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RESEARCH ARTICLE - MATHEMATICS

Linear Formulas for the Methods of Estimating for Exponentiated Inverse Rayleigh Distribution

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Abstract

The two parameters (μ, ω) and the reliability function of the Exponentiated Inverse Rayleigh distribution (EIRD) were estimated in this paper by using four estimation methods which are: Maximum Likelihood (MLE),White (W), Modified White (MW) and Linear Regression (REG) methods. Using simulation to generate the required data on three experiments(E_1, E_2, E_3) of the default value of the parameters, with different sample sizes (m=10,50,100) and repeated 1000 times. The results were compared by using the mean square error criterion, the result appears as follows: The Maximum Likelihood estimator is the best for estimating the exponentiated parameter with reliability in most of the experiments that were used and the Linear Regression estimator is the best in most of the experiments that were used to estimating the scale parameter.

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1. Introduction

In (1880) Rayleigh introduced The Rayleigh distribution. The Rayleigh distribution is one of the important continuous statistical distributions in life applications. It has been studied by many researchers like: In (2014), Merovci introduced transmuted generalized Rayleigh distribution [1]. In (2020), Reyah and Kareema introduced truncated Rayleigh Pareto distribution [2]. Gadde Srinivasa Rao and Sauda Mbwambo [3] introduced and studied the Exponentiated Inverse Rayleigh distribution. The probability density function (p.d.f), the cumulative density function (c.d.f), reliability, and hazard function of the Exponentiated Inverse Rayleigh distribution is given by[3]:

$$f(x;\mu,\omega) = \frac{2\mu\omega^2}{x^3} e^{-(\omega/x)^2}\left(1-e^{-(\omega/x)^2}\right)^{\mu-1} \quad (1)$$

$$F(x;\mu,\omega) = 1-\left(1-e^{-(\omega/x)^2}\right)^{\mu}; x \geq 0\,, \mu > 0\,, \omega > 0 \quad (2)$$

$$R(x) = 1-F(x) = (1-e^{-(\omega/x)^2})^{\mu} \quad (3)$$

$$h(x) = \frac{f(x)}{R(x)} = 2\mu\omega^2 x^{-3} e^{-(\omega/x)^2}(1-e^{-(\omega/x)^2})^{-1} \quad (4)$$

where $\geq 0\ \mu > 0, \omega > 0$, μ and ω are scale and exponentiated parameters.
The Inverse Distribution Function can be obtained from equation (2), as follows:

$$x(F) = \frac{\omega}{\sqrt{-\ln(1-(1-F(x))^{1/\mu})}} \quad (5)$$

2. Estimation methods

In this section, we will derive estimation methods for the Exponentiated Inverse Rayleigh distribution for two parameters and the reliability function.

2.1. Estimates of initial values for parameters

This method is based on equating the median of the generated sample with the median of the distribution,
as follows: [4]
The median formula for the (EIRD)[3] is:

$$x_{med} = \frac{\omega}{\sqrt{-ln\left(1-(0.5)^{1/\mu}\right)}} \quad (6)$$

from equation (6), the initial formulas can be obtained as follows:

$$\widehat{\omega}_0 = \sqrt{-x_{med}^2 ln(1-(0.5)^{1/\mu})} \quad (7)$$

Now,

$$ln(1-(0.5)^{1/\mu}) = -(\frac{\omega}{x_{med}})^2 \quad (8)$$

By taking exponential for equation (8), getting:

$$(0.5)^{1/\mu} = 1 - e^{-(\omega/x)^2} \quad (9)$$

Then

$$\hat{\mu}_0 = \frac{ln(0.5)}{ln\left(1-e^{-(\omega/x_{med})^2}\right)} \quad (10)$$

So $\widehat{\omega}_0$ and $\hat{\mu}_0$ in equations (7) and (10) are the initial values for parameters**,** and x_{med} it is the median value of the generated sample.

