T-Semi Connected Spaces

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Abstract

In this paper, we introduce a new concept, namely T-semi connected space, where T is an operator associated with the topological T defined on a non-empty set X. Several properties of this concept are proved.

Keywords: Semi connected, T-semi connected.

الفضاءات شبه المتصلة-T

الخلاصة

في هذا البحث، قدمنا مفهوماً جديداً إلا وهو مفهوم الفضاء شبه المتصل-T حيث T هو مؤثر مرتبط بالتبولوجي τ المعرف على مجموعة غير خالية X. قد برهنت عدة خصائص لهذا النوع من الفضاءات.

1- Introduction:

In [4], the concept of semi connected set is given. In this paper, we introduce the concept of T-semi connected set and show that it generalizes the concept of semi connected set when the operator T is the identity operator.

Throughout this paper, we use the following notations: cl(A) denotes the usual closure and int(A) denotes the interior of a set A.

2- Basic Definitions and Results:

In this section, we recall and introduce the basic definitions needed in this work.

Definition (2.1):

Let (X, τ) be a topological space and let T be an operator associated with τ, (X, τ, T) is called an operator topological space, [2].

Let $A \subseteq X$, we say that A is T-semi open if there exists an open set $U \in \tau$, such that:

$U \subseteq A \subseteq T(U)$

The complement of a T-semi open set is called a T-semi-closed set.

Remarks (2.2):

(i) If T is the closure operator (T(A) = cl(A)), the above definition agrees with the definition of semi-open set which is given by Levin in [1].

(ii) If T is the identity operator (T(A) = A) the definition of T-semi-open set agrees with the definition of open set.

(iii) For any operator T, each open set is T-semi-open set.

Example

Let (X, τ, T) be an operator topological space, where T is the closure operator and T – semi open set will be the usual semi – open set (the usual semi open set

$G \subseteq W \subseteq \bar{G} = CL(G)$ and T-semi open set $G \subseteq W \subseteq T(G)$)

Consider X = (IR, tu) and A= [0,1) is the T – semi open set (T is the

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closure operator) but A is not open set.

Before we state the next proposition, we recall the following definition:

Definition (2.3), [2]:

Let (X, τ, T) be an operator topological space. We say that T is a monotone operator if for every pair of open sets U and V, such that $U \subset V$, we have that $T(U) \subset T(V)$.

The closure operator is a monotone operator.

Proposition (2.4):

Let (X, τ, T) be an operator topological space, where T is a monotone operator. Let $A \subseteq X$, A is T-semiopen set if and only if $A \subseteq T(\text{int}(A))$.

Proof:

Suppose that $A \subseteq T(\text{int}(A))$

Notice that:

$\text{int}(A) \subseteq A \subseteq T(\text{int}(A))$

which means that A is T-semiopen set

Now, suppose that A is T-semiopen set

Then there exists an open set U, such that $U \subseteq A \subseteq T(U)$

Now, $U \subseteq A \longrightarrow U \subseteq \text{int}(A)$ [where int $U \subseteq \text{int}(A)$ but U is open set, $U=$ int $U \longrightarrow U \subseteq \text{int}(A)$]

So, $T(U) \subseteq T(\text{int}(A))$, since T is a monotone operator

So, $A \subseteq T(\text{int}(A))$. <

Remark (2.5):

If T is the closure operator, then A is T-semiopen set if and only if $A \subseteq cl(\text{int}(A))$, and this agrees with the equivalence given by Levin in [1].

Proposition (2.6):

Let (X, τ, T) be an operator topological space, where T is the closure operator and let $S \subseteq X$. Then S is T-semi closed if and only if there exists a closed subset F of X, such that $\text{int}(F) \subseteq S \subseteq F$.

Proof:

Suppose that:

$\text{Int}(F) \subseteq S \subseteq F$ (F is closed)

Then $F^c \subseteq S^c \subseteq (\text{int}(F))^c$

Now, $(\text{int}(F))^c = cl(F^c)$

So, $F^c \subseteq S^c \subseteq cl(F^c)$

Which means that S^c is T-semiopen. Hence S is T-semiclosed.

Conversely, if S is T-semiclosed

Then S^c is T-semiopen

Therefore, there exists an open set U, such that:

$U \subseteq S^c \subseteq cl(U) = T(U)$

So $(cl(U))^c \subseteq S \subseteq U^c$

But $(cl(U))^c = \text{int}(U^c)$

So, $\text{int}(U^c) \subseteq S \subseteq U^c$. <

Remark (2.7): [3]

Let (X, τ, T) be an operator topological space, where T is the closure operator. Then a subset S of X is T-semiclosed if and only if $\text{int}(T(S)) \subseteq S$.

Definition (2.8):[3]

Let (X, τ, T) be an operator topological space, where T is a monotone operator and let $S \subseteq X$, then the union of all T-semi open sets contained in the set S is T-semi open, and it is denoted by T-sInt(S).

