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RESEARCH ARTICLE - MATHEMATICS

On Fuzzy Nano ζ -Generalized Closed Set in Fuzzy Nano Symmetric Topology

Gazwan Haider Abdulhusein[1*] , Dalia raad abd[2]

[1] Department of Mathematics, Open Educational College, Ministry of Education, Baghdad, Iraq

[2] Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq

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Abstract

In this paper we introduced new types of fuzzy nano open sets is said to fuzzy ζ- nano open set and studied some properties of fuzzy ζ- nano generalized closed set in fuzzy nano symmetric topology also we studied some separation axioms is called (FN $\zeta - T_0$S , FN $\zeta - T_{\frac{1}{2}}$S, FN $\zeta - T_1$S) and we studied the relationship between of them under the condition of fuzzy nano ζ-symmetric topology and proof the converse relations which was studied for the first time, and obtained several important properties.

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***Keywords*: :_** fuzzy ζ-nano open set (FN ζ-OS) , fuzzy ζ- nano generalized closed set (FNG ζ-CS), fuzzy nano symmetric topology (FNSYT), FN $\zeta - T_0$S , FN $\zeta - T_{\frac{1}{2}}$S, FN $\zeta - T_1$S

1. Introduction

Nano topology is substantial science in engineering also in medical science. had become applications of Nano science and Nano structures very important. This concept introduced by L.Thivagar and C.Richard **[1] [2]** in 2013. On the other hand C. L. Chang **[2]** **[3]**in 1968 , introduced and study the topological spaces based on the fuzzy set introduced by Zadeh in its paper . In (2011) S.S.Benchalli and Kandil, R. lowen **[4]**, **[5]**, **[6]** introduce the definition of fuzzy regular open set in fuzzy topological space. And in (2014) Bhuvaneswari K **[7]**, **[8]**, **[9]**[**[10]** introduced the nano generalized closed set in nano topological space. So in this work we extended the fuzzy topological group to fuzzy Nano topological group and we introduced new types of fuzzy nano open set is called fuzzy **ζ** - nano open set and studied some properties of fuzzy **ζ**-nano generalized closed set in fuzzy nano symmetric topology also we study some separation axioms is called (**FN** $\boldsymbol{\zeta - T_0}$**S** , **FN** $\boldsymbol{\zeta - T_{\frac{1}{2}}}$**S, FN** $\boldsymbol{\zeta - T_1}$**S**) and we studied the relationship between of them under the condition of fuzzy nano **ζ**-symmetric topology and proof the converse relations which was studied for the first time, and obtained several important properties.

2. Some Definitions:

Definition 2.1 [9], [12]:

Let $X \neq \emptyset$, a fuzzy set $\hat{S}$ in X is characterize by a function $\mu_{\hat{S}} : X \rightarrow I$, where $I = [0,1]$ and write as $\hat{S} = \{(x , \mu_{\hat{S}}(x)) : x \in X , 0 \leq \mu_{\hat{S}}(x) \leq 1\}$, the collection of all fuzzy set in X refer as I^X, that is $I^X = \{ \hat{S} : \hat{S}$ is a fuzzy sets inX$\}$ where $\mu_{\hat{S}}$ is called the memberships functions .

Definition 2.2 [10], [14]**:**

Let $\hat{Q}$ and $\widehat{W}$ be fuzzy sets in X with memberships μ_Q and μ_W , then ∀ x ∈ X:

1. $\hat{Q} \subseteq \widehat{W} \Leftrightarrow \mu_{\tilde{Q}}(x) \leq \mu_{\widehat{W}}(x)$.
2. $\hat{Q} = \widehat{W} \Leftrightarrow \mu_{\tilde{Q}}(x) = \mu_{\widehat{W}}(x)$.
3. $\hat{S} = \hat{Q} \cap \widehat{W} \Leftrightarrow \mu_{\hat{S}}(x)=$ mini{ $\mu_{\tilde{Q}}(x)$, $\mu_{\widehat{W}}(x)$ }.
4. $\hat{E} = \hat{Q} \cup \widehat{W} \Leftrightarrow \mu_{\hat{E}}(x)=$ maxi{ $\mu_{\tilde{Q}}(x)$, $\mu_{\widehat{W}}(x)$}.

