approximation is good. The curve, which represents the approximate solution almost coincide with the analytic solution.

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Table (1) The results of Example(1)

X	*Exact* $y(x)$	Approximation $y(x)$ of degree(n=2)	$Error = (y_{Exact}(x) - y_{Approximation}(x))^2$
0	1	1	0
0.1	0.9900	0.9728	0.000296
0.2	0.9608	0.9374	0.000548
0.3	0.9139	0.8938	0.000404
0.4	0.8521	0.8420	0.000102
0.5	0.7788	0.7821	1.09E-05
0.6	0.6977	0.7138	0.000259
0.7	0.6126	0.6374	0.000615
0.8	0.5273	0.5528	0.00065
0.9	0.4449	0.4600	0.000228
1	0.3679	0.3590	7.92E-05
L.S.E			0.003192

Table (2) The results of Example (2)

X	*Exact* $y(x)$	Approx-Imation of degree (n=1)	Approx-mation of degree (n=2)	$Error = (y_{Ex}(x) - y_{App}(x))^2$
0	0	0	0	0
0.1	0.0998	0.0857	0.1051	2.8117e-005
0.2	0.1987	0.1714	0.2056	4.7813e-005
0.3	0.2955	0.2571	0.3013	3.3917e-005
0.4	0.3894	0.3428	0.3924	8.9860e-006
0.5	0.4794	0.4285	0.4788	3.9130e-007
0.6	0.5646	0.5143	0.5605	1.7193e-005
0.7	0.6442	0.6000	0.6375	4.5074e-005
0.8	0.7174	0.6857	0.7098	5.6732e-005
0.9	0.7833	0.7714	0.7775	3.4468e-005
1	0.8415	0.8571	0.8404	1.1470e-006
			L.S.E	2.7384e-004

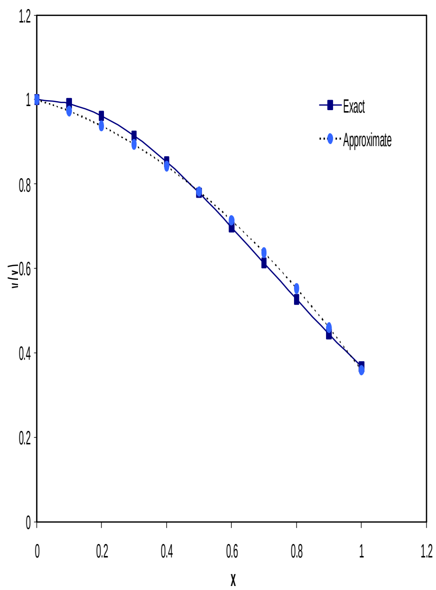

Figure (1)
Approximation and Exact solution of linear Volterra integral equation of Example1

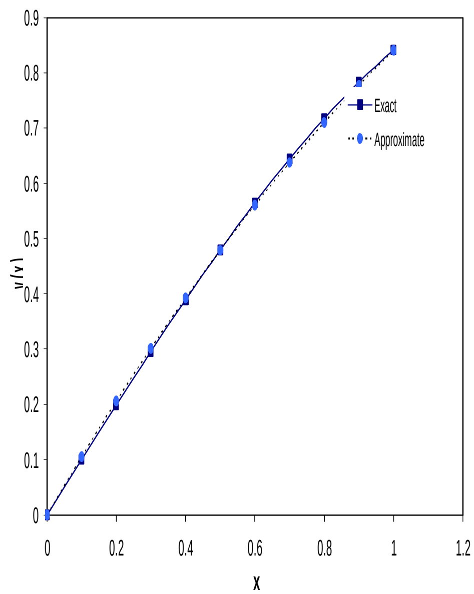

Figure (2)
Approximation and Exact solution of linear Volterra integral equation of Example2