2.2. Maximum Likelihood Method (MLE):

In (1922) R.A. Fisher introduced method of maximum likelihood[5], which estimates parameters by using the probability density function and taking its likelihood and natural logarithm. Let $x_1, x_2, \ldots, x_m$ represent a random sample of size m drawn from the EIR distribution, then the likelihood function for the distribution can be calculated [3]:

$$Lf = 2^m \mu^m \omega^{2m} (\prod_{i=1}^m x_i^{-3}) e^{-\sum_{i=1}^m (\omega/x_i)^2} \prod_{i=1}^m (1-e^{-(\omega/x_i)^2})^{\mu-1} \quad (11)$$

The natural logarithm for equation (11) is

$$lnLf = m\, ln\, 2 + m\, ln\, \mu + 2m\, ln\, \omega - 3\sum_{i=1}^m ln\, xi - \sum_{i=1}^m \left(\frac{\omega}{x_i}\right)^2 + (\mu-1)\sum_{i=1}^m ln(1-e^{-(\omega/x_i)^2})$$

(12)

Now taking the partial derivative of equation (12) concerning μ, ω and equating to zero, we obtain

$$\frac{\partial lnLf}{d\mu} = \frac{m}{\mu} + \sum_{i=1}^m ln(1-e^{-(\omega/x_i)^2}) = 0$$

$$\hat{\mu}_{MLE} = \frac{-m}{\sum_{i=1}^m ln(1-e^{-(\omega/x_i)^2})} \quad (13)$$

$$\frac{\partial lnLf}{d\omega} = \frac{2m}{\omega} - 2\omega \sum_{i=1}^m x_i^{-2} + 2\omega(\mu-1)\sum_{i=1}^m \frac{e^{-(\omega/x_i)^2}}{x_i^2(1-e^{-(\omega/x_i)^2})}$$

$$\widehat{\omega}_{MLE} = \frac{\frac{m}{\omega} + \omega(\mu-1)\sum_{i=1}^m \frac{e^{-(\omega/x_i)^2}}{x_i^2(1-e^{-(\omega/x_i)^2})}}{\sum_{i=1}^m x_i^{-2}} \quad (14)$$

Substituting equations (13) and (14) into equation (3), getting:

$$\hat{R}_{MLE} = (1-e^{-(\widehat{\omega}_{MLE}/t)^2})^{\hat{\mu}_{MLE}} \quad (15)$$

2.3. White Method (W)

In (1980) Hilbert White proposed the White method [6]. The idea of this method depends on converting the reliability function $R(x)$ to a linear regression formula as follows[7]:

Since, $(x-y)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} x^{n-k} y^k$

So, $(1-e^{-(\omega/x)^2})^\mu = \sum_{k=0}^\mu (-1)^k \binom{\mu}{k} (e^{-(\omega/x)^2})^k$

$$\Rightarrow (1-e^{-(\omega/x)^2})^{\mu} = 1-\mu e^{-(\omega/x)^2} + \sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k$$

Then equation (3), become:

$$R(x) = 1-\mu e^{-(\omega/x)^2} + \sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k \quad (16)$$

$$\Rightarrow 1+\sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k - R(x) = \mu e^{-(\omega/x)^2} \quad (17)$$

By taking the natural logarithm for equation (17), getting:

$$\ln(1+\sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}\left(e^{-(\omega/x)^2}\right)^k - R(x)) = \ln\mu - \frac{\omega^2}{x^2} \quad (18)$$

The plotting position formula is:

$$p_i = \frac{i}{m+1}, i = 1,2,\dots,m \quad (19)$$

Equating equation (2) with equation (19), getting:

$$F(x_i) = \frac{i}{m+1}, i = 1,2,\dots,m$$

Since,

$$R(x_i) = 1 - F(x_i)$$

Then:

$$R(x_i) = 1-\frac{i}{m+1}, i = 1,2,\dots,m \quad (20)$$

The formula for linear regression is:

$$\gamma_i = \alpha + \delta\varphi_i + \epsilon_i \quad (21)$$

By comparing the equation (18) with the equation (21), getting:

$$\gamma_i = \ln\left(1+\sum_{k=2}^{\mu_0}(-1)^k\binom{\mu_0}{k}\left(e^{-\left(\omega_0/x_{(i)}\right)^2}\right)^k - R\left(x_{(i)}\right)\right), \bar{\gamma} = \frac{\sum_{i=1}^{m}\gamma_i}{m}, i = 1,2,\dots,m \quad (22)$$

$$\alpha = \ln\mu \rightarrow \mu = e^{\alpha} \quad (23)$$

$$\delta = \omega \quad (24)$$

$$\varphi_i = \frac{-\omega}{x_{(i)}{}^2}, i = 1,2,\dots,m \quad (25)$$

$$\hat{\varphi}_i = \frac{-\omega_0}{x_{(i)}{}^2}, \bar{\varphi} = \frac{\sum_{i=1}^{m}\hat{\varphi}_i}{m}, i = 1,2,\dots,m \quad (26)$$