Definition 2.3 [12], [13] [14]**:**

A fuzzy points x_p is a fuzzy sets such that:

$\mu_{x_p}(y) = p > 0$ if $x \doteq y$, for all y ∈ X ,and

$\mu_{x_p}(y) = 0$ if $x \neq y$, for all y ∈ X

The collections of fuzzy points of X will be sympliezed By FP(X).

Definition 2.4 [15]**:**

Let $\mathfrak{U} \neq \emptyset$ be a finite of obj. said to be universes and $\widehat{R}$ be fuzzy equivalent relations on U said to fuzzy discernibility's relation. Then the pair $(\mathfrak{U}, \widetilde{R})$ is called fuzzy approximations space .

Remark 2.5 :

Element belonging the same fuzzy equivalence class are said to be fuzzy indiscernibility with one another

Definition 2.6 :

Let $\widehat{X}$ be a fuzzy set and $\widehat{R}[x]$ be the fuzzy equivalence class determine by x∈ U, then

1- The fuzzy lower approximation of $\widehat{X}$ is defined by $L_R(\widehat{X})$ ={ $\max_{x \in U}\{\widehat{R}[x] : \mu_{\widehat{R}[x]} \leq \mu_{\widehat{X}}(x)\}$

2- The fuzzy upper approximation of $\widehat{X}$ is defined by $U_R(\widehat{X}) = \max_{x \in U}\{\widehat{R}[x] : \min\{\mu_{\widehat{R}[x]} , \mu_{\widehat{X}}(x) \neq 0\}$

3- The fuzzy boundary region of $\widehat{X}$ is defined by $\acute{B}_R(\widehat{X}) = U_R(\widehat{X}) - L_R(\widehat{X})$

Definition 2.7 :

take $\mathfrak{U}$ be a nonempty finite sets, $\widehat{R}$ be fuzzy equivalences relations on $\mathfrak{U}$, $\widehat{X} \leq \mathfrak{U}$ be a fuzzy subset then

$\tilde{T}_R(\widehat{X}) = \{\widehat{X} , \tilde{\phi}, L_R(\widehat{X}), U_R(\widehat{X}), B_R(\widehat{X})\}$ is named be fuzzy nanotopology on $\widehat{X}$ if hold

1. $\acute{X}$, $\acute{\phi} \in \tilde{T}_R(\tilde{x})$
2. If $\acute{Q}$, $\widehat{M} \in \tilde{T}_R(\tilde{x})$, then $Q \cap \acute{M} \in \tilde{T}_R(\tilde{x})$
3. If $\acute{Q}_\alpha \in \tilde{T}_R(\tilde{x})$, then $\cup_\alpha \acute{Q}_\alpha \in \tilde{T}_R(\tilde{x})$

$(\hat{X}, \tilde{T}_R(\tilde{x}))$ is said to be Fuzzy nano topological space (FNTS) and every element of $\tilde{T}_R(\tilde{x})$ is said to be fuzzy nano open set (FNOS) in $\hat{X}$ and its complement is a fuzzy nanoclosed set (FNCS) .

Definition 2.8 [16] [17] [18] [19]
suppos Q , Ḿ be a fuzzy nano set in a FNTS $(\hat{X}, \tilde{T}_R(\tilde{x}))$ then :

- A fuzzy points x_p is called quasi coincidente (QC) with the fuzzy nanoset Q if $x_p + Q > 1$ and write as x_p q Q , and if $x_r + Q \leq 1$, then x_p isn't quasi coincidente with a fuzzy nano set Q and write as $x_p \tilde{\bar{q}}$ Q.
- A fuzzy nano set Q is said to be quasi coincidente with a fuzzy nano set Ḿ if Q + Ḿ $>$ 1 and write as Q q Ḿ , and if Q + Ḿ $\leq$ 1, then Q isn't quasi coincidente with a fuzzy nano set Ḿ and write as Q $\tilde{\bar{q}}$ Ḿ.

3. Fuzzy Nano ζ-Generalized Closed Set

Definition 3.1 :

A fuzzy subset $\hat{\mathfrak{D}}$ in a FNTS $(\hat{X}, \tilde{T}_R(\tilde{x}))$ is called fuzzy nano regular open set (FNROS) if

$\hat{\mathfrak{D}}$ = NInt(NCl($\hat{\mathfrak{D}}$)) and the fuzzy nano complement of fuzzy nano regular open set is fuzzy nano regular close set (FNRCS).