We show this as follows :

$F = \{ w_\alpha : \alpha \in U\}$

w_α is the T- semi open

$G_\alpha \subseteq w_\alpha \subseteq T(G_\alpha)$

$\bigcup G_\alpha \subseteq \bigcup w_\alpha \subseteq \bigcup T(G_\alpha)$,

$[\bigcup T(G_\alpha) = T(\bigcup G_\alpha) = T(G)]$

$G \subseteq \bigcup w_\alpha \subseteq T(G)$

Remark (2.9):[3]

If T is a monotone operator, the intersection of all T-semi closed sets of X containing the set S is T-semi closed. It is called the T-semi closure of S and is denoted by T-scl(S).

So, S is T-semi closed if and only if T-scl(S) = S.

3. Main Results:

In this section, several properties and characterizations of T-semi connected spaces are given.

Definition (3.1):

[See Willard [4]and Def (3.1) is analogous to the definition given in Willard]

(i) Let (X, τ, T) be an operator topological space.
Let $\varnothing \neq A \subseteq X$, $\varnothing \neq B \subseteq X$
We say that A and B are T-semi separated if:
$(T - scl(A)) \cap B = A \cap (T - scl(B)) = \varnothing$

(ii) Let $W \subseteq X$, we say that W is T-semi disconnected if W can be expressed as the union of two T-semi separated sets

(iii) Let $W \subseteq X$, we say that W is T-semi connected if W is not T-semi disconnected, that is, if W cannot be expressed as the union of two T-semi separated sets.

(iv) (X, τ, T) is said to be T-semi connected if and only if X is T-semi connected.

Remark (3.2):

Let (X, τ, T) be an operator topological space, where T is the identity operator (In this case we write (X, τ) instead of (X, τ, T)) let $A \subseteq X$, $B \subseteq X$. The definition of T-semi separated sets agrees with the definition of separated sets in the usual sense

$cl(A) \cap B = A \cap cl(B) = \varnothing$

and therefore the definition of T-semi connected set generalizes the definition of connected set.

Example:

Let (X, τ, T) be an operator topological space consider X= (IR , t_u) and $Y = [0,1]$ **U** $[2,3]$ then Y is not connected but Y is T- semi connected
(where T is the closure operator).

Theorem (3.3):

Let (X, τ, T) be an operator topological space. Then X is T-semi connected if and only if the only subsets of X that are both T-semi open and T-semi closed in X are the empty set and X itself.

Proof:

Let $A \subseteq X$ be a non-empty proper subset of X, which is both T-semi open and T-semi closed in X, then the sets $U = A$ and $V = A^c$ constitute a T-semi separation of X. So X will be T-semi disconnected.

Conversely, if U and V form T-semi separation of X and $X = U \cup V$, then U is a non-empty and different from X, since:

$U \cap V \subseteq U \cap (T\text{–}scl(V))$
$= (T\text{–}scl(U)) \cap V = \varnothing$

We obtain that both sets U and V are T-semi open and T-semi closed. <

Theorem (3.4):

Let (X, τ, T) be an operator topological space, if $A \subseteq X$ is a T-semi connected and $A \subseteq C \cup D$, where C and D are T-semi separated sets, then either $A \subseteq C$ or $A \subseteq D$.

Proof:

$A = A \cap (C \cup D)$
$= (A \cap C) \cup (A \cap D)$

Since C and D are T-semi separated sets

$C \cap (T\text{–}scl(D)) = \varnothing$

so

$(A \cap C) \cap (T\text{–}scl(A)) \cap (T\text{–}scl(D)) \subseteq C \cap (T\text{–}scl(D)) = \varnothing$

So if both $A \cap C \neq \varnothing$ and $A \cap D \neq \varnothing$, then A is T-semi disconnected

This shows that either $A \cap C = \varnothing$, or $A \cap D = \varnothing$

So $A \subseteq C$, or $A \subseteq D$. <

The proof of the following theorem is clear:

Theorem (3.5):

Let (X, τ, T) be an operator topological space, then the union E of any family $\{C_i \mid i \in I\}$ of T-semi connected sets having a non-empty intersection is T-semi connected.

Theorem (3.6):

Let (X, τ, T) be an operator topological space, and let $C \subseteq X$, $E \subseteq X$. If C is T-semi connected and

$C \subset$ (T-scl(E)) $\subseteq$ T-scl(C)

Then (T-scl(E)) is T-semi connected.

Proof:

Suppose that T-scl(E) is T-semi disconnected

So T-scl(E) $= A \cup B$, where $\varnothing \neq A$, $\varnothing \neq B$

$A \cap$ (T-scl(B)) $= B \cap$ (T-scl(A)) $= \varnothing$

Now, $C \subseteq A \cup B$ and C is T-semi connected, so:

$C \subseteq A$ or $C \subseteq B$

Let us assume that $C \subseteq A$

Now, T-scl(C) $\subseteq$ T-scl(A)

Therefore:

T-scl(C)$\cap$B $\subseteq$ T-scl(A) $\cap$ B $=\varnothing$

But B $\subseteq$ T-scl(E) $\subseteq$ T-scl(C)

So $B = \varnothing$

Hence, T-scl(E) is T-semi connected. <

Corollary (3.7):

Let (X, τ, T) be an operator topological space, let $C \subseteq X$. If C is T-semi connected, then T-scl(C) is also T-semi connected.

Proof:

C $\subseteq$ T-scl(C) $\subseteq$ T-scl(C)

So by theorem (3.6), we get that T-scl(C) is T-semiconnected. <

4. References:

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