The formula to find the approximate value of (δ) by the White method is:

$$\hat{\delta}_W = \frac{\sum_{i=1}^{m}(\hat{\varphi}_i-\bar{\varphi})(\gamma_i-\bar{\gamma})}{\sum_{i=1}^{m}(\hat{\varphi}_i-\bar{\varphi})^2} \quad (27)$$

Now substituting equations (22) and (26) into equation (27), getting:

$$\hat{\omega}_W = \hat{\delta}_W \quad (28)$$

From equation (21), getting:

$$\hat{\alpha}_W = \bar{\gamma} - \hat{\delta}_W\bar{\varphi} \quad (29)$$

Substituting equation (29) into (23), getting:

$$\hat{\mu}_W = e^{\bar{\gamma}-\hat{\delta}_W\bar{\varphi}} \quad (30)$$

By substituting equations (28) and (30) into equation (3), getting:

$$\hat{R}_W = (1 - e^{-(\hat{\omega}_W/t)^2})^{\hat{\mu}_W} \quad (31)$$

2.4. Modified White Method(MW)

Modified White method suggested by Makki (2006) [8]. This method depends on converting the hazard function formula $h(x)$ into an equation like the linear regression formula[9].

From equation (4), getting:

$$\frac{1}{h(x)} = \frac{x^3}{2\mu\omega^2} e^{(\omega/x)^2}(1 - e^{-(\omega/x)^2}) \quad (32)$$

$$\frac{1}{h(x)} = \frac{x^3}{2\mu\omega^2} e^{(\omega/x)^2} - \frac{x^3}{2\mu\omega^2}$$

$$\frac{1}{h(x)} + \frac{x^3}{2\mu\omega^2} = \frac{x^3}{2\mu\omega^2} e^{(\omega/x)^2}$$

$$\frac{1}{\frac{1}{h(x)} + \frac{x^3}{2\mu\omega^2}} = 2\mu\omega^2 x^{-3} e^{-(\omega/x)^2} \quad (33)$$

Now by taking the natural logarithm for equation (33), getting:

$$\ln\left(\frac{1}{\frac{1}{h(x)} + \frac{x^3}{2\mu\omega^2}}\right) = \ln(2) + \ln(\mu) + 2\ln(\omega) - 3\ln(x) - \frac{\omega^2}{x^2} \quad (34)$$

$$\ln\left(\frac{1}{\frac{1}{h(x)} + \frac{x^3}{2\mu\omega^2}}\right) - \ln(2) - 2\ln(\omega) + 3\ln(x) = \ln(\mu) - \frac{\omega^2}{x^2} \quad (35)$$

By comparing equation (35) with equation (21), getting:

$$\gamma_i = \ln\left(\frac{1}{\frac{1}{h_0\left(x_{(i)}\right)} + \frac{{x_{(i)}}^3}{2\mu_0{\omega_0}^2}}\right) - \ln(2) - 2\ln(\omega_0) + 3\ln\left(x_{(i)}\right), \bar{\gamma} = \frac{\sum_{i=1}^m \gamma_i}{m}, i = 1,2,\dots,m \quad (36)$$

Whereas: $h_0\left(x_{(i)}\right) = 2\mu_0\omega_0^2 x_{(i)}^{-3}(1 - \left(1 - \frac{i}{m+1}\right)^{1/\mu_0})(\frac{1}{\left(1-\frac{i}{m+1}\right)^{1/\mu_0}})$

$$\alpha = \ln(\mu) \;\rightarrow\; \mu = e^{\alpha} \quad (37)$$

$$\delta = \omega \quad (38)$$

$$\varphi_i = \frac{-\omega}{{x_{(i)}}^2} \quad , i = 1,2,\dots,m \quad (39)$$

$$\hat{\varphi}_i = \frac{-\omega_0}{{x_{(i)}}^2}, \bar{\varphi} = \frac{\sum_{i=1}^m \hat{\varphi}_i}{m}, i = 1,2,\dots,m \quad (40)$$