Definition 3.2 :

If $\hat{\mathfrak{D}}$ be a fuzzy nano in a FNTS $(\hat{X}, \tilde{T}_R(\tilde{x}))$

Nano R-Int ($\hat{\mathfrak{D}}$) = $\bigcup\{ \hat{\mathfrak{J}} \in$ (FNROS) : $\hat{\mathfrak{J}} \subseteq \hat{\mathfrak{D}}\}$ is called the fuzzy nano regular interior of $\hat{\mathfrak{D}}$ (FNR-Int ($\hat{\mathfrak{D}}$)) and Nano R-Cl ($\hat{\mathfrak{D}}$) = $\bigcap\{ \hat{\mathfrak{J}} \in$ (FNRCS) : $\hat{\mathfrak{J}} \supseteq \hat{\mathfrak{D}}\}$ is called the fuzzy nano regular closure of $\hat{\mathfrak{D}}$ (FNR-Cl ($\hat{\mathfrak{D}}$))

Definition 3.3 :

A fuzzy subset $\hat{\mathfrak{D}}$ in a FNTS $(\hat{X}, \tilde{T}_R(\tilde{x}))$ is called fuzzy nano ζ-open set (FN ζ-OS) if

$\hat{\mathfrak{D}} \subseteq$ FNInt(FNCl(FNR-Int($\hat{\mathfrak{D}}$))) and the fuzzy nano complements of fuzzy nano ζ-open set is fuzzy nano ζ-close set (FN ζ-CS).

Definition 3.4 :

A $\hat{\mathfrak{D}}$ be a fuzzy nano in a FNTS $(\hat{X}, \tilde{T}_R(\hat{X}))$

Nano ζ-Int ($\hat{\mathfrak{D}}$) = $\bigcup\{ \hat{\mathfrak{J}} \in$ (FN ζ-OS) : $\hat{\mathfrak{J}} \subseteq \hat{\mathfrak{D}}\}$ is called the fuzzy nano ζ-interior of $\hat{\mathfrak{D}}$ and

Nano ζ-Cl ($\hat{\mathfrak{D}}$) = $\bigcap\{ \hat{\mathfrak{J}} \in$ (FN ζ-CS) : $\hat{\mathfrak{J}} \supseteq \hat{\mathfrak{D}}\}$ is called the fuzzy nano ζ-closure of $\hat{\mathfrak{D}}$ and denoted by

FN ζ-Int ($\hat{\mathfrak{D}}$) (resp. FN ζ-Cl ($\hat{\mathfrak{D}}$))

Theorem 3.5 :

Let $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ be a FNTS and let $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{D}}$ be two fuzzy nano sets in $\widehat{X}$.

Then these hold;

1- $\widehat{\mathfrak{B}} \subseteq$ FN ζ-Cl $(\widehat{\mathfrak{B}})$

2- $\widehat{\mathfrak{B}} \in$ FN ζ-CS if and only if $\widehat{\mathfrak{B}}$ = FN ζ-Cl $(\widehat{\mathfrak{B}})$

3- if $\widehat{\mathfrak{B}} \subseteq \widehat{\mathfrak{D}}$ then FN ζ-Cl $(\widehat{\mathfrak{B}}) \subseteq$ FN ζ-Cl $(\widehat{\mathfrak{D}})$

4- FN ζ-Cl $(\widehat{\mathfrak{B}} \cup \widehat{\mathfrak{D}})$ = FN ζ-Cl $(\widehat{\mathfrak{B}}) \cup$ FN ζ-Cl $(\widehat{\mathfrak{D}})$

5- FN ζ-Cl $(\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}}) \subseteq$ FN ζ-Cl $(\widehat{\mathfrak{B}}) \cap$ FN ζ-Cl $(\widehat{\mathfrak{D}})$

Theorem 3.6 :

Let $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ be a FNTS and let $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{D}}$ be fuzzy nano ζ-open sets in $\widehat{X}$, then $\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}}$ is fuzzy nano ζ-open set

Proof:-

Let $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{D}}$ be a fuzzy nano ζ-open sets, then

$\widehat{\mathfrak{B}} \subseteq$ FNInt(FNCl(FNR-Int($\widehat{\mathfrak{B}}$) and $\widehat{\mathfrak{D}} \subseteq$ FNInt(FNCl(FNR-Int($\widehat{\mathfrak{D}}$)))

$\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}} \subseteq$ FNInt(FNCl(FNR-Int($\widehat{\mathfrak{B}}$) $\cap$ FNInt(FNCl(FNR-Int($\widehat{\mathfrak{D}}$)))