Now substituting equations (36) and (40) into equation (27), getting:

$$\hat{\omega}_{MW} = \hat{\delta}_{MW} \quad (41)$$

By using equation (21), getting:

$$\hat{\alpha}_{MW} = \bar{\gamma} - \hat{\delta}_{MW}\bar{\varphi} \quad (42)$$

Substituting equation (42) into equation (37), getting:

$$\hat{\mu}_{MW} = e^{\bar{\gamma}-\hat{\delta}_{MW}\bar{\varphi}} \quad (43)$$

By substituting equations (41) and (43) into equation (3), getting:

$$\hat{R}_{MW} = (1 - e^{-(\hat{\omega}_{MW}/t)^2})^{\hat{\mu}_{WM}} \quad (44)$$

2.5. Linear Regression method (REG)

This method is based on converting the cumulative distribution function formula $F(x)$ into a linear regression formula [10].

Since, $(x - y)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{n-k}y^k$

So, $(1 - e^{-(\omega/x)^2})^{\mu} = \sum_{k=0}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k$

$\Rightarrow (1 - e^{-(\omega/x)^2})^{\mu} = 1 - \mu e^{-(\omega/x)^2} + \sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k$

Then equation (2), become:

$1 - F(x) = 1 - \mu e^{-(\omega/x)^2} + \sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k$

$$\Rightarrow F(x) + \sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k = \mu e^{-(\omega/x)^2} \quad (45)$$

Now by taking the natural logarithm for equation (45), getting:

$$\ln(F(x) + \sum_{k=2}^{\mu}(-1)^k\binom{\mu}{k}(e^{-(\omega/x)^2})^k) = \ln\mu - \frac{\omega^2}{x^2} \quad (46)$$

By comparing equation (46) with equation (21), getting:

$$\gamma_i = \ln\left(F(x) + \sum_{k=2}^{\mu_0}(-1)^k\binom{\mu_0}{k}\left(e^{-(\omega_0/x_{(i)})^2}\right)^k\right), \bar{\gamma} = \frac{\sum_{i=1}^{m}\gamma_i}{m} \quad i = 1,2,\dots,m \quad (47)$$

$$\alpha = \ln\mu \rightarrow \mu = e^{\alpha} \quad (48)$$

$$\delta = \omega^2 \rightarrow \omega = \delta^{1/2} \quad (49)$$

$$\varphi_i = \frac{-1}{{x_{(i)}}^2},\ i = 1,2,\dots,m \quad (50)$$

$$\hat{\varphi}_i = \frac{-1}{{x_{(i)}}^2}, \bar{\varphi} = \frac{\sum_{i=1}^{m}\hat{\varphi}_i}{m} \quad i = 1,2,\dots,m \quad (51)$$

The formula to find the approximate value of (δ) by Linear Regression method is:

$$\hat{\delta}_{REG} = \frac{\sum_{i=1}^{m}(\hat{\varphi}_i - \bar{\varphi})(\gamma_i)}{\sum_{i=1}^{m}(\hat{\varphi}_i - \bar{\varphi})^2} \quad (52)$$

Now substituting equations (47) and (51) into equation (52), getting:

$$\hat{\omega}_{REG} = \hat{\delta}_{REG}^{1/2} \quad (53)$$

From equation (21), getting:

$$\hat{\alpha}_{REG} = \bar{\gamma} - \hat{\delta}_{REG}\bar{\varphi} \quad (54)$$

Substituting equation (54) into (48), getting:

$$\hat{\mu}_{REG} = e^{\bar{\gamma}-\hat{\delta}_{REG}\bar{\varphi}} \quad (55)$$

By substituting equations (53) and (55) into equation (3), getting:

$$\hat{R}_{REG} = (1 - e^{-(\hat{\omega}_{REG}/t)^2})^{\hat{\mu}_{REG}} \quad (56)$$

3. Simulation

In this section focuses on estimating the two parameters and reliability function of the Exponentiated Inverse Rayleigh distribution (EIRD) as follows:

- Choose different sample sizes: small, medium and large as m=10,50 and100. With repeated 1000 times.
- To find the value of two parameters and reliability estimated through the equations numbered (13,14,28,30,41,43,53 ,55 ,15,31,44 and 56).
- Table 1 shows the experiments for the default values of the two parameters.

Table 1. The default values of the parameters.

Exp. → Par.↓	E_1	E_2	E_3
μ	1	4	0.5
ω	4	1	0.5

- Finally comparing the results by using mean squares error (MSE), as follows:
  $MSE(\hat{\rho}) = \frac{\sum_{i=1}^{m}(\hat{\rho}_i-\rho)^2}{M}$, where ρ is any parameter and $M = 1000$.

4. Numerical Results

The results of the estimators are shown in the tables below

Table 2. MSE values using(E_1).

Estimators	m	MLE	W	MW	REG	Best
$\hat{\mu}$	10	0.058118	0.087943	0.185114	0.049919	REG
$\hat{\omega}$		2.801728	2.680485	2.527448	0.733260	REG
$\hat{R}$		0.000846	0.006765	0.005100	0.000586	REG
$\hat{\mu}$	50	0.011522	0.017136	0.036926	0.015861	MLE
$\hat{\omega}$		0.487690	0.714628	0.705000	0.183507	REG
$\hat{R}$		0.000114	0.001552	0.001325	0.000193	MLE
$\hat{\mu}$	100	0.005850	0.009986	0.019878	0.008508	MLE
$\hat{\omega}$		0.236477	0.383628	0.382079	0.084864	REG
$\hat{R}$		0.000030	0.000815	0.000771	0.000062	MLE

Table 3. MSE values using(E_2).

Estimators	m	MLE	W	MW	REG	Best
$\hat{\mu}$	10	0.887711	5.550171	5.894006	5.602888	MLE
$\hat{\omega}$		0.204880	0.344797	0.098466	0.151013	MW
$\hat{R}$		0.009111	0.126469	0.009652	0.021780	MLE
$\hat{\mu}$	50	0.212428	5.510028	1.198727	5.522177	MLE
$\hat{\omega}$		0.028223	0.332780	0.027980	0.145112	MW
$\hat{R}$		0.000776	0.104841	0.002376	0.018948	MLE
$\hat{\mu}$	100	0.108324	5.495092	0.572319	5.432257	MLE
$\hat{\omega}$		0.015223	0.322028	0.013520	0.142359	MW
$\hat{R}$		0.000291	0.097359	0.000798	0.017351	MLE

Table 4. MSE values using(E_3).

Estimators	m	MLE	W	MW	REG	Best
$\hat{\mu}$	10	0.014263	0.092208	0.049922	0.089605	MLE
$\hat{\omega}$		0.070554	0.110506	0.077696	0.023332	REG
$\hat{R}$		0.000474	0.005741	0.005236	0.008230	MLE
$\hat{\mu}$	50	0.003162	0.065056	0.010049	0.062005	MLE
$\hat{\omega}$		0.012770	0.025282	0.020111	0.004435	REG
$\hat{R}$		0.000095	0.005616	0.001093	0.006786	MLE
$\hat{\mu}$	100	0.001425	0.061692	0.004894	0.060967	MLE
$\hat{\omega}$		0.005453	0.015346	0.010750	0.002794	REG
$\hat{R}$		0.000023	0.005281	0.000523	0.006661	MLE

- The results of the first experiment when the parameter value of μ is less than ω we note that: REG is better in most of the sample size, $\hat{\mu}_{REG}$ is better in (m=10), but $\hat{\mu}_{MLE}$ is better in other samples, $\hat{\omega}_{REG}$ is better in all samples and $\hat{R}_{REG}$ is better in (m=10), but $\hat{R}_{MLE}$ is better in other samples.
- The results of the second experiment when the parameter value of ω is less than μ we note that: MLE is better in most of the sample size, $(\hat{\mu}_{MLE})$ is better in all samples, $(\hat{\omega}_{MW})$ is better in all samples and $(\hat{R}_{MLE})$ is better in all samples.
- The results of the third experiment when the parameter values are equal we note that: MLE is better in most of the sample size, $(\hat{\mu}_{MLE})$ is better in all samples, $(\hat{\omega}_{REG})$ is better in all samples and $(\hat{R}_{MLE})$ is better in all samples.

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