$\subseteq$ FNInt(FNInt(FNCl(FNR-Int($\widehat{\mathfrak{B}}$) $\cap$ (FNCl(FNR-Int($\widehat{\mathfrak{D}}$))))

$\subseteq$ FNInt(FNCl(FNInt(FNCl(FNR-Int($\widehat{\mathfrak{B}}$))) $\cap$ (FNR-Int($\widehat{\mathfrak{D}}$)))))

$\subseteq$ FNInt(FNCl(FNR-Int(FNR-Cl(FNR-Int($\widehat{\mathfrak{B}}$))) $\cap$ (FNR-Int($\widehat{\mathfrak{D}}$)))))

= FNInt(FNCl(FNR-Int(FNR-Cl(FNR-In($\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}}$)))))

= FNInt(FNCl(FNR-Int($\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}}$)))

Hence $\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}} \subseteq$ FNInt(FNCl(FNR-Int($\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}}$))) and $\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{D}}$ is fuzzy nano ζ-open set

Definition 3.7 :

A fuzzy nano subset $\widehat{\mathfrak{D}}$ of FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is called fuzzy nano ζ-Symmetric if for every fuzzy points x_p , y_t in $\widehat{X}$, $x_p \in$ FN ζ-Cl(y_t) implies $y_t \in$ FN ζ-Cl(x_p) (i.e $x_p \in$ FN ζ-Cl(y_t) implies FN ζ-Cl(y_t) = FN ζ-Cl(x_p))

Definition 3.8 :

A fuzzy nano subset $\widehat{\mathfrak{D}}$ of FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is called fuzzy nano ζ-generalized closed set (FNζ-gC) in $\widehat{X}$, if FN ζ-Cl($\widehat{\mathfrak{D}}$) ⊆ $\widehat{U}$, whenever $\widehat{\mathfrak{D}} \subseteq \widehat{U}$ and $\widehat{U}$ ∈ FN ζ-OS.

Theorem 3.9 :

A FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is fuzzy nano ζ-Symmetrics if and only if for every fuzzy point x_p in $\widehat{X}$ is fuzzy nano ζ-generalized close set.

Proof:-

let $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is fuzzy nano ζ-Symmetric and suppose that $x_p \in \widehat{U}$ ∈ FN ζ-OS

And FN ζ-Cl(x_p) ⊈ $\widehat{U}$. This yiled that there is fuzzy point y_t in $\widehat{X}$ such that

y_t ∈ FN ζ-Cl(x_p) ∩ $\widehat{U}^c$, then y_t ∈ FN ζ-Cl(x_p) and $y_t \in \widehat{U}^c$, that is

FN ζ-Cl(y_t) ⊆ FN ζ-Cl($\widehat{U}^c$) = $\widehat{U}^c$, since $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is fuzzy nano ζ-Symmetric and

y_t ∈ FN ζ-Cl(x_p), then x_p ∈ FN ζ-Cl(y_t) ⊆ $\widehat{U}^c$. But this contradiction with $x_p \in \widehat{U}$

Hence FN ζ-Cl(x_p) ⊆ $\widehat{U}$

let for any fuzzy point x_p in $\widehat{X}$ is fuzzy nano ζ-generalized close set.

Suppose that x_p ∈ FN ζ-Cl(y_t) and y_t ∉ FN ζ-Cl(x_p), that is y_t ∈ 1- FN ζ-Cl(x_p)

Since 1- FN ζ-Cl(x_p) ∈ FN ζ-OS, and y_t fuzzy nano ζ-generalized closed set, then

FN ζ-Cl(y_t) ⊆ 1- FN ζ-Cl(x_p). this implies that x_p ∈ 1- FN ζ-Cl(x_p) ∈ 1- x_p, this is a contradiction, hence $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is fuzzy nano ζ-Symmetrics.

Theorem 3.10 :

If $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ be a FNTS and let $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{D}}$ be two fuzzy nano ζ-generalized closed sets in $\widehat{X}$, then $\widehat{\mathfrak{B}} \cup \widehat{\mathfrak{D}}$ is fuzzy nano ζ-generalized closed sets

Proof:-

let $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{D}}$ be two fuzzy nano ζ-generalized close sets in $\widehat{X}$, then FN ζ-Cl($\widehat{\mathfrak{B}}$) ⊆ $\widehat{U}$, whenever $\widehat{\mathfrak{B}} \subseteq \widehat{U}$ and

$\widehat{U}$ ∈ FN ζ-OS and FN ζ-Cl($\widehat{\mathfrak{D}}$) ⊆ $\widehat{U}$, whenever $\widehat{\mathfrak{D}} \subseteq \widehat{U}$ and $\widehat{U}$ ∈ FN ζ-OS.

Since $\widehat{\mathfrak{B}}$ and $\widehat{\mathfrak{D}}$ are subset of $\widehat{U}$, $\widehat{\mathfrak{B}} \cup \widehat{\mathfrak{D}}$ are subset of $\widehat{U}$ and $\widehat{U}$ ∈ FN ζ-OS

Then FN ζ-Cl($\widehat{\mathfrak{B}} \cup \widehat{\mathfrak{D}}$) = FN ζ-Cl($\widehat{\mathfrak{B}}$) ∪ FN ζ-Cl($\widehat{\mathfrak{D}}$) ⊆ $\widehat{U}$, which implies that $\widehat{\mathfrak{B}} \cup \widehat{\mathfrak{D}}$ is fuzzy nano ζ-generalized closed sets.

Theorem 3.11 :

Let ($\widehat{X}$, $\tilde{T}_R(\widehat{X})$) be a FNTS and let $\widehat{\mathfrak{B}}$ be fuzzy nano ζ-generalized closed set and $\widehat{\mathfrak{B}} \subseteq \widehat{\mathfrak{D}} \subseteq$ FN ζ-Cl($\widehat{\mathfrak{B}}$) , then $\widehat{\mathfrak{D}}$ is fuzzy nano ζ-generalized closed set

Proof:-

$\widehat{\mathfrak{D}} \subseteq \widehat{U}$ and $\widehat{U} \in$ FN ζ-OS in a fuzzy nano topological space ($\widehat{X}$, $\tilde{T}_R(\widehat{X})$), then $\widehat{\mathfrak{B}} \subseteq \widehat{\mathfrak{D}}$ implies

$\widehat{\mathfrak{B}} \subseteq \widehat{U}$, since $\widehat{\mathfrak{B}}$ be fuzzy nano ζ-generalized closed, FN ζ-Cl($\widehat{\mathfrak{B}}$) ⊆ $\widehat{U}$, and $\widehat{\mathfrak{D}} \subseteq$ FN ζ-Cl($\widehat{\mathfrak{B}}$)

Then FN ζ-Cl($\widehat{\mathfrak{D}}$) ⊆ FN ζ-Cl($\widehat{\mathfrak{B}}$), thus FN ζ-Cl($\widehat{\mathfrak{D}}$) ⊆ $\widehat{U}$, $\widehat{U} \in$ FN ζ-OS, hence $\widehat{\mathfrak{D}}$ is FN ζ-gC.

Theorem 3.12 :

In a fuzzy nano topological space ($\widehat{X}$, $\tilde{T}_R(\widehat{X})$) every FN ζ-CS is FN ζ-gC

Proof:-

Let $\widehat{\mathfrak{B}} \subseteq \widehat{U}$ and $\widehat{U} \in$ FN ζ-OS, since $\widehat{\mathfrak{B}}$ is FN ζ-CS, then FN ζ-Cl($\widehat{\mathfrak{B}}$) ⊆ $\widehat{\mathfrak{B}}$, that is

FN ζ-Cl($\widehat{\mathfrak{B}}$) ⊆ $\widehat{\mathfrak{B}} \subseteq \widehat{U}$, hence $\widehat{\mathfrak{B}}$ is FN ζ-gC.

4. Fuzzy Nano ζ-T_i , i = 0 , 1 , 1/2 , In Fuzzy Nano ζ-Symmetric Topology

Definition 4.1 :

A FNTS ($\widehat{X}$, $\tilde{T}_R(\widehat{X})$) is said to be

1- fuzzy nano ζ-T_0 space (FN ζ-T_0S) if for every fuzzy points x_p, y_p in $\widehat{X}$, $x_p \neq y_p$

∃ $\widehat{\mathfrak{B}} \in$ FN ζ-OS such that x_p q $\widehat{\mathfrak{B}} \leq 1- y_p$ or y_p q $\widehat{\mathfrak{B}} \leq 1- x_p$

2- fuzzy nano ζ-T_1 space (FN ζ-T_1S) if for every fuzzy points x_p, y_p in $\widehat{X}$, $x_p \neq y_p$

∃ $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{D}} \in$ FN ζ-OS such that x_p q $\widehat{\mathfrak{B}} \leq 1- y_p$ and y_p q $\widehat{\mathfrak{D}} \leq 1- x_p$

3- fuzzy nano ζ-$T_{1/2}$ space (FN ζ-$T_{1/2}$S) if every FN ζ-gC is FN ζ-C

Theorem 4.2 :

Let ($\widehat{X}$, $\tilde{T}_R(\widehat{X})$) be FNTS if ($\widehat{X}$, $\tilde{T}_R(\widehat{X})$) is FN ζ-T_iS then it is FN ζ-T_{i-1}S where i=0,1

Proof;- Clear.

Theorem 4.3 :

A FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_0S iff for every pair of distance fuzzy points x_p, y_p in $\widehat{X}$, FN ζ-Cl(x_p) ≠ FN ζ-Cl(y_p)

Proof;- Obviouss

Theorem 4.4 :

A FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1S iff for every singleton fuzzy points x_p,in $\widehat{X}$, FN ζ-CS

Proof:-

⇒ Suppose$(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1S, let $y_p \leq$ 1- x_p , then ∃ $\widehat{\mathfrak{B}}, \widehat{\mathfrak{D}}$ ∈ FN ζ-OS such that x_p q $\widehat{\mathfrak{B}} \leq$ 1- y_p and y_p q $\widehat{\mathfrak{D}} \leq$ 1- x_p, if y_p q $\widehat{\mathfrak{D}} \leq$ 1- x_p, we have $\widehat{\mathfrak{D}} \leq$ 1- x_p, let $\hat{Q} = \cup \{\widehat{\mathfrak{D}} : y_p$ q 1- $x_p\}$ hence $\hat{Q}$ = 1- x_p , and 1- x_p FN ζ-OS , and x_p FN ζ-CS

⇐ let $x_p \neq y_p$ and x_p, y_p are FN ζ-CS, then 1- x_p, 1- y_p FN ζ-OS, hence y_p q 1- $x_p \leq$ 1- x_p and

x_p q 1- $y_p \leq$ 1- y_p , therefore $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1S.

Proposition 4.5 :

If a FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1 then it is fuzzy nano ζ-Symmetric

Proof:-

Let $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1then by theorem 3.4 every fuzzy point is FN ζ-CS,

and by theorem 2.12 It is FN ζ-gC and by theorem 2.9 $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is fuzzy nano ζ-Symmetric

Proposition 4.6 :

A FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is fuzzy nano ζ-Symmetric and FN ζ-T_0 if and only if it is FNζ-T_1S

Proof:-

⇒ If $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1S, then by theorem 3.2 and proposition 3.5 it is fuzzy nano ζ-Symmetric and FN ζ-T_0

⇐ suppose that $x_p \neq y_p$ be a fuzzy point in a FNTS $(\widehat{X}, \tilde{T}_R(\widehat{X}))$, and since it is FN ζ-$T_0$

Then ∃ $\widehat{\mathfrak{B}}$ ∈ FN ζ-OS such that x_p q $\widehat{\mathfrak{B}} \leq$ 1- y_p, $x_p \notin$ FN ζ-Cl(y_p), and these implies that by fuzzy nano ζ-Symmetric $y_p \notin$ FN ζ-Cl(x_p), hence y_p q (1- FN ζ-Cl(x_p)) ≤ 1- FN ζ-Cl(x_p) ≤ 1- x_p, hence $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1S

Theorem 4.7 :

For fuzzy nano ζ-Symmetrics topological space $(\widehat{X}, \tilde{T}_R(\widehat{X}))$, the these properties are equivalents:

1- $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_0S

2- $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-$T_{1/2}$S

3- $(\widehat{X}, \tilde{T}_R(\widehat{X}))$ is FN ζ-T_1S

Proof:- Obvious.

Conclusions

This paper investigates three objectives, the first objective we introduced new types of fuzzy nano open set is called fuzzy ζ- nano open set, Also we studied fuzzy ζ-nano generalized closed set , The second objective we introduced new concept is called fuzzy nano ζ-Symmetric topological space, And the third objective we proof the equivalent relation between fuzzy nano separation axioms (FN ζ-T_0S, FN ζ-$T_{1/2}$S, FN ζ-T_1S) under the condition of fuzzy nano ζ-Symmetric topological space which was studied for the first time.

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